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INFN - Roma Tre

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Challenges in Semileptonic B decays
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$|V_{cb}|$ and $R(D_{(s)}^{(*)})$ using lattice QCD and unitarity

in collaboration with:

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outline of the talk

* **the Dispersion Matrix approach:** an attractive way to implement unitarity and Lattice QCD calculations in the analysis of exclusive semileptonic decays of hadrons [PRD '21 (2105.02497), PRD '21 (2105.07851), PRD '22 (2105.08674)]

* results for $B \to D^{(*)}\ell\nu_\ell$ decays: extraction of $|V_{cb}|$ and theoretical determination of $R(D^{(*)})$ [2105.08674, 2109.15248]

* results for $B_s \to D_s^{(*)}\ell\nu_\ell$ decays: extraction of $|V_{cb}|$ and theoretical determination of $R(D_s^{(*)})$ [2204.05925]

* results for $|V_{ub}|$ from $B \to \pi\ell\nu_\ell$ and $B_s \to K\ell\nu_\ell$ decays [2202.10285] → Ludovico's slides in the discussion session

motivations

* two critical issues in semileptonic $B \to D^{(*)}\ell\nu_\ell$ decays

- **exclusive/inclusive $|V_{cb}|$ puzzle:**

<u>exclusive</u> (FLAG '21): $|V_{cb}|(BGL) \cdot 10^3 = 39.36\ (68)$

<u>inclusive</u> (HFLAV '21): $|V_{cb}| \cdot 10^3 = 42.19\ (78)$

difference of $\sim 2.7\ \sigma$

$|V_{cb}| \cdot 10^3 = 42.16\ (50)$

(Bordone et al. 2107.00604)

- **$R(D^{(*)})$ anomalies:**

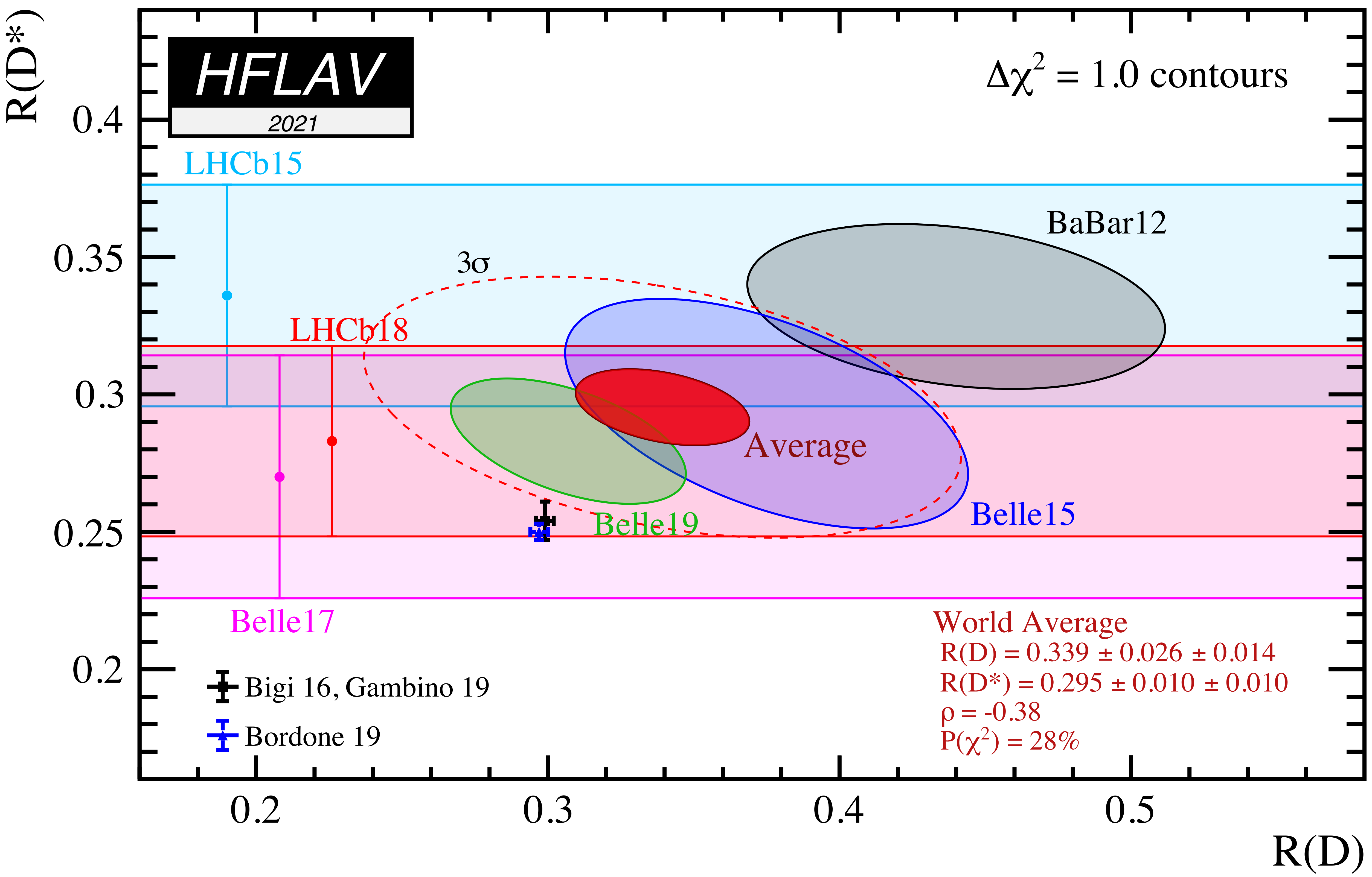
$$R(D) = \frac{\mathscr{B}(B \to D\tau\nu_\tau)}{\mathscr{B}(B \to D\ell\nu_\ell)}$$

$$R(D^*) = \frac{\mathscr{B}(B \to D^*\tau\nu_\tau)}{\mathscr{B}(B \to D^*\ell\nu_\ell)}$$

$\ell = e, \mu$

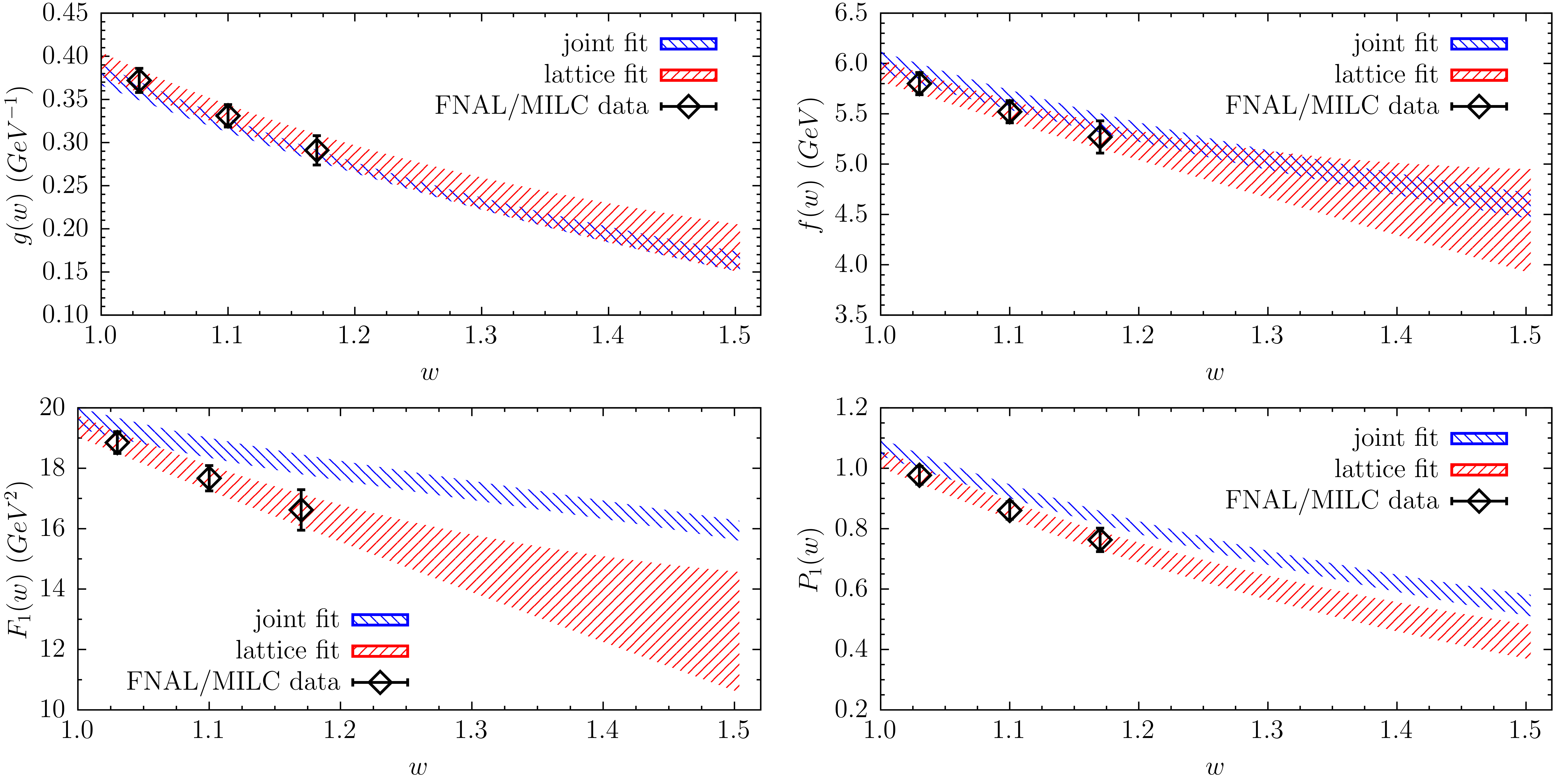
differences of $\sim 3.4\sigma$ between exp.'s and "SM"

" " = mix of theoretical calculations and experimental



important news: LQCD form factors for $B \to D^* \ell \nu_\ell$ decays from **FNAL/MILC (arXiv:2105.14019)**

synthetic data points at 3 non-zero values of the recoil (w - 1)



joint fit:
BGL fit of LQCD points + Belle + BaBar exp. data

$|V_{cb}| \cdot 10^3 = 38.40 \pm 0.74$

$R(D^*) = 0.2483 \pm 0.0013$

lattice fit:
quadratic BGL fit of LQCD points only

$|V_{cb}| > |V_{cb}|^{\text{joint fit}}$?

