Using noise fields to probe supersymmetry

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1. Work based on arXiv : 1302.2361[hep-th], 1405.0820[hep-th], 1606.08284[hep-th],1712.07045[hep-th], 1912.12925[hep-th] and in progress.

Outline

From scalar fields to noise fields

Scalar fields–along with the particles they describe and the corresponding fluctuations–appear in two ways (at least) in particle physics :

- As the Higgs boson of the Standard Model.
- As the inflaton in early Universe cosmology.

Learning to control their fluctuations is crucial for understanding the physics they describe. The idea is that the degrees of freedom that can describe the fluctuations can be written as the superpartners of the scalars. This, in turn, implies identities for the correlation functions of the scalars that can be subject to measurements. This talk is a summary of numerical experiments that provide evidence for this idea. (For inflation, cf. arXiv :1912.05358[hep-th] by G. Moreau and J. Serreau.)

The fluctuations as degrees of freedom

The classical, Euclidian, action of these scalars is, typically, given by the expression

$$S_{\mathrm{E}}[\phi] = \int d^d x \left\{ \frac{1}{2} (\partial\phi)^2 + V(\phi) \right\}$$

and is used in the partition function

$$Z_{\mathrm{E}} = \int [\mathscr{D}\phi]\, e^{-S_{\mathrm{E}}[\phi]}$$

to compute correlation functions. Indeed this expression implies that the scalar fields are in equilibrium with a bath. So we can ask, whether it is possible to describe this bath in terms of particles and what might their properties be.

White additive noise

A useful starting point is a Gaussian process $\{\eta(\tau)\}$:

$$\langle\eta(\tau)\rangle = 0$$
$$\langle\eta(\tau_1)\eta(\tau_2)\rangle = \nu\delta(\tau_1 - \tau_2)$$
$$\langle\eta(\tau_1)\eta(\tau_2)\cdots\eta(\tau_{2n})\rangle = \sum_\pi \left\langle\eta(\tau_{\pi(1)})\eta(\tau_{\pi(2)})\right\rangle \cdots \left\langle\eta(\tau_{\pi(2n-1)})\eta(\tau_{\pi(2n)})\right\rangle$$

$$\nu = \begin{cases} \hbar & \text{quantum} \\ k_B T & \text{thermal} \\ \sigma & \text{annealed disorder} \end{cases}$$

Choose units such that $1 = \hbar\,(= k_B T)\,(= \sigma)$.

The noise fields

Now consider $\eta(\tau)$ as the following expression, in terms of the scalar(s) :

$$\eta(\tau) = \frac{d\phi}{d\tau} + \frac{dW}{d\phi(\tau)}$$
$$\eta^I(\tau^A) = \boldsymbol{s}_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I}$$

In the second expression we have written the generalization to a $d-$dimensional worldvolume, $A = 1, \ldots, d$; the values for I, J depend on the properties of the $\boldsymbol{s}_A$.

Noncommutative worldvolume

When $[\boldsymbol{s}_A, \boldsymbol{s}_B] \neq 0$, the worldvolume can't be reduced to one dimension. A particularly interesting choice for the matrices $\boldsymbol{s}_A$ is to have them generate a Clifford algebra, e.g.

$$\{\boldsymbol{s}_A, \boldsymbol{s}_B\} = 2\delta_{AB}$$

The simplest such example is when $\boldsymbol{s}_A \equiv \sigma_A$, the Pauli matrices and the worldvolume taken to be two–dimensional, Euclidian, space, the global invariance is $SO(2)$ and the two scalars, ϕ^I, label the target space. The reason we would like to choose the $\boldsymbol{s}_A$ this way is because we would like to describe target space Lorentz invariance.

