Gravitational form factor constraints for states of arbitrary spin

Peter Lowdon

(Ecole Polytechnique)
Outline

1. The Gravitational form factors
2. Constraints for massive states
3. Arbitrary state generalisation
4. Summary & outlook
1. The Gravitational form factors

• The energy-momentum tensor (EMT) matrix elements encode many different dynamical properties, including:
  – Quantum corrections to the gravitational motion of particles
  – Distribution of mass and angular momentum within hadrons

• The EMT matrix elements can be decomposed into a series of form factors → these fully parametrise the non-perturbative information

**Define on-shell states:**
\[
|p, m; M\rangle = \delta_M^{(+)}(p)|p, m\rangle \equiv 2\pi \theta(p^0) \delta(p^2 - M^2)|p, m\rangle
\]

\[
\langle p', m'; M|T^{\mu\nu}(0)|p, m; M\rangle = \bar{\eta}_{m'}(p')O^{\mu\nu}(p', p)\eta_m(p)\delta^{(+)}_M(p')\delta^{(+)}_M(p),
\]

**“Generalised” polarisation tensor (GPT)**
\[
O^{\mu\nu}(p', p) = \bar{p}^{\{\mu} p^{\nu\}} A(q^2) + ip^{\{\mu} S^{\nu\}} p q \rho G(q^2) + \cdots
\]

Lorentz generator
\[
q = p' - p
\]
\[
\bar{p} = \frac{1}{2}(p' + p)
\]
1. The Gravitational form factors

→ What are the constraints on these form factors?

• Most previous studies chose to focus on massive (canonical spin) states with lower spin, in particular spin 0, \( \frac{1}{2} \) or 1

• Analyses often suffered from technical issues, such as the incorrect treatment of non-normalisable states or boundary terms, as detailed in: [Bakker, Leader, Trueman, hep-ph/0406139].

• An approach was developed in [PL, Chiu, Brodsky, 1707.06313] for the spin-\( \frac{1}{2} \) case in which the EMT matrix elements were treated rigorously, using their properties as distributions

→ Established that the \( q \to 0 \) limit of \( A(q^2) \) and \( G(q^2) \) are fixed by the Poincaré transformation properties of the states alone

Central question: What about states with arbitrary spin?
2. Constraints for massive states

- Use “distributional matching procedure” [Cotogno, Lorcé, PL, 1905.11969]:

**Step 1:** Construct rigorous expressions for the Lorentz charge operators in terms of the EMT components

\[
J^i = \frac{1}{2} \epsilon^{ijk} \lim_{d \to 0} \lim_{R \to \infty} \int d^4x f_{d,R}(x) \left[ x^j T^{0k}(x) - x^k T^{0j}(x) \right],
\]

\[
K^i = \lim_{d \to 0} \lim_{R \to \infty} \int d^4x f_{d,R}(x) \left[ x^0 T^{0i}(x) - x^i T^{00}(x) \right],
\]

*Need smearing with appropriate test functions*

**Step 2:** Use these definitions, together with the EMT form factor decomposition, to write the rotation and boost generator matrix elements in terms of these form factors

\[
\langle p', m'; M | J^i | p, m; M \rangle = -i \epsilon^{ijk} p^k (2\pi)^4 \delta^{(+)}_M (\vec{p}) \left[ \delta_{m'm} \partial^j \delta^4(q) - \partial^j [\bar{\eta}_{m'}(p') \eta_m(p)] \right]_{q=0} \delta^4(q) A(q^2)
\]

\[
+ \frac{1}{2} \epsilon^{ijk} (2\pi)^4 \delta^{(+)}_M (\vec{p}) \left[ \bar{\eta}_{m'}(\vec{p}) S^{jk} \eta_m(\vec{p}) \right] \delta^4(q) G(q^2)
\]

\[
\langle p', m'; M | K^i | p, m; M \rangle = i (2\pi)^4 \delta^{(+)}_M (\vec{p}) \left[ \delta_{m'm} (\vec{p}^0 \partial^i - \vec{p}^i \partial^0) \delta^4(q) - \vec{p}^0 \partial^i [\bar{\eta}_{m'}(p') \eta_m(p)] \right]_{q=0} \delta^4(q) A(q^2)
\]

\[
+ (2\pi)^4 \delta^{(+)}_M (\vec{p}) \left[ \bar{\eta}_{m'}(\vec{p}) S^{0i} \eta_m(\vec{p}) \right] \delta^4(q) G(q^2)
\]
2. Constraints for massive states

**Step 3:** Use the transformation properties of on-shell states under rotations and boosts...

\[ U(\alpha)|p, k; M\rangle = \sum_l D^{(s)}_{lk}(\alpha)|\Lambda(\alpha)p, l; M\rangle, \]

...to write an arbitrary spin representation for the rotation and boost matrix elements in terms of the rest frame spin

\[ \Sigma^i_{m'm}(k) = \bar{\eta}^i_{m'}(k) J^i \eta_m(k) \]

\[ \langle p', m'; M|J^i|p, m; M\rangle = (2\pi)^4 \delta^{(+)}_M(\vec{p}) \left[ \Sigma^i_{m'm}(k) - \delta_{m'm} \epsilon^{ijk} \vec{p}^k \frac{\partial}{\partial q_j} \right] \delta^4(q) \]

\[ \langle p', m'; M|K^i|p, m; M\rangle = (2\pi)^4 \delta^{(+)}_M(\vec{p}) \left[ -\epsilon^{ijk} \vec{p}^j \frac{\partial}{\partial q_i} + \delta_{m'm} i \left( \vec{p}^0 \frac{\partial}{\partial q_i} - \vec{p}^i \frac{\partial}{\partial q_0} \right) \right] \delta^4(q) \]