$R(D^*) = 0.265 \pm 0.013$

simultaneous fit of the lattice points and experimental data to determine the shape of the FFs and to extract $|V_{cb}|$

*** slope differences between exp's and theory $\to$ bias on $|V_{cb}|^{\text{joint fit}}$? ***

aim of the talk

to show the relevant, attractive features of the Dispersion Matrix (DM) approach [arXiv:2105.02497], which is a rigorously model-independent tool for describing the hadronic form factors (FFs) in their whole kinematical range

- entirely based on first principles (i.e. lattice QCD simulations of 2- and 3-point Euclidean correlators)
- independent on any assumption about the momentum dependence of the FFs
- unitarization of the input theoretical data (including also kinematical constraints)
- **no mixing among theoretical calculations and experimental data to describe the shape of the FFs**

* results for $B_{(s)} \to D^{(*)}_{(s)} \ell \nu_\ell$ decays: extraction of $|V_{cb}|$ and theoretical determination of $R(D^{(*)}_{(s)})$ using LQCD results for the FFs (from FNAL/MILC and HPQCD) [2105.08674, 2109.15248, 2204.05925]

decay	$\lvert V_{cb}\rvert^{DM} \cdot 10^3$	inclusive [2107.00604]	exclusive [FLAG 21]
$B \to D$	41.0 ± 1.2		
$B \to D^*$	41.3 ± 1.7		
$B_s \to D_s$	42.4 ± 2.0		
$B_s \to D_s^*$	41.4 ± 2.6		
average	41.4 ± 0.8	42.16 ± 0.50	39.36 ± 0.68
difference		$\simeq 0.8\,\sigma$	$\simeq 1.9\,\sigma$

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.340 (27) (13)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (11) (8)	$\simeq 1.3\,\sigma$
$R(D_s)$	0.298 (5)		
$R(D_s^*)$	0.250 (6)		

*** reduced tensions in both $|V_{cb}|$ and $R(D^{(*)})$ ***

Dispersion Matrix (DM) approach

[arXiv:2105.02497]

* reappraisal and improvement of the method originally proposed by Bourrely et al. NPB ’81 and Lellouch in NPB ’96

$$
\mathcal{M} = \begin{pmatrix}
< \phi f | \phi f > & < \phi f | g_t > & < \phi f | g_{t_1} > & \cdots & < \phi f | g_{t_N} > \\
< g_t | \phi f > & < g_t | g_t > & < g_t | g_{t_1} > & \cdots & < g_t | g_{t_N} > \\
< g_{t_1} | \phi f > & < g_{t_1} | g_t > & < g_{t_1} | g_{t_1} > & \cdots & < g_{t_1} | g_{t_N} > \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
< g_{t_N} | \phi f > & < g_{t_N} | g_t > & < g_{t_N} | g_{t_1} > & \cdots & < g_{t_N} | g_{t_N} >
\end{pmatrix}
$$

inner product: $< g | h > \equiv \dfrac{1}{2\pi i} \displaystyle\int_{|z|=1} \dfrac{dz}{z}\, \overline{g}(z)\, h(z)$

$$g_t(z) \equiv \frac{1}{1 - \overline{z}(t)\, z}$$

$$< g_t | \phi f > \equiv \phi(z, q_0^2) f(z) \qquad < g_t | g_{t_m} > = \frac{1}{1 - \overline{z}(t_m)\, z(t)}$$

$t_1, t_2, \ldots, t_N$ are the N values of the squared 4-momentum transfer where the form factor *f* has been computed and t is its value where we want to compute *f*(t)

unitarity bound: $< \phi f | \phi f > \equiv \dfrac{1}{2\pi i} \displaystyle\int_{|z|=1} \dfrac{dz}{z} |\phi(z, q_0^2) f(z)|^2 \leq \chi(q_0^2)$

in the case of interest $z_i \equiv z(t_i)$ and $\phi_i f_i \equiv \phi(z_i, q_0^2) f(t_i)$ are real numbers and the positivity of the inner product implies:

$$
\det[\overline{\mathcal{M}}] = \begin{vmatrix}
\chi(q_0^2) & \phi f & \phi_1 f_1 & \cdots & \phi_N f_N \\
\phi f & \frac{1}{1 - z^2} & \frac{1}{1 - z z_1} & \cdots & \frac{1}{1 - z z_N} \\
\phi_1 f_1 & \frac{1}{1 - z_1 z} & \frac{1}{1 - z_1^2} & \cdots & \frac{1}{1 - z_1 z_N} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_N f_N & \frac{1}{1 - z_N z} & \frac{1}{1 - z_N z_1} & \cdots & \frac{1}{1 - z_N^2}
\end{vmatrix} \geq 0
$$

* the explicit solution is a band of values: $\beta - \sqrt{\gamma} \leq f(z) \leq \beta + \sqrt{\gamma}$

$$\beta = \frac{1}{d(z)\,\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_j} \qquad \gamma = \frac{1}{d^2(z)\,\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

χ, f_i : nonperturbative input quantities, $\phi(z), d(z), \phi_i, d_i$: kinematical coefficients depending on z_i

* unitarity is satisfied when $\gamma \geq 0$, which implies: $\chi \geq \chi_{\{f\}}^{DM} \equiv \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$

*** parameterization-independent “unitarization” of the input data $\{f\}$ ***

select only events with $\chi \geq \chi_{\{f\}}^{DM}$

* important feature: when $z \to z_j$ one has $\beta \to f_j$ and $\gamma \to 0$, i.e. the DM band collapses to f_j for $z = z_j$

↓

for any given set of input data the DM approach reproduces exactly the known data and it allows to extrapolate the form factor in the whole kinematical range in a parameterization-independent way providing a band of values representing the results of all possible BGL fits satisfying unitarity and passing through the known points

* the DM band represents a uniform distribution which is combined with the multivariate distribution of the input data $\{f_j\}$ to generate the final band for the FF $f(z)$

* kinematical constraint(s) can be easily and rigorously implemented in the DM approach [2105.02497, 2105.08674, 2109.15248]

nonperturbative determination of the susceptibilities

* lattice QCD simulations can provide a first-principle determination of the unitarity bounds [arXiv:2105.02497]

time-momentum representation (Q = Euclidean 4-momentum)

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^+}(t)\ ,$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4 \frac{j_1(Qt)}{Qt}\ C_{1^-}(t)\ ,$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2 \Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\ t^2 j_0(Qt)\ C_{0^-}(t)\ ,$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2 \Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\ t^4 \frac{j_1(Qt)}{Qt}\ C_{1^+}(t)\ ,$$

2-point Euclidean correlation functions

$$C_{0^+}(t) = \widetilde{Z}_V^2 \int d^3x \langle 0| T\left[\bar{q}_1(x)\gamma_0 q_2(x)\ \bar{q}_2(0)\gamma_0 q_1(0)\right]|0\rangle\ ,$$

$$C_{1^-}(t) = \widetilde{Z}_V^2\ \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0| T\left[\bar{q}_1(x)\gamma_j q_2(x)\ \bar{q}_2(0)\gamma_j q_1(0)\right]|0\rangle\ ,$$

$$C_{0^-}(t) = \widetilde{Z}_A^2 \int d^3x \langle 0| T\left[\bar{q}_1(x)\gamma_0\gamma_5 q_2(x)\ \bar{q}_2(0)\gamma_0\gamma_5 q_1(0)\right]|0\rangle\ ,$$

$$C_{1^+}(t) = \widetilde{Z}_A^2\ \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0| T\left[\bar{q}_1(x)\gamma_j\gamma_5 q_2(x)\ \bar{q}_2(0)\gamma_j\gamma_5 q_1(0)\right]|0\rangle\ ,$$

* in arXiv:2105.02497, 2105.07851 and 2202.10285 we have calculated the $\chi's$ for the $c \to s$, $b \to c$ and $b \to u$ transitions at $Q^2 = 0$ using the $N_f = 2+1+1$ gauge ensembles generated by ETMC

$b \to c$

- subtraction of discretization effects evaluated in perturbation theory at order $\mathcal{O}(\alpha_s^0)$
- implementation of WI for the 0^+ and 0^- channels to avoid exactly contact terms
- use of the ETMC ratio method (hep-lat/0909.3187) to reach the physical b-quark point

applicable also at $Q^2 \neq 0$

$b \to c$ transition (arXiv:2105.07851)

channel	nonPT	with GS subtr.	NNLO PT	with GS subtr.
0^+ [10^{-3}]	7.58 (59)	—	6.204 (81)	—
1^- [10^{-4} GeV^{-2}]	6.72 (41)	5.88 (44)	6.486 (48)	5.131 (48)
0^- [10^{-2}]	2.58 (17)	2.19 (19)	2.41	1.94
1^+ [10^{-4} GeV^{-2}]	4.69 (30)	—	3.894	—

GS = ground state

perturbative

Bigi, Gambino PRD ’16
Bigi, Gambino, Schacht PLB ’17
Bigi, Gambino, Schacht JHEP ’17

* differences with NNLO PT ~ 4% for 1^-, ~7% for 0^-, ~20 % for 0^+ and 1^+

$c \to s$ transition (arXiv:2105.02497)

channel	nonPT	with GS subtr.
0^+ [10^{-2}]	0.929 (64)	0.433 (133)
1^- [10^{-3} GeV^{-2}]	7.88 (41)	4.19 (36)
0^- [10^{-2}]	2.48 (15)	0.942 (91)
1^+ [10^{-3} GeV^{-2}]	4.89 (29)	3.74 (56)

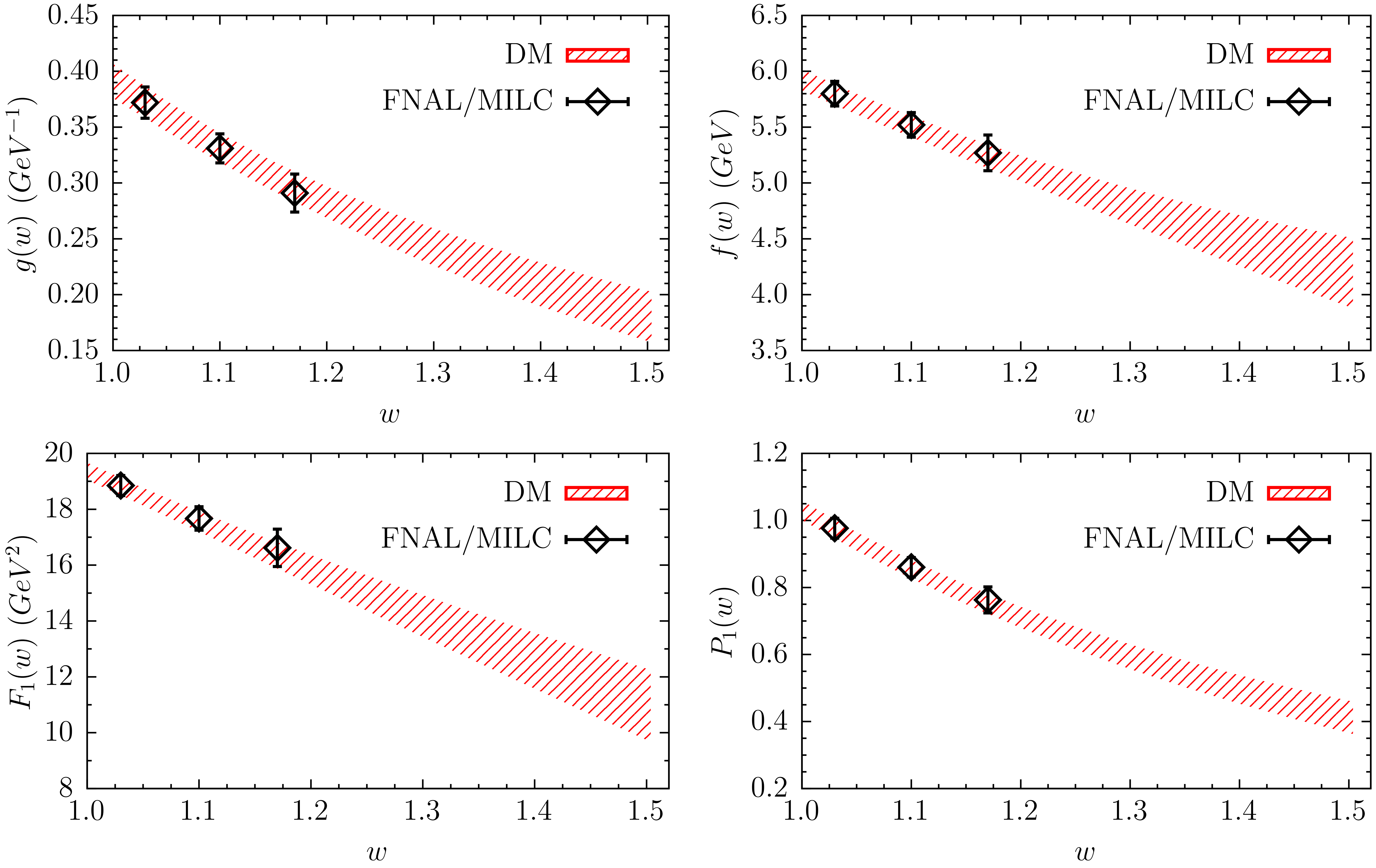
$b \to \ell$ transition (arXiv:2202.10285)

channel	nonPT	with GS subtr.
0^+ [10^{-2}]	2.04 (20)	—
1^- [10^{-4} GeV^{-2}]	4.88 (1.16)	4.45 (1.16)
0^- [10^{-2}]	2.34 (13)	—
1^+ [10^{-4} GeV^{-2}]	4.65 (1.02)	—

form factors for $B \to D^* \ell \nu_\ell$ decays

[arXiv:2109.15248]