The (Langevin) partition function

Cf. G. Parisi and N. Sourlas Nucl. Phys. B206 (1982) 321.

$$Z = 1 \equiv \int [\mathscr{D}\eta(\tau)]\, e^{-\int d\tau\, \frac{1}{2}\eta(\tau)^2} \int [\mathscr{D}\phi(\tau)]\delta\left(\frac{d\phi(\tau)}{d\tau} + \frac{dW}{d\phi(\tau)} - \eta(\tau)\right)$$
$$= \int [\mathscr{D}\phi(\tau)]\, e^{-\int d\tau\, \frac{1}{2}\left(\frac{d\phi}{d\tau}+\frac{dW}{d\phi(\tau)}\right)^2} \left|\det\left(\delta(\tau-\tau')\left[\frac{d}{d\tau} + \frac{d^2 W}{d\phi(\tau)^2}\right]\right)\right|$$

These are not well–defined expressions ! Use lattice techniques to *define* them (cf. arXiv :1302.2361, arXiv :1405.0820 and 1606.08284).
These expressions define the "Langevin" partition function, Z_{L}. It describes the consistent closure of the system, if it can be well defined.

Surface terms and their avatars

When $[s_A, s_B] = 0$, the s_A can all be diagonalized simultaneously : $s_A^{IJ} = \lambda_A \delta^{IJ}$. When we expand out the action, we find

$$\frac{1}{2}\left(\lambda_A \delta^{IJ} \dot{\phi}^I + \frac{\partial W}{\partial \phi_I}\right)^2 = \frac{1}{2}\lambda_A^2 \left(\dot{\phi}^I\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial \phi_I}\right)^2 + \lambda_A \delta^{IJ} \dot{\phi}^I \frac{\partial W}{\partial \phi_I}$$

In this case we notice that the last term,

$$\lambda_A \delta^{IJ} \dot{\phi}^I \frac{\partial W}{\partial \phi_I} = \lambda_A \frac{d}{d\tau^A} W(\phi)$$

is a total derivative ; if we impose periodic boundary conditions, it doesn't contribute.

Surface terms and their avatars–II

When $[s_A, s_B] \neq 0$, the s_A can't all be diagonalized simultaneously. In that case

$$s_A^{IJ} \frac{\partial \phi^J}{\partial \tau^A} \frac{\partial W}{\partial \phi_I}$$

isn't, manifestly, a total derivative, so does contribute, apparently, to the classical equations of motion, even if we impose periodic boundary conditions.
The worldvolume is a non–commutative manifold and this term resembles a spin–orbit coupling, that describes long–range interactions. But could these, in fact, be an illusion ?

How holomorphy eliminates the spin–orbit term

When $s_A = \sigma_A$, something interesting happens : The cross–term takes the form

$$\sigma_x^{IJ}\frac{\partial\phi^J}{\partial x}\frac{\partial W}{\partial\phi_I} + \sigma_z^{IJ}\frac{\partial\phi^J}{\partial y}\frac{\partial W}{\partial\phi_I} = \left(\frac{\partial\phi^2}{\partial x} + \frac{\partial\phi^1}{\partial y}\right)\frac{\partial W}{\partial\phi_1} + \left(\frac{\partial\phi^1}{\partial x} - \frac{\partial\phi^2}{\partial y}\right)\frac{\partial W}{\partial\phi_2}$$

and, if the $\phi^{1,2}$ satisfy the Cauchy–Riemann equations, the term, in fact, vanishes. It, also, does so, if W is a holomorphic function. The question is, whether the fluctuations respect this, too. Let's have a look at them–they're described by the insertion of the absolute value of the determinant in the expectation values. If this isn't well–defined, this is a manifestation of the holomorphic anomaly.

The determinant

Assumption : W is ultra local :

$$\frac{\partial^2 W}{\partial\phi(\tau)\phi(\tau')} = \delta(\tau-\tau')\frac{d^2 W}{d\phi(\tau)^2}$$

Then, in one dimension (or a worldvolume with abelian isometries)

$$\left|\det\left(\delta(\tau-\tau')\frac{d}{d\tau}+\frac{\partial^2 W}{\partial\phi(\tau)\partial\phi(\tau')}\right)\right| = \left|\det\delta(\tau-\tau')\right|\left|\det\left(\frac{d}{d\tau}+\frac{d^2 W}{d\phi(\tau)^2}\right)\right|$$

The first factor can be taken outside the path integral and contributes the same constant to numerator and denominator, to any expectation value–it can be dropped.

