The momentum spectrum versus the probability flux direction in quantum mechanics

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Abstract

Simple examples that the direction of the momentum, defined by its spectral support in a given state, and the direction of the probability flux may be locally uncorrelated in 1+1 dimensional space-time are given for the Schrödinger and the Dirac particle with non-zero mass. For the zero-mass Dirac particle, they are always correlated.

1. Introduction

Direction of a particle motion

classical mechanics: direction of the momentum

quantum mechanics:

(1) direction of the momentum mean value

(2) spectral support of the momentum

(3) direction of the probability flux

Characteristics (2) and (3) need not be locally correlated.

Motivation

Scattering of particles along a thin infinite channel, modeled by a delta interaction supported by a curve. Direction of the asymptotic motion need to be defined. A temporary attempt to use (3) failed as it does not follow from (2). We demonstrate that on a simple example here.

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Scattering of particles bounded to an infinite planar curve,
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2. Schrödinger equation

A free quantum particle moving on a line $\mathbb{R}$,
Hilbert space of states $L^2(\mathbb{R})$,
Hamiltonian $H = -\partial_x^2$ with the domain $\mathcal{D}(H) = H^{2,2}(\mathbb{R})$.
Units where the Planck constant $\hbar = 1$ and the particle mass $m = \frac{1}{2}$.

The momentum operator $P = -i\partial_x$, $\mathcal{D}(P) = H^{1,2}(\mathbb{R})$.
After the Fourier transformation, multiplication operators

$$\hat{P} = p \quad , \quad \hat{H} = p^2 \quad .$$

The spectral projectors of P acts as

$$(\varphi, E_P(M)\psi) = \int_M \overline{\hat{\varphi}(p)}\hat{\psi}(p)dp \ ,$$

the time evolution of the state $\phi(0,x) = \varphi(x)$

$$\phi(t,x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{i(px-p^2t)} \hat{\varphi}(p) dp \quad .$$

The support of the measure $(\varphi, E_P(\cdot)\varphi)$ is the essential support of $\hat{\varphi}$. If contained in $[0,+\infty)$, we may consider the particle as moving to the right and also

$$(\phi(t,\cdot), P\phi(t,\cdot) = (\varphi, P\varphi) > 0 \quad .$$

The momentum mean value is constant in time as the momentum P commutes with the free Hamiltonian H; a special case of the Ehrenfest theorem.

For $\varphi \in \mathcal{D}(H)$, we can define probability density ρ and probability flux j

$$\rho(t,x) = |\phi(t,x)|^2 \quad , \quad j(t,x) = 2\Im\left(\overline{\phi(t,x)}\,\partial_x \phi(t,x)\right)$$

satisfying the equation of continuity

$$\partial_t \rho(t,x) + \partial_x j(t,x) = 0 \ .$$

We construct an example of φ such that

$$\mathrm{supp}\,(\varphi, E_P(\cdot)\varphi) \subset [0,\infty) \quad \text{but} \quad j(t,x) < 0$$

for some range of variables t and x.
Let us choose (normalization skipped)

$$\hat{\varphi}(p) = \sqrt{2\pi}\chi_{(M_1,M_2)}(p) + i\sqrt{2\pi}\chi_{(N1,N2)}(p)$$

where $0 \le M_1 < M_2 < N_1 < N_2 < \infty$ and $\chi_{(m_1,m_2)}$ is the characteristic function of the interval (m_1, m_2).
In the Figures, $M_1 = 0$, $M_2 = 1$, $N_1 = 2$, $N_2 = 3$.