* lattice QCD form factors from **FNAL/MILC arXiv:2105.14019**: synthetic data points at 3 (small) values of the recoil w
* nonperturbative susceptibilities from arXiv:2105.07851 (resonances from Bigi et al., arXiv:1707.09509)



three unitarity bounds: χ_{1^-} for g, χ_{1^+} for f and $\mathcal{F}_1$, χ_{0^-} for P_1

unitarity + two kinematical constraints

$$w = 1 : \qquad \mathcal{F}_1(1) = m_B(1-r)f(1)$$

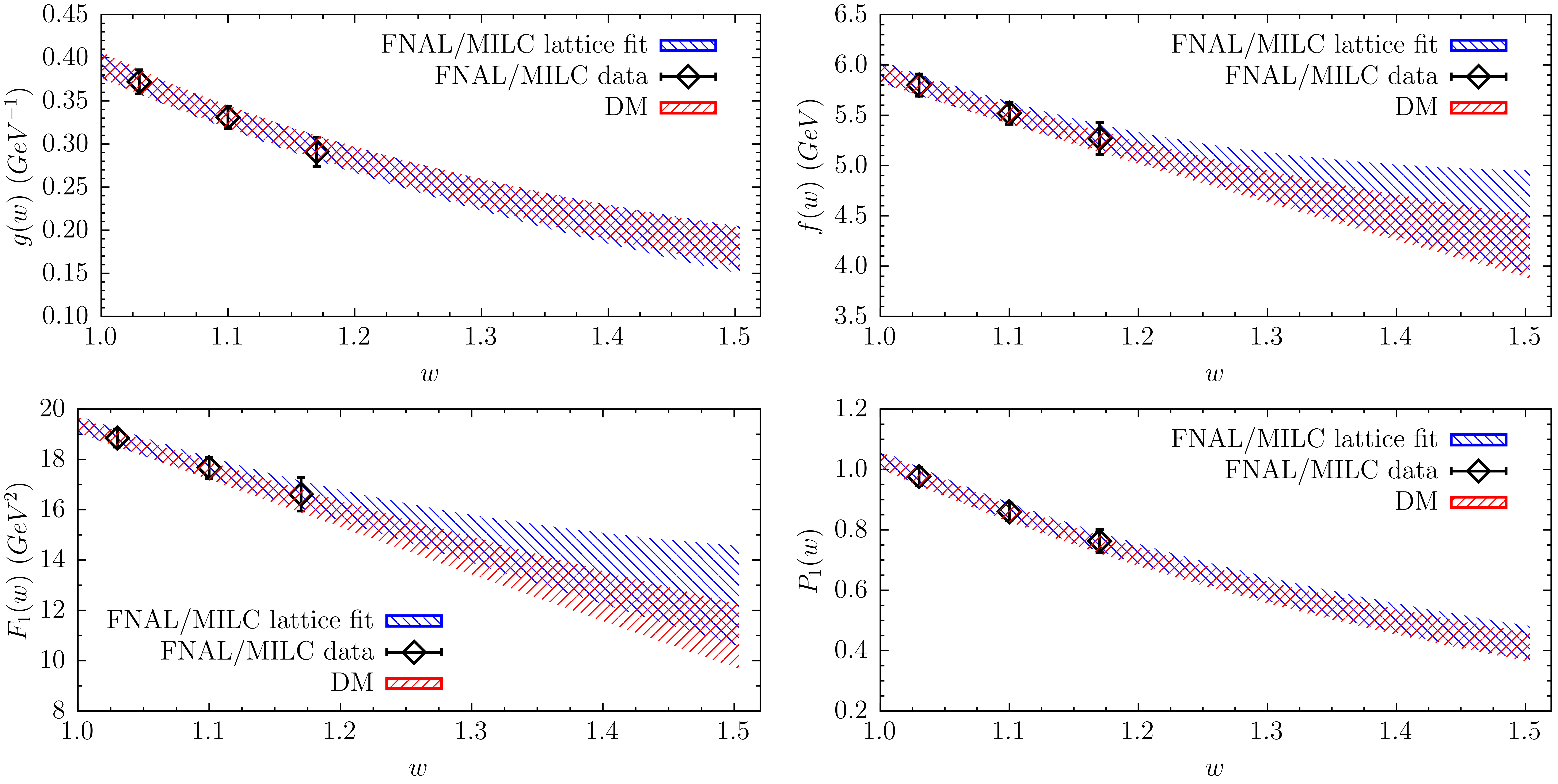
$$w = w_{max} : \qquad P_1(w_{max}) = \frac{\mathcal{F}_1(w_{max})}{m_B^2(1+w_{max})(1-r)\sqrt{r}}$$

extrapolation at maximum recoil

$$f(w_{max}) = 4.19 \pm 0.31 \text{ GeV} \,,$$
$$g(w_{max}) = 0.180 \pm 0.023 \text{ GeV}^{-1} \,,$$
$$\mathcal{F}_1(w_{max}) = 11.0 \pm 1.3 \text{ GeV}^2 \,,$$
$$P_1(w_{max}) = 0.411 \pm 0.048 \,.$$

LCSR: $\mathcal{F}_1(w_{max}) = 16.0 \pm 2.1$ GeV^2 (arXiv:1811.00983)

* comparison with FNAL/MILC "lattice fit" from arXiv:2105.14019 → blue bands: quadratic BGL fit of LQCD points only



blue bands

$$\sum_i a_i^2 \leq 1 \qquad 68\,\% \quad (g)$$

$$\sum_i \left(b_i^2 + c_i^2\right) \leq 1 \qquad 94\,\% \; (f + \mathcal{F}_1)$$

$$\sum_i d_i^2 \leq 1 \qquad 67\,\% \quad (P_1)$$

43 % of events satisfy unitarity

KC at w=1: OK

KC at w=w_{max}: not applied

red bands (DM)

100 % of events satisfying unitarity

KC at w=1: OK

KC at w=w_{max}: OK

(after iterative procedure)

* overall consistency, differences hidden in the correlations among the FFs at different values of w

* some differences for $\mathcal{F}_1(w_{max})$: some impact on $R(D^*)$

$$R(D^*) = 0.265 \pm 0.013$$

$$R(D^*) = 0.275 \pm 0.008$$

extraction of $|V_{cb}|$ from $B \to D^* \ell \nu_\ell$ decays

[arXiv:2109.15248]

*** we do not mix theoretical calculations with experimental data to describe the shape of the FFs ***

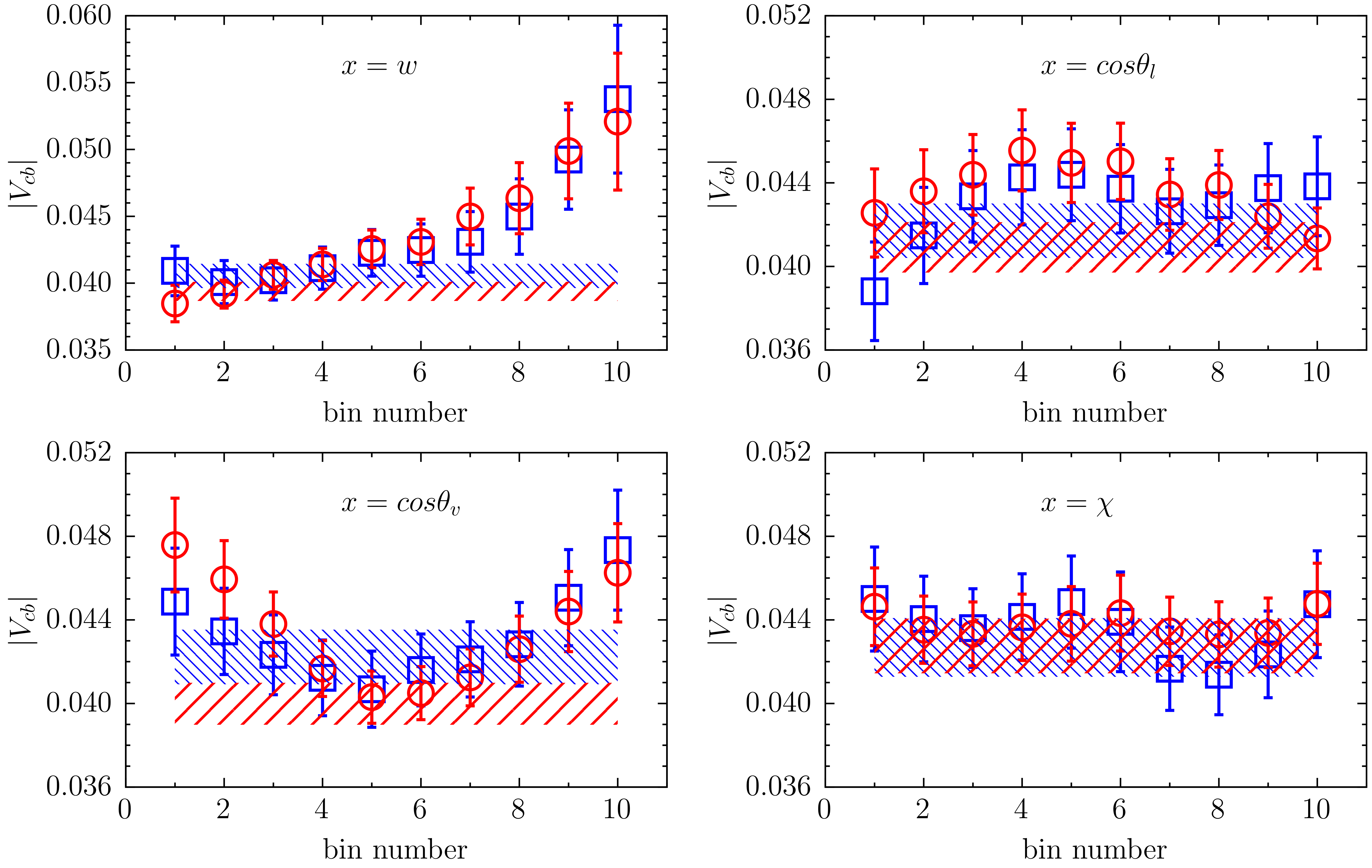
$$|V_{cb}|_i \equiv \sqrt{\frac{(d\Gamma/dx)_i^{exp}}{(d\Gamma/dx)_i^{th}}} \qquad i = 1, \ldots, N_{bins}$$

four different differential decay rates $d\Gamma/dx$ where $x = \{w, \cos\theta_v, \cos\theta_\ell, \chi\}$:

- 10 bins for each variable
- total of 80 data points

blue data: Belle 1702.01521

red data: Belle 1809.03290



bands are (correlated) weighted averages

$$|V_{cb}| = \frac{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij} |V_{cb}|_j}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} ,$$

$$\sigma^2_{|V_{cb}|} = \frac{1}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} ,$$

	experiment	$\lvert V_{cb}\rvert(x=w)$	$\lvert V_{cb}\rvert(x=\cos\theta_l)$	$\lvert V_{cb}\rvert(x=\cos\theta_v)$	$\lvert V_{cb}\rvert(x=\chi)$
Belle 1702.01521	Ref. [11]	0.0398 (9)	0.0422 (13)	0.0421 (13)	0.0426 (14)
Belle 1809.03290	Ref. [12]	0.0395 (7)	0.0405 (11)	0.0402 (10)	0.0430 (13)

averaging procedure

$$\mu_x = \frac{1}{N}\sum_{k=1}^{N} x_k \ ,$$

$$\sigma_x^2 = \frac{1}{N}\sum_{k=1}^{N}\sigma_k^2 + \frac{1}{N}\sum_{k=1}^{N}(x_k-\mu_x)^2,$$