The determinant

The absolute value of the determinant can be expressed as a product of its value and its phase, in general (assuming the determinant is real, the phase is, just, its sign) :

$$\left|\det\left(\frac{d}{d\tau}+\frac{d^2W}{d\phi(\tau)^2}\right)\right| = e^{-\mathrm{i}\phi_{\mathrm{det}}}\det\left(\frac{d}{d\tau}+\frac{d^2W}{d\phi(\tau)^2}\right)$$

$$\left|\det\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2W}{\partial\phi_I\partial\phi_J}\right)\right| = e^{-\mathrm{i}\phi_{\mathrm{det}}}\det\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2W}{\partial\phi_I\partial\phi_J}\right)$$

We may, thus, write

$$1 = Z_{\mathrm{L}} = \left\langle e^{-\mathrm{i}\phi_{\mathrm{det}}}\right\rangle_{\mathrm{SUSY}} Z_{\mathrm{SUSY}}$$

assuming that Z_{SUSY} exists, and that the average displayed does too. The reason for the qualifier SUSY will be explained presently.

Grassmann variables for a local action

The determinant–which is a non–local quantity–can be introduced in the action in a local way, using Grassmann fields, $\psi_I(\tau^A), \chi_J(\tau^A)$:

$$\det\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)=\int[\mathscr{D}\psi_I(\tau^A)][\mathscr{D}\chi_J(\tau^A)]\, e^{\int d^D\tau\,\psi_I\left(s_A^{IJ}\frac{\partial}{\partial\tau^A}+\frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)\chi_J}$$

(In Euclidian signature the fields ψ_I and χ_J are independent.)

SUSY

Assuming that $\{s_A, s_B\} = 2\delta_{AB}$, the Euclidian action

$$S_{\mathrm{E}} = \int d^D\tau \left[\frac{1}{2}\left(\frac{\partial\phi^I}{\partial\tau^A}\right)^2 - \frac{1}{2}F_I^2 + F_I\frac{\partial W}{\partial\phi_I} - \psi_I\left(s_A^{IJ}\frac{\partial}{\partial\tau^A} + \frac{\partial^2 W}{\partial\phi_I\partial\phi_J}\right)\chi_J \right]$$

can be shown to be invariant–up to total derivatives–under supersymmetric transformations. The practical question is how to test this–as well as the equivalence between the three partition functions that seem to describe the same system.

How to use the lattice regularization

We seem, therefore, to have three partition functions, that describe the same system : Z_{L}, Z_{SUSY} and Z_{QM} ; the latter is defined by

$$Z_{\mathrm{QM}} = \int [\mathscr{D}\phi^I(\tau^A)]\, e^{-\int d^D\tau \left\{ \frac{1}{2}\left(\frac{\partial\phi^I}{\partial\tau^A}\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial\phi_I}\right)^2 \right\}}$$

It's clear that Z_{QM} is the easiest to study numerically–and has, indeed, been studied ; only not from this angle.
The appropriate observables seem to be the noise fields :

$$\eta^I(\tau^A) = s_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I}$$

The observables : general considerations

These should satisfy

$$\left\langle \eta^I(\tau^A)\eta^J(\tau'^B)\right\rangle_{\mathrm{QM}} = 2\delta^{IJ}\delta(\tau^A - \tau'^B)$$

while the higher connected correlation functions should be compatible with zero.
If the 1–point function vanishes, SUSY is realized in the Wigner mode ; if it's non–zero, SUSY is realized in the Nambu–Goldstone mode.
These properties have, already, been checked for the case where $[\boldsymbol{s}_A, \boldsymbol{s}_B] = 0$. The question is to understand, whether there could appear any obstruction to their holding, also, in the case where $\{\boldsymbol{s}_A, \boldsymbol{s}_B\} = 2\delta_{AB}$.