**Step 4:** Compare the two different representations!

\[ \rightarrow \text{Implies the constraint: } A(q^2) \delta^4(q) = G(q^2) \delta^4(q) = \delta^4(q) \]

...which is simply: \(A(0)=1\) and \(G(0)=1\)
2. Constraints for massive states

• Identical form factor constraints obtained from boost and rotation generators → **not generator specific!**

• In fact, one can instead use the covariant operator basis
  
  – *Pauli-Lubanski*, $W^\mu \rightarrow$ implies: $G(0)=1$
  
  – *Covariant boost*, $B^\mu = \frac{1}{2}(S^{\nu\mu}P_\nu + P_\nu S^{\nu\mu}) \rightarrow$ implies: $A(0)=1$

The constraints are non-perturbative and **independent of both the spin and internal structure of the states** in the matrix elements

→ *fixed purely from Poincaré covariance of states*

**Implications**

→ Spin universality of GPD sum rules

→ AGM $B(0)=G(0)-A(0)$ vanishes for particles of **any** spin
3. Arbitrary state generalisation

- Relativistic spin states are **convention dependent**

  \[ |p, \sigma \rangle = U(L(p)) |k, \sigma \rangle \quad \quad \Lambda(L(p)) k = p \]

- Defined by choice of Lorentz transformation and reference vector
  
  (i) "Canonical spin state" → \( k = (m,0,0,0) \), \( L_c(p) = \) pure boost
  
  (ii) "Wick helicity state" → \( k = (\kappa,0,0,\kappa) \), \( L_w(p) = z\)-boost & rotation

- Results derived in most of the literature, including [Cotogno, Lorcé, PL, 1905.11969], assumed massive canonical spin states

  → **What happens for arbitrary spin-states?**

- It turns out [Lorcé, PL, 1908.02567] that one can apply an analogous matching procedure

  \[ \eta_\sigma(p) = D(L(p)) \eta_\sigma(k) \]

  **Key:** need to take derivative wrt to momentum components of \( D(L(p)) \)
For the matrix element
\[ \langle p', \sigma'; M | \tilde{U}(J^i) | p, \sigma; M \rangle = (2\pi)^4 \delta_M^{(+)}(\tilde{p}) J_{\sigma' \sigma}^i(\tilde{p}, q), \]
one can write this in the state-independent form:

\[ J_{\sigma' \sigma}^i(\tilde{p}, q) = -i \epsilon^{ijk} \tilde{p}^k \delta_{\sigma' \sigma} \partial \delta^4(q) A(q^2) + i \epsilon^{ijk} \tilde{p}^k \bar{\eta}_{\sigma'}(k) \tilde{D} \left( \frac{\partial L^{-1}(\tilde{p})}{\partial \tilde{p}_j} L(\tilde{p}) \right) \eta_{\sigma}(k) \delta^4(q) A(q^2) \]
\[ + \bar{\eta}_{\sigma'}(k) \tilde{D} \left( L^{-1}(\tilde{p}) J^i L(\tilde{p}) \right) \eta_{\sigma}(k) \delta^4(q) G(q^2), \]

**Lie algebra representation of D**

**3. Arbitrary state generalisation**

Similarly, the transformation properties of the states under **rotations** implies the general representation:

\[ J_{\sigma' \sigma}^i(\tilde{p}, q) = -i \epsilon^{ijk} \tilde{p}^k \delta_{\sigma' \sigma} \partial \delta^4(q) + i \delta^4(q) \frac{d}{d\beta} \bigg|_{\beta=0} \bar{\eta}_{\sigma'}(k) D(W(\mathcal{R}_i, \tilde{p})) \eta_{\sigma}(k). \]

**“Wigner rotation”**

\[ W(\mathcal{R}_i, \tilde{p}) = L^{-1}(\Lambda(\mathcal{R}_i, \tilde{p}) e^{-i\delta^i J^i} L(\tilde{p}). \]

\[ J_{\sigma' \sigma}^i(\tilde{p}, q) = -i \epsilon^{ijk} \tilde{p}^k \delta_{\sigma' \sigma} \partial \delta^4(q) + i \epsilon^{ijk} \tilde{p}^k \bar{\eta}_{\sigma'}(k) \tilde{D} \left( \frac{\partial L^{-1}(\tilde{p})}{\partial \tilde{p}_j} L(\tilde{p}) \right) \eta_{\sigma}(k) \delta^4(q) \]
\[ + \bar{\eta}_{\sigma'}(k) \tilde{D} \left( L^{-1}(\tilde{p}) J^i L(\tilde{p}) \right) \eta_{\sigma}(k) \delta^4(q). \]

→ Comparing these expressions implies: \( A(0) = 1 \) and \( G(0) = 1 \)
4. Summary & outlook

• By adopting a distributional approach one can prove on a non-perturbative level that Poincaré symmetry alone is responsible for the $q \to 0$ behaviour of $A(q^2)$ and $G(q^2)$.

• These constraints hold independently of the internal properties of the states (internal structure, spin convention, mass, spin representation).

→ GPD spin sum rules are state universal

→ AGM vanishes for any particle

• More generally, these results are relevant for understanding the spin structure of gravitational scattering amplitudes.
4. Summary & outlook

• One can in fact classify all the possible terms appearing in the form factor decomposition of the EMT matrix elements for (massive) states with arbitrary spin [Cotogno, Lorcé, PL, Morales, 1912.08749]

→ Enables one to count the total number of independent gravitational form factors for states of a given spin

• This classification can also be generalised to massless states (work in progress!)

→ Provides an alternative approach for understanding the constraints imposed by conformal symmetry

→ New insights in understanding the interplay between gauge symmetry and the structure of massless amplitudes