→ $|V_{cb}|_{excl.} \cdot 10^3 = 41.3 \pm 1.7$

$|V_{cb}|_{incl.} \cdot 10^3 = 42.16 \pm 0.50$ (Bordone et al: arXiv:2107.00604)

exclusive/inclusive tension reduced to less than 1σ

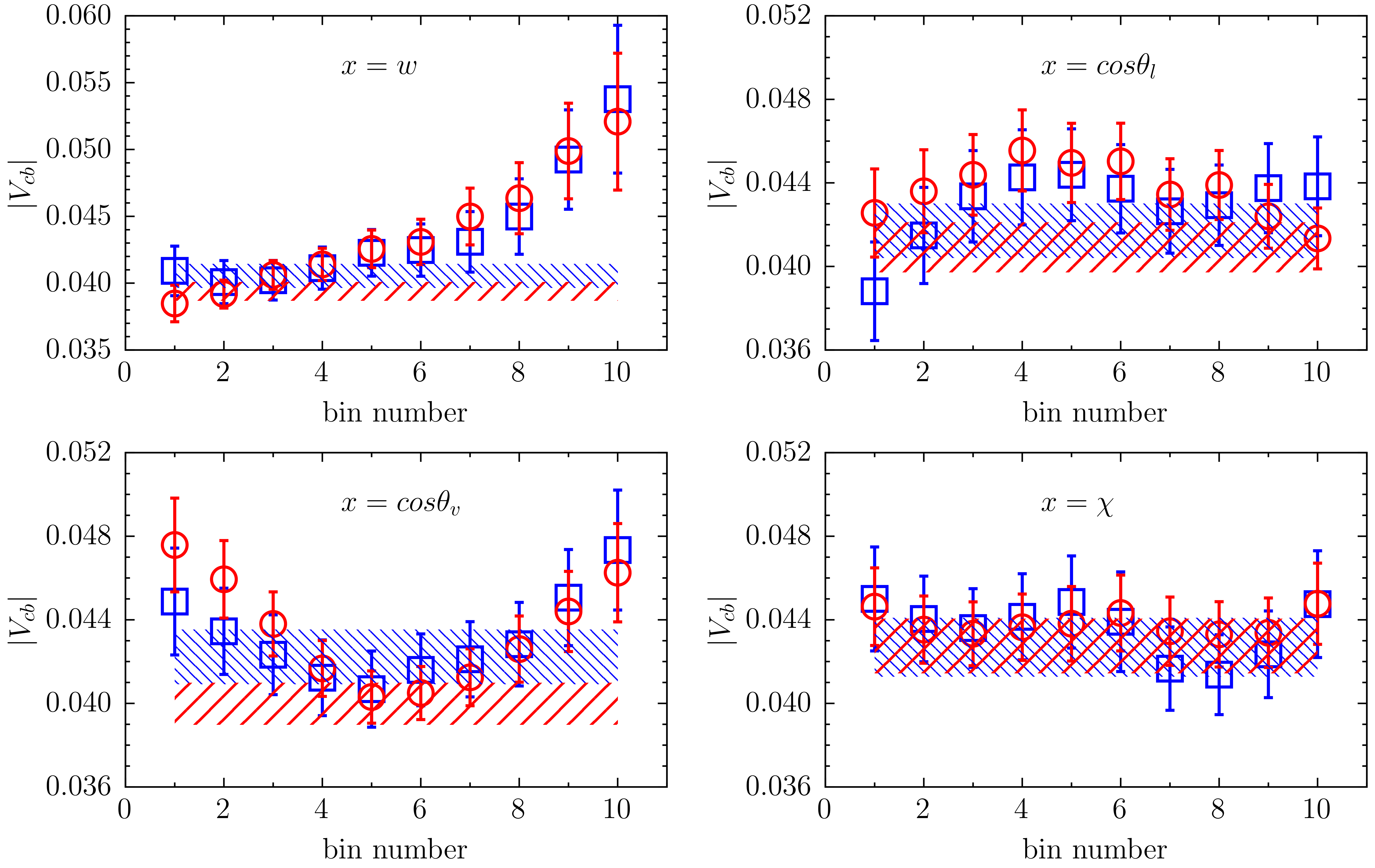
the use of exp. data to describe the shape of the FFs leads to smaller errors, but it produces a bias on the extracted value of $|V_{cb}|$

←

$|V_{cb}|_{excl.} \cdot 10^3 = 39.6^{+1.1}_{-1.0}$ Gambino et al., arXiv:1905.08209

$|V_{cb}|_{excl.} \cdot 10^3 = 39.56^{+1.04}_{-1.06}$ Jaiswal et al., arXiv:2002.05726

$|V_{cb}|_{excl.} \cdot 10^3 = 38.86 \pm 0.88$ FLAG '21, arXiv:2111.09849



Remark 1

The value of $|V_{cb}|$ exhibits some dependence on the specific w-bin. The value obtained adopting a constant fit is dominated by the bins at small values of the recoil w, where direct lattice data are available and the lenght of the momentum extrapolation is limited.

Remark 2

The value of $|V_{cb}|$ deviates from a constant fit for $x = \cos(\theta_v)$. If we try a quadratic fit of the form

$$|V_{cb}|\left[1 + \delta B\ \cos^2(\theta_v)\right]$$

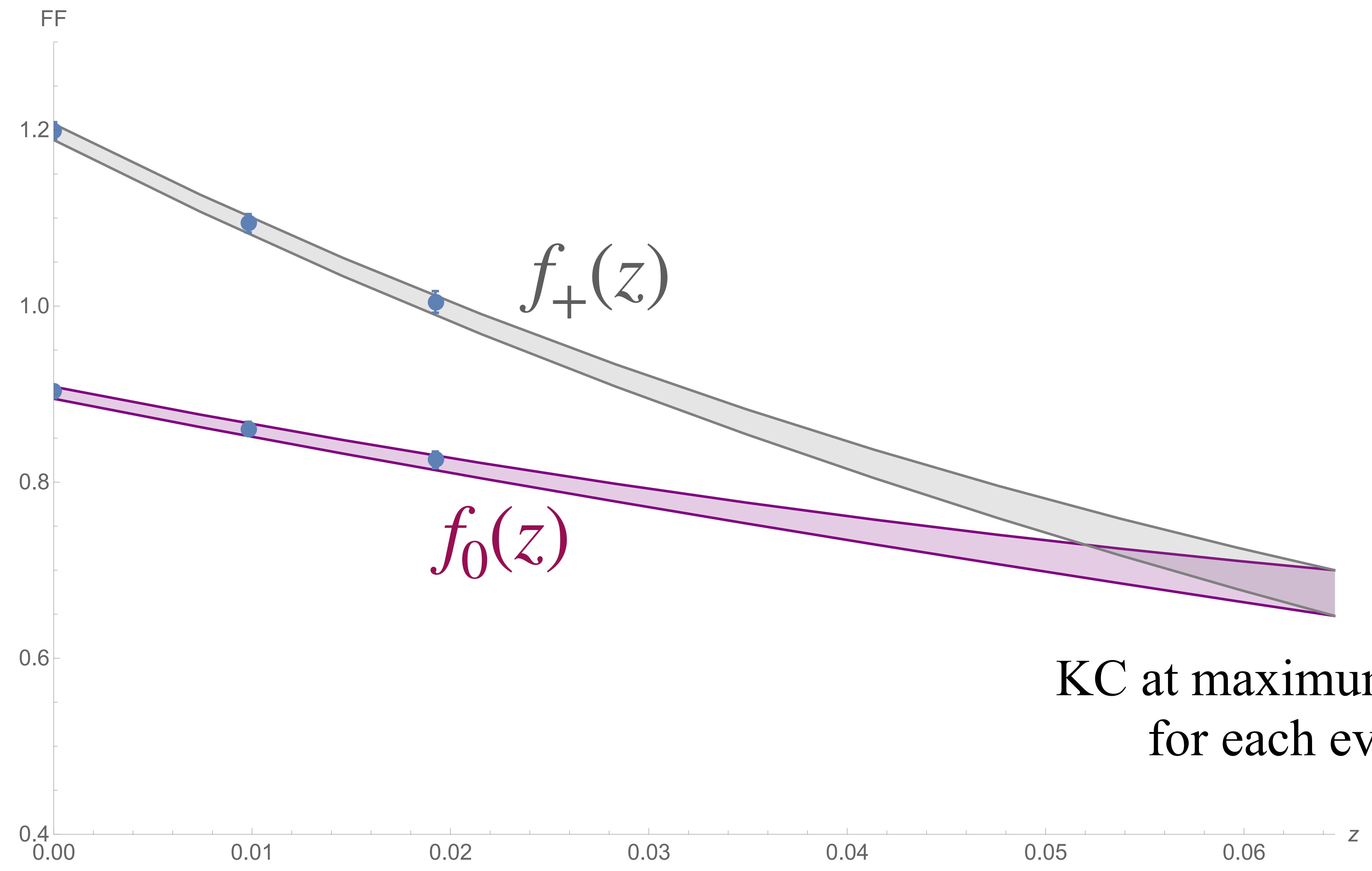
we get $\delta B \neq 0$ (2-3σ level) and $|V_{cb}|$ more consistent between the two sets of Belle data, but still in agreement with the value of $|V_{cb}|$ obtained with a constant fit

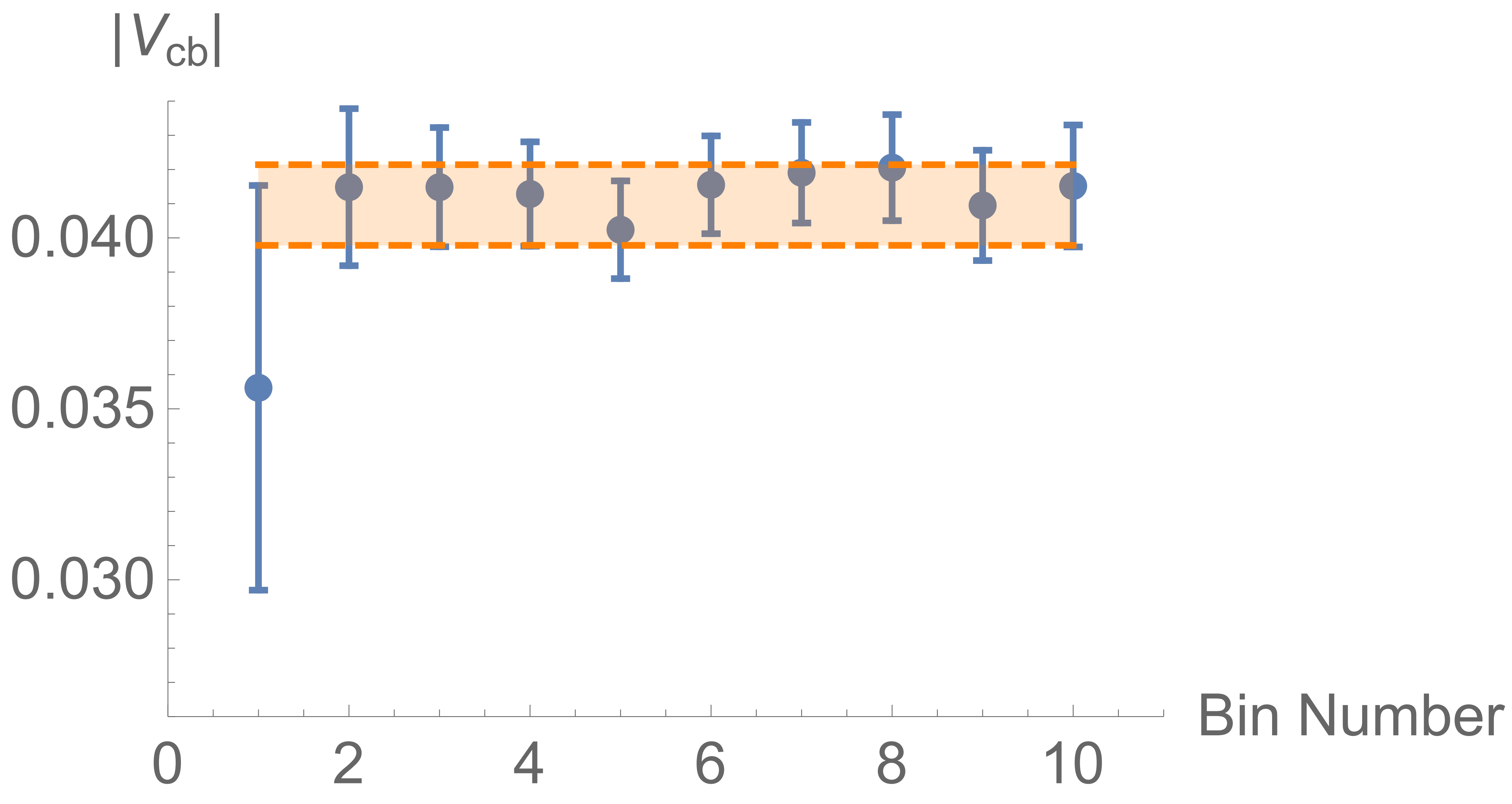
Both remarks appear to be related to a different w-slope of the theoretical FFs based on the lattice results from FNAL/MILC with respect to the Belle experimental data. This crucial issue (a kind of ***slope puzzle***) needs to be further investigated by forthcoming calculations of the FFs at non-zero recoil expected from the JLQCD Collaboration (see Kaneko's talk) as well as by future improvements of the precision of the experimental data.