The observables : general considerations

For two–dimensional models such an obstruction, naïvely isn't expected. The reason is that tunneling, also, occurs, as in the one–dimensional case. What can occur is that the superpotential can change quite dramatically, due to the quantum fluctuations, since the multiple classical minima will, always, be eliminated.
A caveat seems to be that it is, rather, the infrared divergences of massless particles, rather than tunneling, that may be the relevant property–and, whether a Berezinskii–Kosterlitz–Thouless (i.e. topology–changing) transition can occur and how it might be detected from the correlations of the noise field, remains to be clarified.

The lattice action

The lattice action is given by the expression

$$S_{\mathrm{latt}} = -\sum_{n_1,n_2} \Big[\phi_{n_1,n_2} \cdot (\phi_{n_1+1,n_2} + \phi_{n_1-1,n_2} + \phi_{n_1,n_2+1} + \phi_{n_1,n_2-1}) + \frac{1}{2}||\phi_{n_1,n_2}||^2 + g_{\mathrm{latt}}^2 v(||\phi_{n_1,n_2}||^2) \Big]$$

where $g_{\mathrm{latt}} \equiv ga$ and it can be readily verified that it is equal to $||\eta_{n_1,n_2}||^2/2$ up to surface terms and terms that are suppressed by positive powers of the lattice spacing–as occurs in $D = 1$, as well.

Noise on the lattice

The discretization of the noise field on the lattice is given by the expressions

$$\eta^1_{n_1,n_2} = \frac{1}{2}\left(\phi^2_{n_1+1,n_2} - \phi^2_{n_1-1,n_2} + \phi^1_{n_1,n_2+1} - \phi^1_{n_1,n_2-1}\right) + g_{\mathrm{latt}}\frac{\partial w}{\partial \phi_1}$$
$$\eta^2_{n_1,n_2} = \frac{1}{2}\left(\phi^1_{n_1+1,n_2} - \phi^1_{n_1-1,n_2} - \phi^2_{n_1,n_2+1} + \phi^2_{n_1,n_2-1}\right) + g_{\mathrm{latt}}\frac{\partial w}{\partial \phi_1}$$

where we have written $W(\phi_1,\phi_2) \equiv gw(\phi_1,\phi_2)$. We remark that, for a cubic superpotential, $\partial w/\partial \phi$ is quadratic in the fields, therefore, generically, its vev doesn't vanish–so SUSY, generically, will be broken, in this case. An open question is, whether it can be restored by linear term of the superpotential–which represents a surface term, in fact.

One–dimensional worldvolume

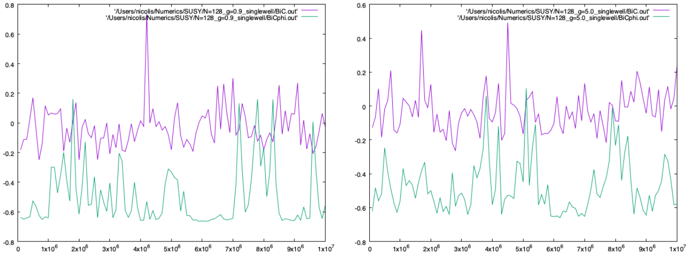
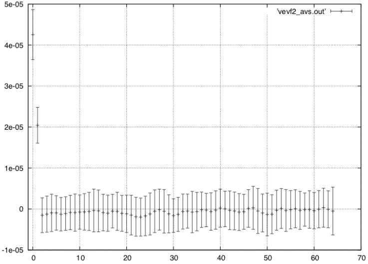


Figure – Time series for the Binder cumulants $-B_1(\eta)$ and $-B_1(\varphi)$, for $g = 0.9$ and $g = 5.0$. They imply that $-B_1(\eta) = -0.28 \pm 0.14$–i.e. consistent with zero–while $-B_1(\varphi) = -0.49677 \pm 0.2018$–i.e. consistent with a non–zero value for $g = 0.9$ and $-B_1(\eta) = -0.026 \pm 0.13$, while $-B_1(\varphi) = -0.47 \pm 0.16$ for $g = 5.0$.