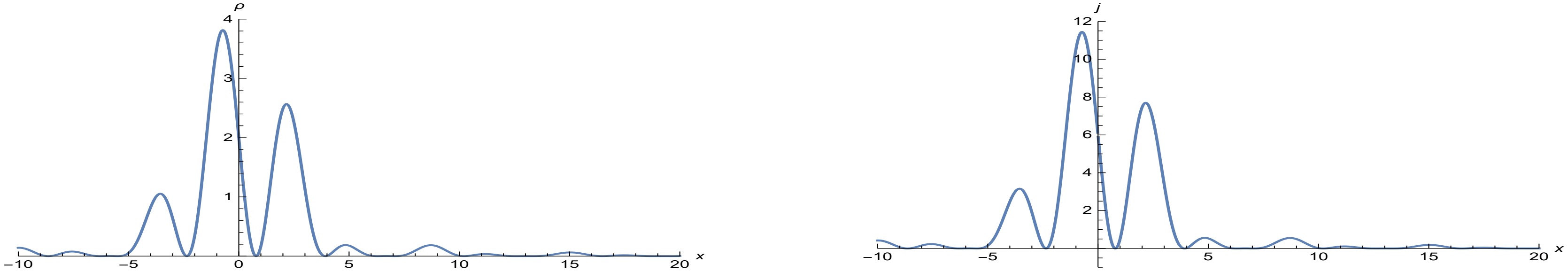

Figure 1: Probability density $\rho(t,x)$ and current $j(t,x)$ at the time t=0.

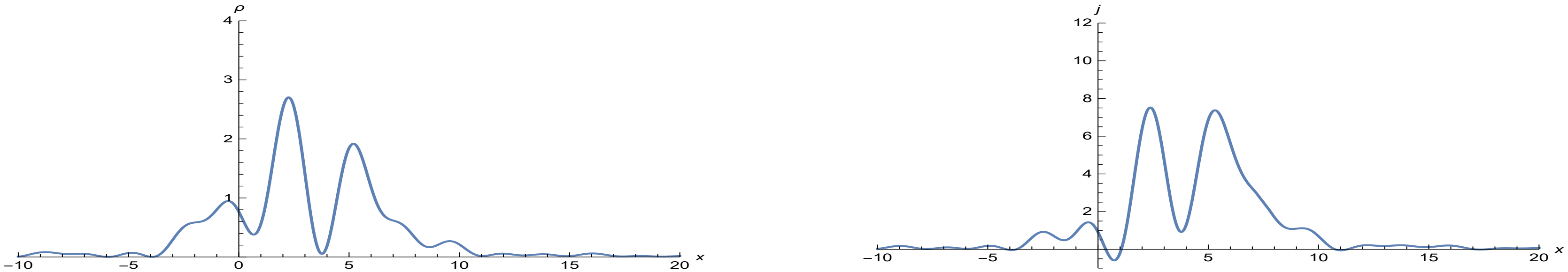

Figure 2: Probability density $\rho(t,x)$ and current $j(t,x)$ at the time t=1.

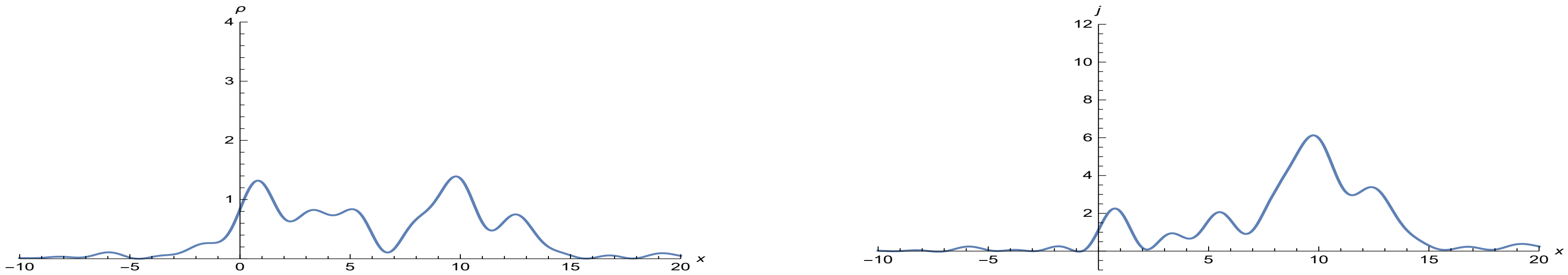

Figure 3: Probability density $\rho(t,x)$ and current $j(t,x)$ at the time t=2.