extraction of $|V_{cb}|$ from $B \to D\ell\nu_\ell$ decays

[arXiv:2105.08674]

* lattice QCD form factors from **FNAL/MILC (arXiv:1503.07237)**: synthetic data points at 3 (small) values of the recoil
* experimental data from Belle collaboration in 10 bins (arXiv:1510.03657)





$$|V_{cb}|_{excl.} \cdot 10^3 = 41.0 \pm 1.2$$

nice consistency with $|V_{cb}|$ from $B \to D^*$

$|V_{cb}|_{excl.} \cdot 10^3 = 40.49 \pm 0.97$ Gambino et al., arXiv:1606.08030

$|V_{cb}|_{excl.} \cdot 10^3 = 41.0 \pm 1.1$ Jaiswal et al., arXiv:1707.09977

$|V_{cb}|_{excl.} \cdot 10^3 = 40.0 \pm 1.0$ FLAG '21, arXiv:2111.09849

no tension between exp and theory shapes

↓

no bias on the extracted value of $|V_{cb}|$

$R(D)$, $R(D^*)$ and polarization observables

* pure theoretical and parameterization-independent determinations within the DM approach

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.339 (26) (14)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (10) (10)	$\simeq 1.2\,\sigma$
$P_\tau(D^*)$	-0.52 (1)	-0.38 (51) $\left({}^{+21}_{-16}\right)$	
$F_L(D^*)$	0.42 (1)	0.60 (8) (4)	$\simeq 2.0\,\sigma$

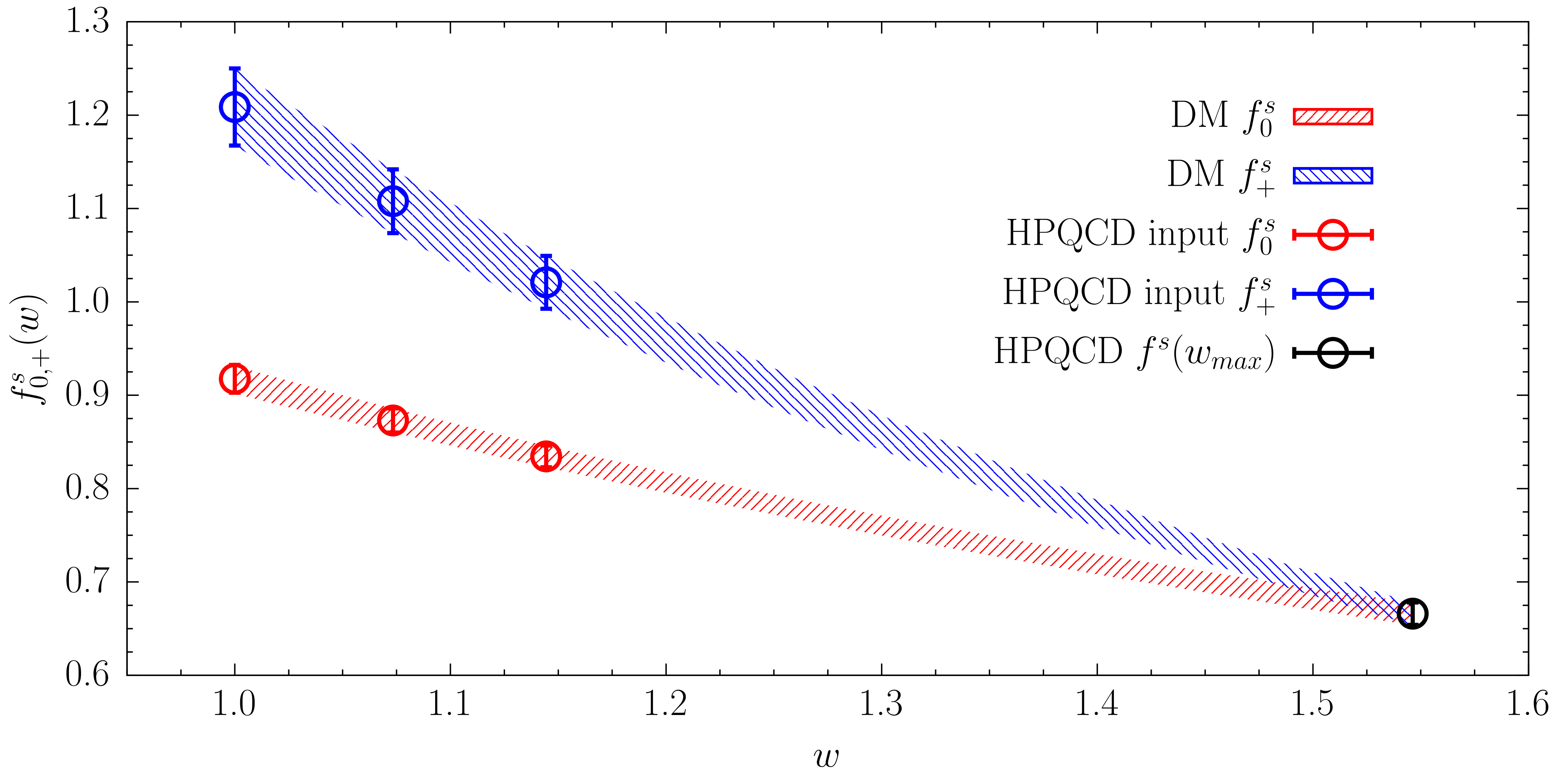
***** exp/SM tension significantly reduced for $R(D^*)$ *****

form factors for $B_s \to D_s^{(*)} \ell \nu_\ell$ decays

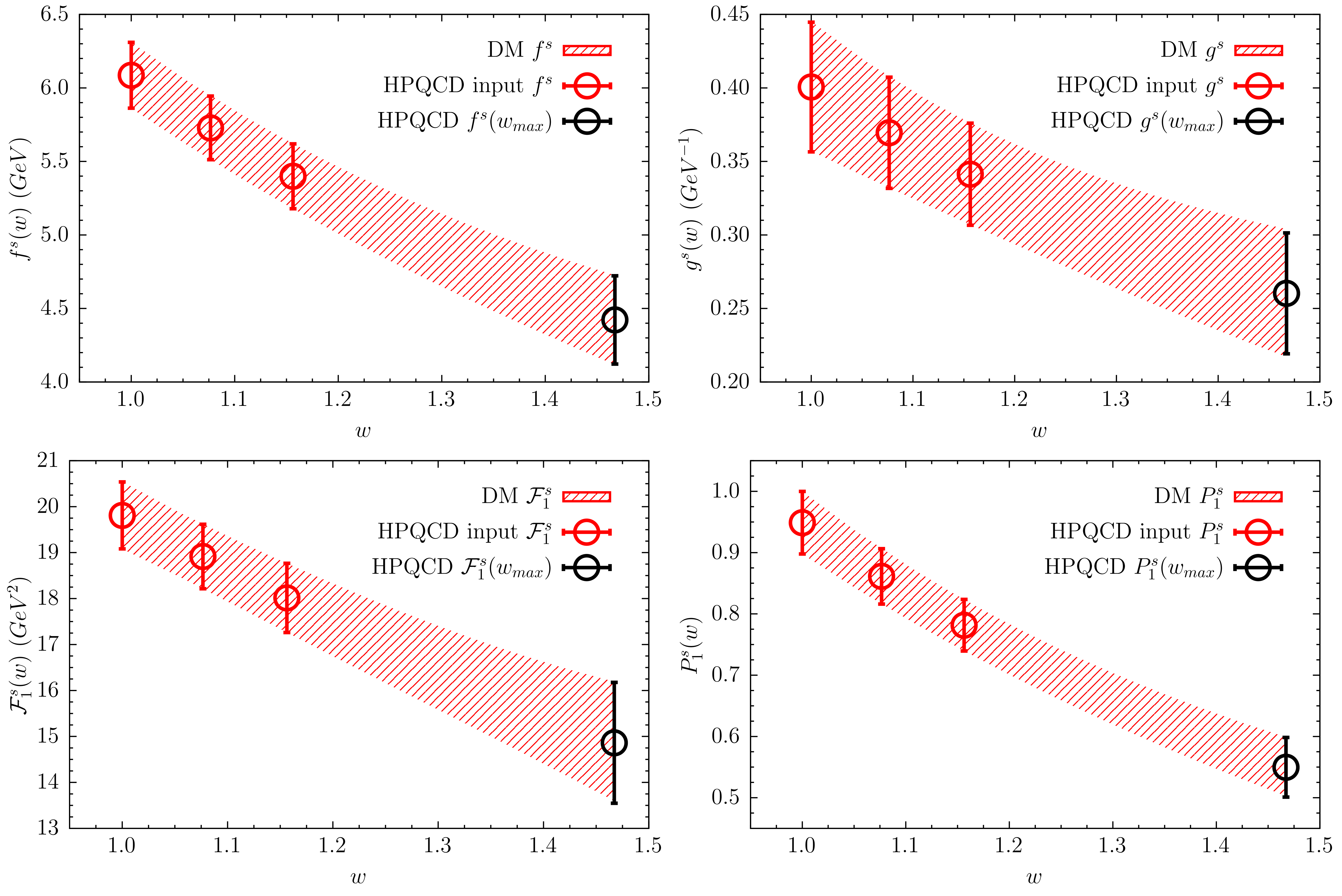
[arXiv:2204.05925]

* lattice QCD form factors from **HPQCD arXiv:1906.00701**($B_s \to D_s$) and **arXiv:2105.11433** ($B_s \to D_s^*$) in the form of BCL fits in the whole kinematical range
* we extract 3 data points for the FFs at small values of the recoil and we apply the DM approach

$B_s \to D_s \ell \nu_\ell$



$B_s \to D_s^* \ell \nu_\ell$



* nice agreement in the whole kinematical range

extraction of |V$_{cb}$| from $B_s \to D_s^{(*)} \ell \nu_\ell$ decays

two different runs at LHC

* two sets of experimental data from LHCb collaboration: arXiv:2001.03225 and arXiv:2003.08453

* first analysis: ratios of branching ratios [2001.03225]

$$\frac{\mathscr{B}(B_s \to D_s \mu \nu_\mu)}{\mathscr{B}(B \to D \mu \nu_\mu)} = 1.09 \pm 0.05_{stat} \pm 0.06_{syst} \pm 0.05_{inputs} = 1.09 \pm 0.09$$

$$\frac{\mathscr{B}(B_s \to D_s^* \mu \nu_\mu)}{\mathscr{B}(B \to D^* \mu \nu_\mu)} = 1.06 \pm 0.05_{stat} \pm 0.07_{syst} \pm 0.05_{inputs} = 1.06 \pm 0.10$$