One–dimensional worldvolume



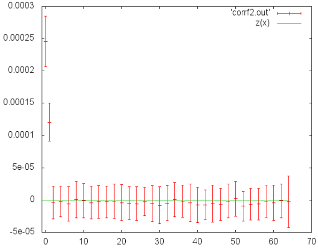


Figure – The connected 2–point function for the noise field, $\langle \eta_{|n-n'|}\eta_0 \rangle - \langle \eta \rangle^2$ for $N = 128$, $g = 0.9$ and $g = 5.0$.The value of $m^2_{\mathrm{latt}} = 0.0001$. We notice that its value, at $|n - n'| = 0$, is, indeed, equal to $g|m^2_{\mathrm{latt}}|/2$, in both cases.

One–dimensional worldlvolume

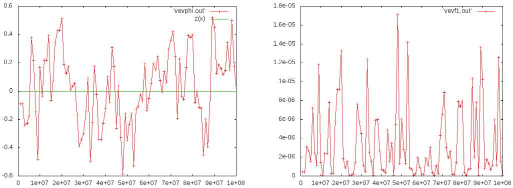


Figure – The 1–point functions, $\langle\phi\rangle$ and $\langle\boldsymbol{\eta}\rangle$, for the cubic superpotential, when the scalar potential has a unique minimum, when $g = 0.9$. We remark that supersymmetry is spontaneously broken–but $\langle\boldsymbol{\eta}\rangle$ is, numerically, very small.

One–dimensional worldvolume

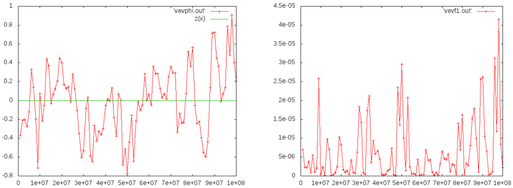


Figure – The 1–point functions, $\langle \phi \rangle$ and $\langle \boldsymbol{\eta} \rangle$, for the cubic superpotential, when the scalar potential has a unique minimum, when $g = 2.0$. We remark that supersymmetry is spontaneously broken–but $\langle \boldsymbol{\eta} \rangle$ is, numerically, very small.

One–dimensional worldvolume

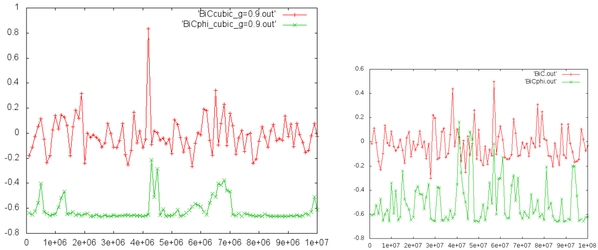


Figure – The time series for the Binder cumulants, $-B_1(\eta)$ and $-B_1(\varphi)$, for the cubic superpotential, when $g = 0.9$ and $g = 2.0$ and the classical scalar potential has a unique minimum. $m_{\mathrm{latt}} = 0.0001$.

Two–dimensional worldvolume

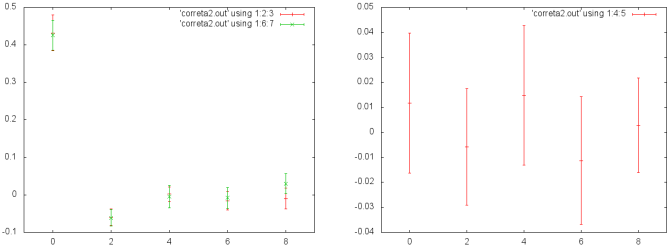


Figure – Typical results for the cubic superpotential : $\langle \eta^I_n \eta^J_{n+d} \rangle$ for $I = J$ (left panel) and $I \neq J$ (right panel) and $d = 0, 2, 4, 6, 8$, on the 17×17 square lattice. $g^2_{\mathrm{latt}} = 0.7$. The diagonal noise term is a $\delta-$function, while the off-diagonal noise term vanishes, to numerical precision.