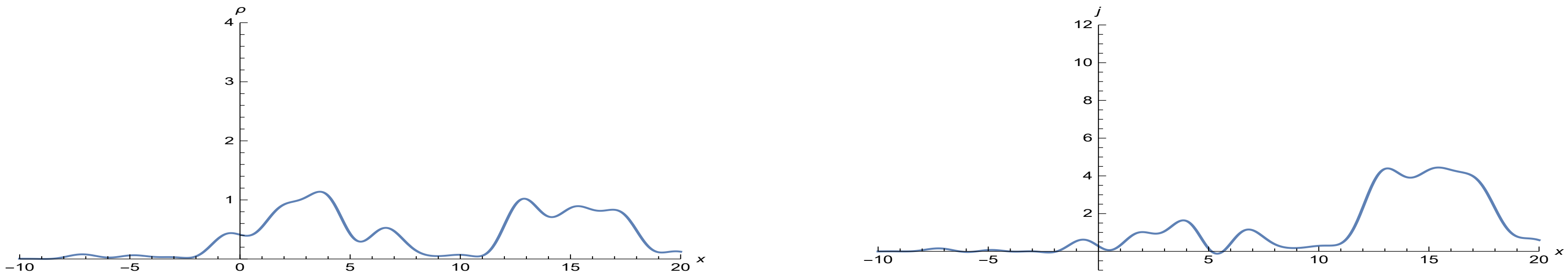

Figure 4: Probability density $\rho(t,x)$ and current $j(t,x)$ at the time t=3.

Apparently, regions with the negative current (i.e., in the direction opposite to the momentum) occur, e.g., at $t = 1$ around $x = 0.7$ (Fig. 2) or at $t = 3$ around $x = 5.4$ (Fig. 4).
Interpreted as a combination the motion of the peaks of the wave packet probability density to the right in the momentum direction but at the same time their spreading to the both sides.

3. Dirac equation (1+1 D, $m \neq 0$)

2-component spinors,
units where $\hbar = 1$ and $c = 1$,
the Dirac 2×2 matrices randomly chosen,
unique up to a similarity transformation.

$$(i\sigma_2\partial_t - \sigma_1\partial_x - m)\psi = 0$$

where $\psi(t,x) \in \mathbb{C}^2$ and σ_1, σ_2, σ_3 are the usual Pauli matrices. Probability density and current

$$\rho(t,x) = \psi(t,x)^+\psi(t,x) \quad , \quad j(t,x) = \psi(t,x)^+\sigma_3\psi(t,x).$$

The simple indefinite form of the current

$$j(t,x) = |\psi_1(t,x)|^2 - |\psi_2(t,x)|^2$$

easily allows for the construction of states with momentum spectral support in $[0,+\infty)$ but the current negative in some regions. Even some states of momentum as well as energy spectral support in $[0,+\infty)$ but somewhere locally negative current exist.

4. Dirac equation (1+1 D, $m = 0$)

Dirac equation reduces to the two uncoupled 1+1 dimensional analogs of the Weyl equations equations for the components of ψ,

$$(\partial_t + \partial_x)\psi_1 = 0 \quad , \quad (\partial_t - \partial_x)\psi_2 = 0.$$

The two components are independent and the current may have arbitrary sign in general.

Considering only states with energy as well as momentum spectral supports fully contained in $[0, +\infty)$ or $(-\infty, 0]$ the direction of the current is the same as the direction
of the momentum for the positive energy states and the opposite for the negative energy states.

The negative energy states are interpreted as the charge conjugated states of the antiparticles in quantum field theory. So the directions of the momentum and the probability current are the same for the massless neutrinos and antineutrinos in 1+1 dimensional space time.

5. Conclusions

Positive/negative momentum spectral support does not induce positive/negative direction of the probability current for 1+1 D Schrödinger and massive Dirac particles.
For the massless Dirac particles, the current direction is positive/negative for the positive/negative momentum spectral support.