- using the PDG values for $\mathscr{B}(B \to D^{(*)} \mu \nu_\mu)$ and the B_s-meson lifetime one gets

$$\Gamma^{\rm LHCb}(B_s \to D_s \mu \nu_\mu) = (1.08 \pm 0.10) \cdot 10^{-14} \ \text{GeV}$$
$$\Gamma^{\rm LHCb}(B_s \to D_s^* \mu \nu_\mu) = (2.34 \pm 0.26) \cdot 10^{-14} \ \text{GeV}$$

to be compared with

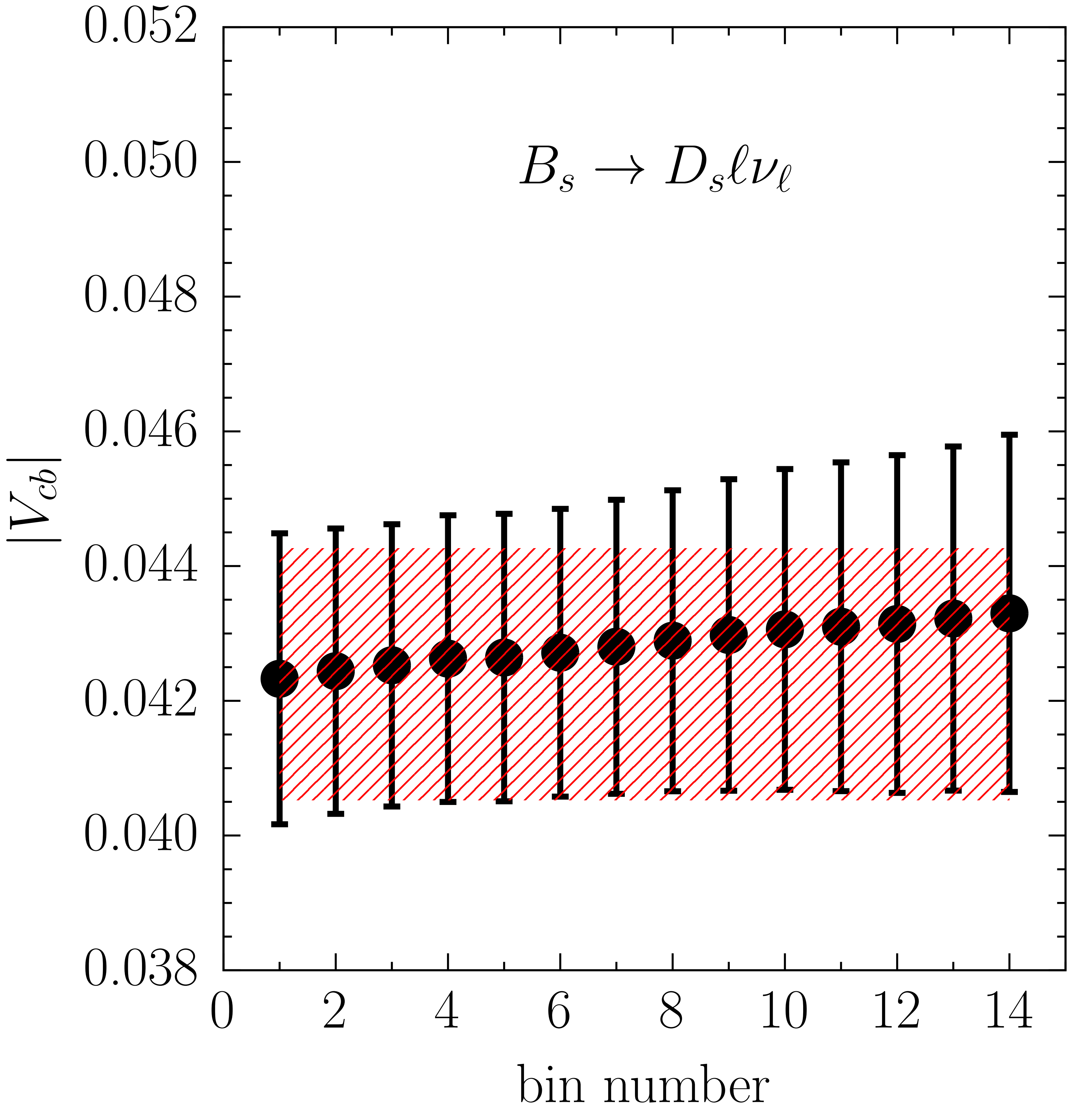
$$\Gamma^{\rm DM}(B_s \to D_s \mu \nu_\mu)/|V_{cb}|^2 = (6.04 \pm 0.23) \cdot 10^{-12} \ \text{GeV}$$
$$\Gamma^{\rm DM}(B_s \to D_s^* \mu \nu_\mu)/|V_{cb}|^2 = (1.39 \pm 0.11) \cdot 10^{-11} \ \text{GeV}$$

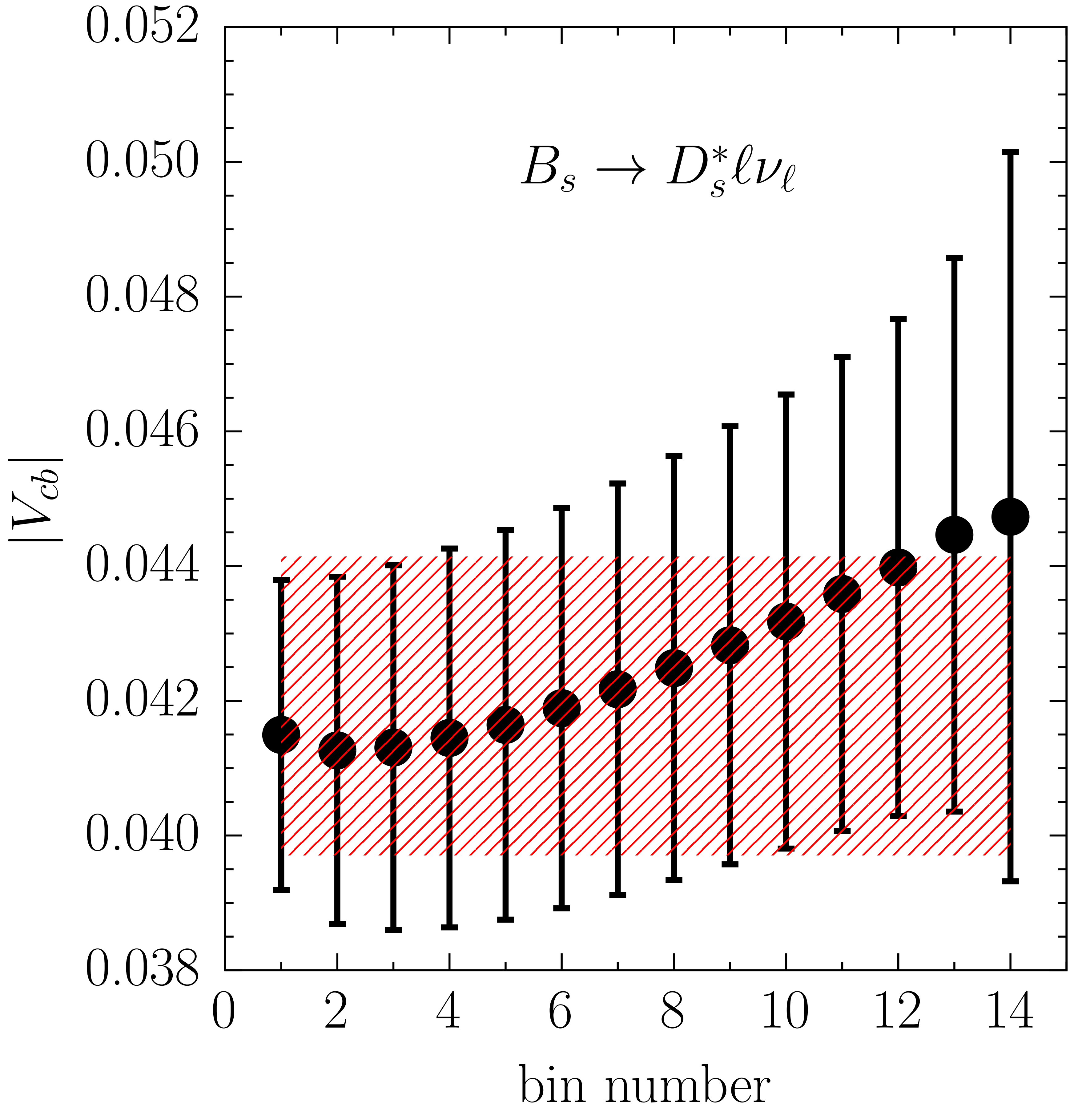
⇓

decays	$\lvert V_{cb} \rvert^{\rm DM} \cdot 10^3$
$B_s \to D_s \ell \nu_\ell$	42.3 ± 2.1
$B_s \to D_s^* \ell \nu_\ell$	41.0 ± 2.8

* second analysis: differential decay rates reconstructed from the LHCb fits of $p_\perp$ distributions (BGL/CLN parameterizations for the FFs) carried out in arXiv:2001.03225

bin-per-bin analysis: $|V_{cb}|_j \equiv \sqrt{\dfrac{d\Gamma^{\rm LHCb}/dw_j}{d\Gamma^{\rm DM}/dw_j}}$ $\quad j = 1,\ldots,N_{bins}$ $\quad$ we adopted $N_{bins} = 14$ w-bins





correlated weighted averages

$$|V_{cb}| = \frac{\sum_{i,j=1}^{N_{\rm bins}} (\mathbf{C}^{-1})_{ij} |V_{cb}|_j}{\sum_{i,j=1}^{N_{\rm bins}} (\mathbf{C}^{-1})_{ij}}$$

$$\sigma^2_{|V_{cb}|} = \frac{1}{\sum_{i,j=1}^{N_{\rm bins}} (\mathbf{C}^{-1})_{ij}}$$

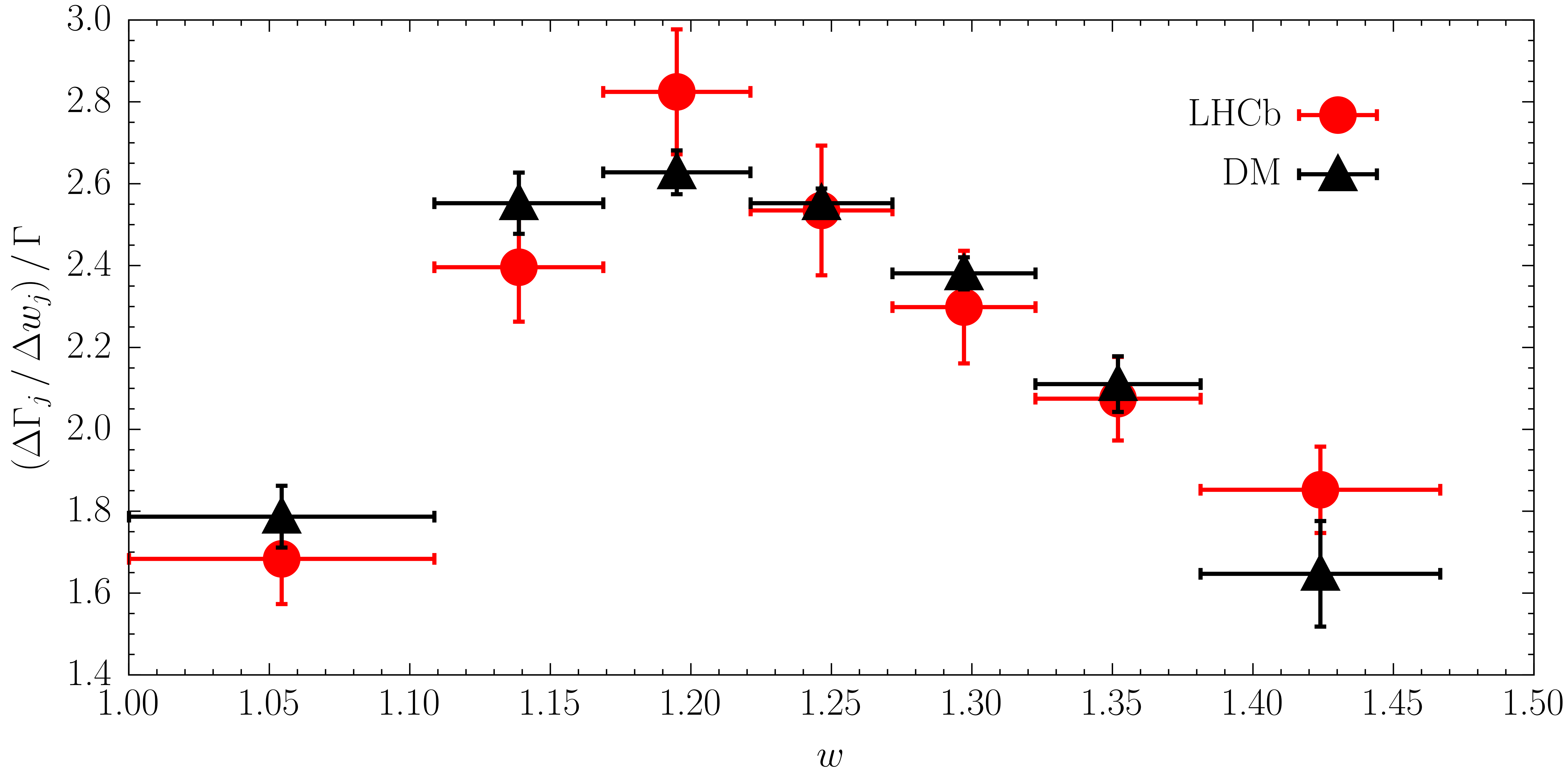
decays	$\|V_{cb}\|^{\rm DM} \cdot 10^3$
$B_s \to D_s \ell \nu_\ell$	42.4 ± 1.9
$B_s \to D_s^* \ell \nu_\ell$	41.9 ± 2.2