Conclusions–Perspectives

- It is possible to describe consistently the fluctuations, to which physical systems are coupled, by quantum, thermal effects, or disorder, in a way that doesn't depend on how the physical degrees of freedom are selected among the fluctuations–and the reason is that the fluctuations and the physical degrees of freedom are but components of the same supermultiplet. The fluctuations are mutually non–local wrt the physical degrees of freedom.

Conclusions–Perspectives

- The difference between quantum, thermal and disorder fluctuations is that, while, in the latter two cases, the effective anticommuting degrees of freedom can be expressed in terms of more fundamental degrees of freedom, that can be commuting at that scale, quantum fluctuations cannot be expressed as local, commuting, degrees of freedom–but as local, anticommuting, degrees of freedom, only.

Conclusions–Perspectives

The most direct building blocks of a quantum field theory, therefore, seem to be the noise fields

$$\eta^I(\tau^A) = \mathbf{s}_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I} \equiv \mathbf{s}_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + F^I$$

This expression is known as the Nicolai map (H. Nicolai Phys.Lett. 89B (1980) 341 ; Nucl.Phys. B176 (1980) 419-428).

From the known properties of the noise it is possible to deduce the properties of the physical fields and, in particular, describe how the bath reacts to the presence of the physical degrees of freedom. A consistent reaction is described by the fact that the 1–point function can take a non–zero value ; and that the potential can change, due to tunneling effects.

Conclusions–Persepctives

- The noise fields, indeed, preserve half the supersymmetries ; that's another way to understand their significance. But they are defined for any theory !
- The framework for studying hypermultiplets by themselves seems well–defined. For studying (super)particles and the corresponding fields in curved target spaces (or with torsion)–relevant for non–linear $\sigma-$models as well as for gauge theories, what's needed is multiplicative noise, e.g.

$$E_M^I(\phi)\eta^M(\tau^A) = s_A^{IJ}\frac{\partial\phi^J}{\partial\tau^A} + \frac{\partial W}{\partial\phi_I}$$

here $E_M^I(\phi)$ is the vielbein, that expresses the fact that the target space is curved. Cf. arXiv : 1610.01622 for an application to magnets.

Conclusions–Perspectives

We have presented arguments for how supersymmetry not only can but must be realized for the description of the dynamics of scalar field theories to be complete. It's an emergent property that provides the content to the statement that a system is in equilibrium with a bath. What does remain to be clarified, via numerical simulations, is what happens in $d = 3$ and $d = 4$ dimensions, where (a) phase transitions do occur and (b) the rotation group is non-abelian. Parisi and Sourlas did identify a possible obstruction, but it wasn't obvious whether non-perturbative effects could affect it and how.

Conclusisons–Perspectives

Scalar fields, typically, interact with other fields. Such an interaction can be described as a "quenched disorder" from the point of view of the scalars :

$$Z = \int [\mathscr{D}h]\, P(h) \int [\mathscr{D}\phi]\, e^{-S[\phi]+\int d^dx\, h(x)\phi(x)} \left|\det\frac{\delta\eta}{\delta\phi}\right|$$

which, of course, doesn't have any reason to equal 1 and every reason not to ; the fluctuations, that are captured by the absolute value of the determinant, aren't "correlated" with the fluctuations of the fields, to which the scalars are coupled ! This means that it's not possible to deduce the fluctuations of the *h* from the fluctuations of the ϕ in a "local" way. How supersymmetry can be used to consistently close the full system remains to be understood.

Conclusions–Perspectives

So monitoring the deviations of the correlation functions of the noise fields $\eta[\phi]$ from the identities they would have satisfied, were the system closed, can provide information of how supersymmetry is broken due to the mismatch–which corresponds, indeed, to an explicit breaking.

It should be stressed that supersymmetry puts fermions and bosons in the same multiplet ; so by storing the configurations of the bosonic fields, it is possible, in principle, to deduce the correlation functions of the fermionic fields ; this was pointed out by Parisi and Sourlas (1982), but wasn't pursued. It definitely warrants a closer look, for making numerical simulations with fermions more efficient.