$|V_{cb}|^{\rm LHCb} \cdot 10^3 = 42.3 \pm 1.7$

* third analysis: LHCb ratios from arXiv:2003.08453

$$\Delta r_j = \frac{\Delta\Gamma_j(B_s \to D_s^* \mu \nu_\mu)}{\Gamma(B_s \to D_s^* \mu \nu_\mu)} \qquad j = 1,\ldots,7$$

j	1	2	3	4	5	6	7
w-bin	1.000 - 1.1087	1.1087 - 1.1688	1.1688 - 1.2212	1.2212 - 1.2717	1.2717 - 1.3226	1.3226 - 1.3814	1.3814 - 1.4667
Δw_j	0.1087	0.0601	0.0524	0.0505	0.0509	0.0588	0.0853
$\Delta r_j^{\mathrm{LHCb}}$	0.183(12)	0.144(8)	0.148(8)	0.128(8)	0.117(7)	0.122(6)	0.158(9)
Δr_j^{DM}	0.1942(82)	0.1534(45)	0.1377(28)	0.1289(18)	0.1212(20)	0.1241(40)	0.1405(110)



consistency within ~1σ

⇩

shape of theoretical FFs is consistent with the one of the experimental data

* to determine $|V_{cb}|$ we evaluate the integrated differential decay rates for each bin

$$\Delta\Gamma_j^{\rm exp} = \Delta r_j^{\rm LHCb} \cdot \Gamma^{\rm LHCb}(B_s \to D_s^* \mu\nu_\mu) \qquad j = 1,\dots,7$$

and the covariance matrix: $\Gamma_{ij}^{\rm exp} = R_{ij}^{\rm LHCb}\left[\overline{\Gamma}^2 + \sigma_{\overline{\Gamma}}^2\right] + \Delta r_i^{\rm LHCb}\Delta r_j^{\rm LHCb}\sigma_{\overline{\Gamma}}^2$

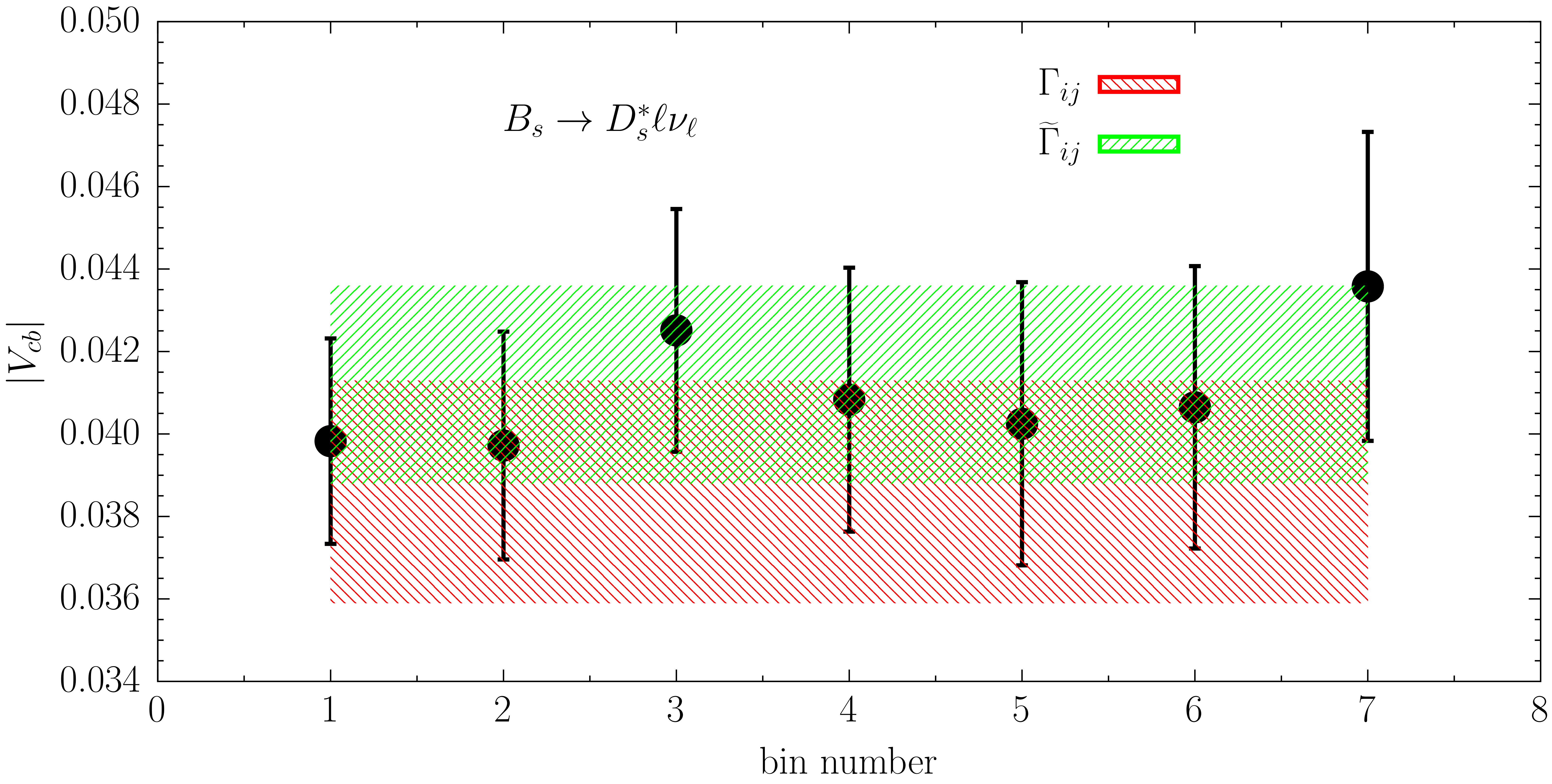
$\Gamma^{\rm LHCb}(B_s \to D_s^* \mu\nu_\mu)$ from arXiv:2001.03225

$$\overline{\Gamma} \pm \sigma_{\overline{\Gamma}} = (2.34 \pm 0.26) \cdot 10^{-14} \text{ GeV}$$

*** uncorrelated with $\Delta r_j^{\rm LHCb}$ ***

general property: $\sum_{i,j=1}^{N_{bins}} \Gamma_{ij}^{\rm exp} = \sigma_{\overline{\Gamma}}^2 \iff \sum_{i=1}^{N_{bins}} \Delta r_i^{\rm LHCb} = 1$ and $\sum_{i,j=1}^{N_{bins}} R_{ij}^{\rm LHCb} = 0$

- D'Agostini effect (NIMA '94): negative bias on constant fits to data affected by an overall normalization uncertainty; it depends upon $\sigma_{\overline{\Gamma}}$ and $\Delta r_i^{\rm LHCb} \neq \Delta r_j^{\rm LHCb}$



<u>modified covariance matrix</u>

$$\widetilde{\Gamma}_{ij}^{\rm exp} = R_{ij}^{\rm LHCb}\left[\overline{\Gamma}^2 + \sigma_{\overline{\Gamma}}^2\right] + \sigma_{\overline{\Gamma}}^2 / N_{bins}^2$$

$$\sum_{i,j=1}^{N_{bins}} \widetilde{\Gamma}_{ij}^{\rm exp} = \sum_{i,j=1}^{N_{bins}} \Gamma_{ij}^{\rm exp} = \sigma_{\overline{\Gamma}}^2$$

correlated weighted averages

$$|V_{cb}| \cdot 10^3 = 38.6 \pm 2.7$$

$$|V_{cb}| \cdot 10^3 = 41.2 \pm 2.4$$

$|V_{cb}|^{\mathrm{DM}} \cdot 10^3$ from $B_s \to D_s^{(*)} \ell \nu_\ell$

analysis	$B_s \to D_s$	$B_s \to D_s^*$
first	42.3 ± 2.1	41.0 ± 2.8
second	42.4 ± 1.9	41.9 ± 2.2
third		41.2 ± 2.4
average	42.4 ± 2.0	41.4 ± 2.6

$$x = \sum_{k=1}^{N} \omega_k x_k$$

$$\sigma^2 = \sum_{k=1}^{N} \omega_k \left[\sigma_k^2 + (x_k - x)^2\right]$$

$$\omega_k = (1/\sigma_k^2) / \sum_{j=1}^{N} (1/\sigma_j^2)$$

summary of $|V_{cb}|^{\mathrm{DM}}$ from $B_{(s)} \to D_{(s)}^{(*)} \ell \nu_\ell$

decay	$\vert V_{cb}\vert^{\mathrm{DM}} \cdot 10^3$	inclusive [2107.00604]	exclusive [FLAG 21]
$B \to D$	41.0 ± 1.2		
$B \to D^*$	41.3 ± 1.7		
$B_s \to D_s$	42.4 ± 2.0		
$B_s \to D_s^*$	41.4 ± 2.6		
average	41.4 ± 0.8	42.16 ± 0.50	39.36 ± 0.68
difference		$\simeq 0.8\,\sigma$	$\simeq 1.9\,\sigma$

summary of $R(D_{(s)})$, $R(D^*_{(s)})$ and polarization observables

observable	DM
$R(D_s)$	0.298 (5)
$R(D_s^*)$	0.250 (6)
$P_\tau(D_s^*)$	−0.520 (12)
$F_L(D_s^*)$	0.440 (16)

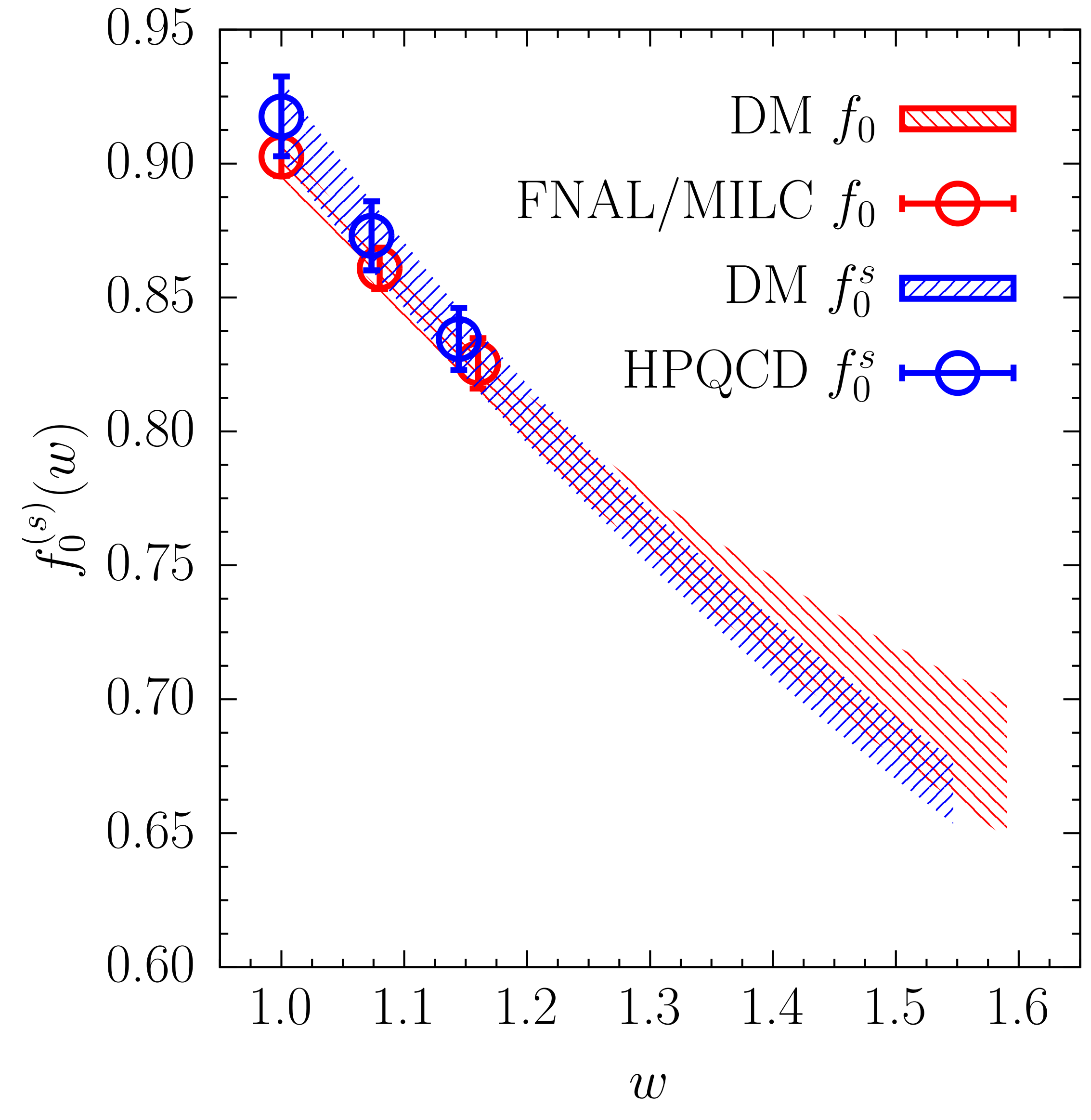
⟷ SU(3)F breaking ?

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.339 (27) (14)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (10) (10)	$\simeq 1.2\,\sigma$
$P_\tau(D^*)$	−0.52 (1)	$-0.38\,(51)\,(^{+21}_{-16})$	
$F_L(D^*)$	0.42 (1)	0.60 (8) (4)	$\simeq 2.0\,\sigma$

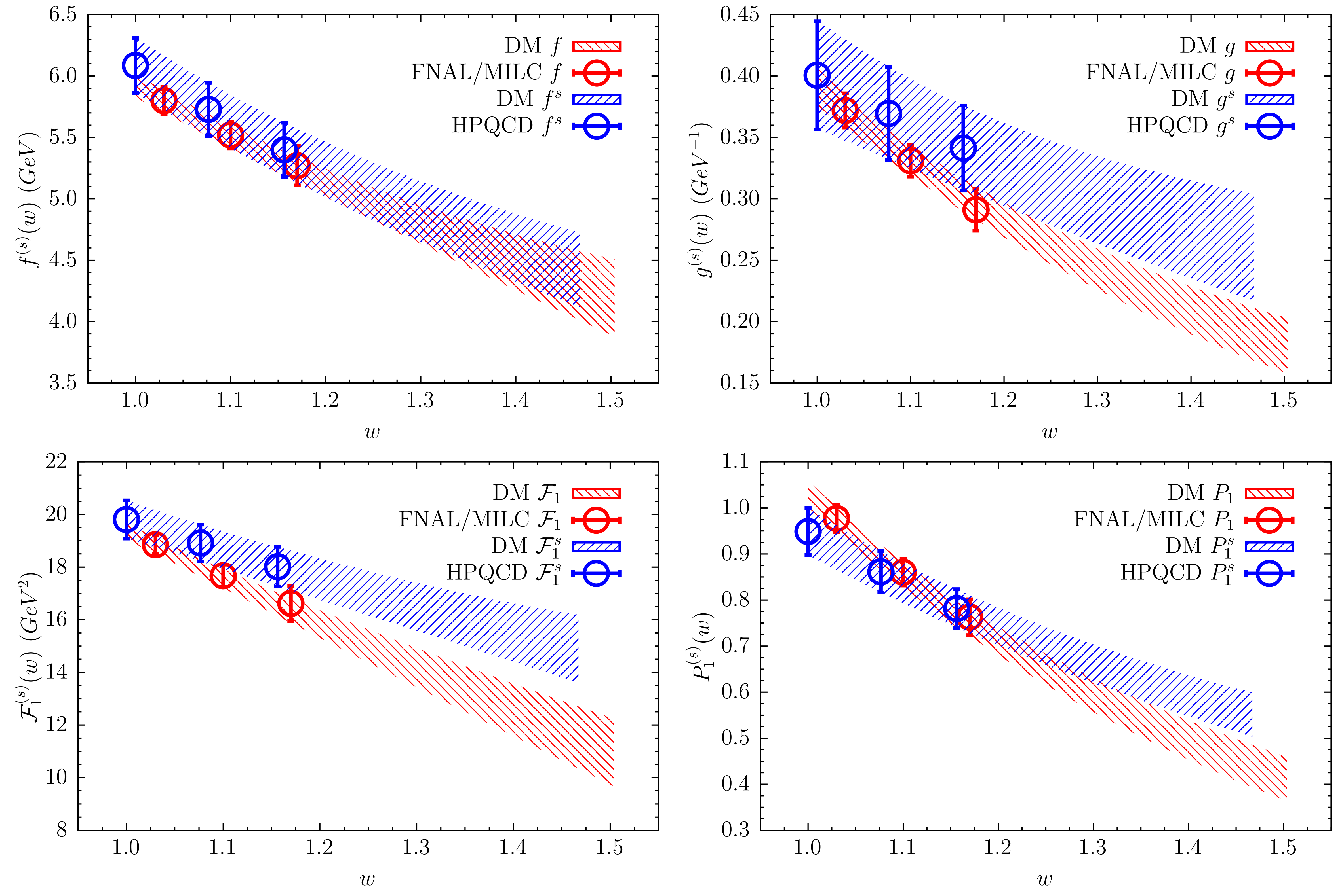
SU(3)$_F$ breaking effects

[arXiv:2204.05925]

$B_{(s)} \to D_{(s)} \ell \nu_\ell$



$B_{(s)} \to D^*_{(s)} \ell \nu_\ell$



red: u/d spectator quark

blue: strange spectator quark

ratios of branching ratios

$$\left.\frac{\mathcal{B}(B_s \to D_s \mu \nu_\mu)}{\mathcal{B}(B \to D \mu \nu_\mu)}\right|_{\text{LHCb}} = 1.09 \pm 0.09 \qquad \left.\frac{\mathcal{B}(B_s \to D_s \mu \nu_\mu)}{\mathcal{B}(B \to D \mu \nu_\mu)}\right|_{\text{DM}} = 1.02 \pm 0.06$$

$$\left.\frac{\mathcal{B}(B_s \to D_s^* \mu \nu_\mu)}{\mathcal{B}(B \to D^* \mu \nu_\mu)}\right|_{\text{LHCb}} = 1.06 \pm 0.10 \qquad \left.\frac{\mathcal{B}(B_s \to D_s^* \mu \nu_\mu)}{\mathcal{B}(B \to D^* \mu \nu_\mu)}\right|_{\text{DM}} = 1.19 \pm 0.11$$

- no SU(3)$_F$ breaking effects in $B_{(s)} \to PS$
- some SU(3)$_F$ breaking effects in $B_{(s)} \to V$

need of more precise exp. and theo. data

Conclusions

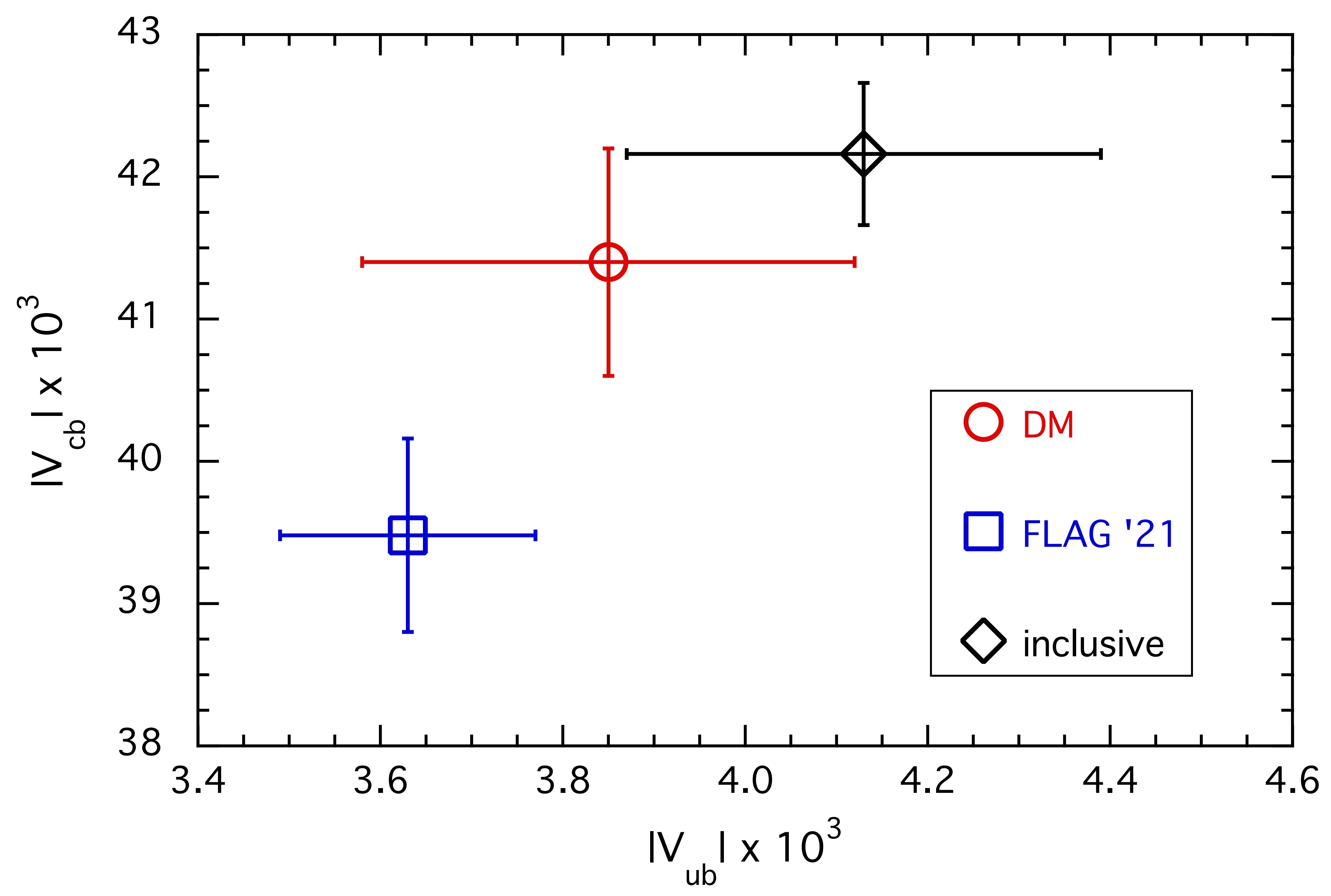
* the Dispersion Matrix approach is an attractive tool to implement unitarity and lattice QCD calculations in the analysis of exclusive semileptonic decays of hadrons. The main features are:

- it does not rely on any assumption about the momentum dependence of the hadronic form factors
- it can be based entirely on first principles using lattice determinations both of the relevant form factors and of the dispersive bounds (the susceptibilities) from appropriate 2-point and 3-point (Euclidean) correlation functions
- it allows to implement unitarity and kinematical constraints in a rigorous and parameterization-independent way
- it predicts band of values that are equivalent to the infinite number of BGL fits satisfying unitarity and KCs and reproducing exactly a given set of data points
- it can be applied to any exclusive semileptonic decay of hadrons

* results for $B_{(s)} \to D^{(*)}_{(s)} \ell \nu_\ell$ decays: extraction of $|V_{cb}|$ and theoretical determination of $R(D^{(*)}_{(s)})$ using LQCD results for the FFs (from FNAL/MILC and HPQCD) [2105.08674, 2109.15248, 2204.05925]

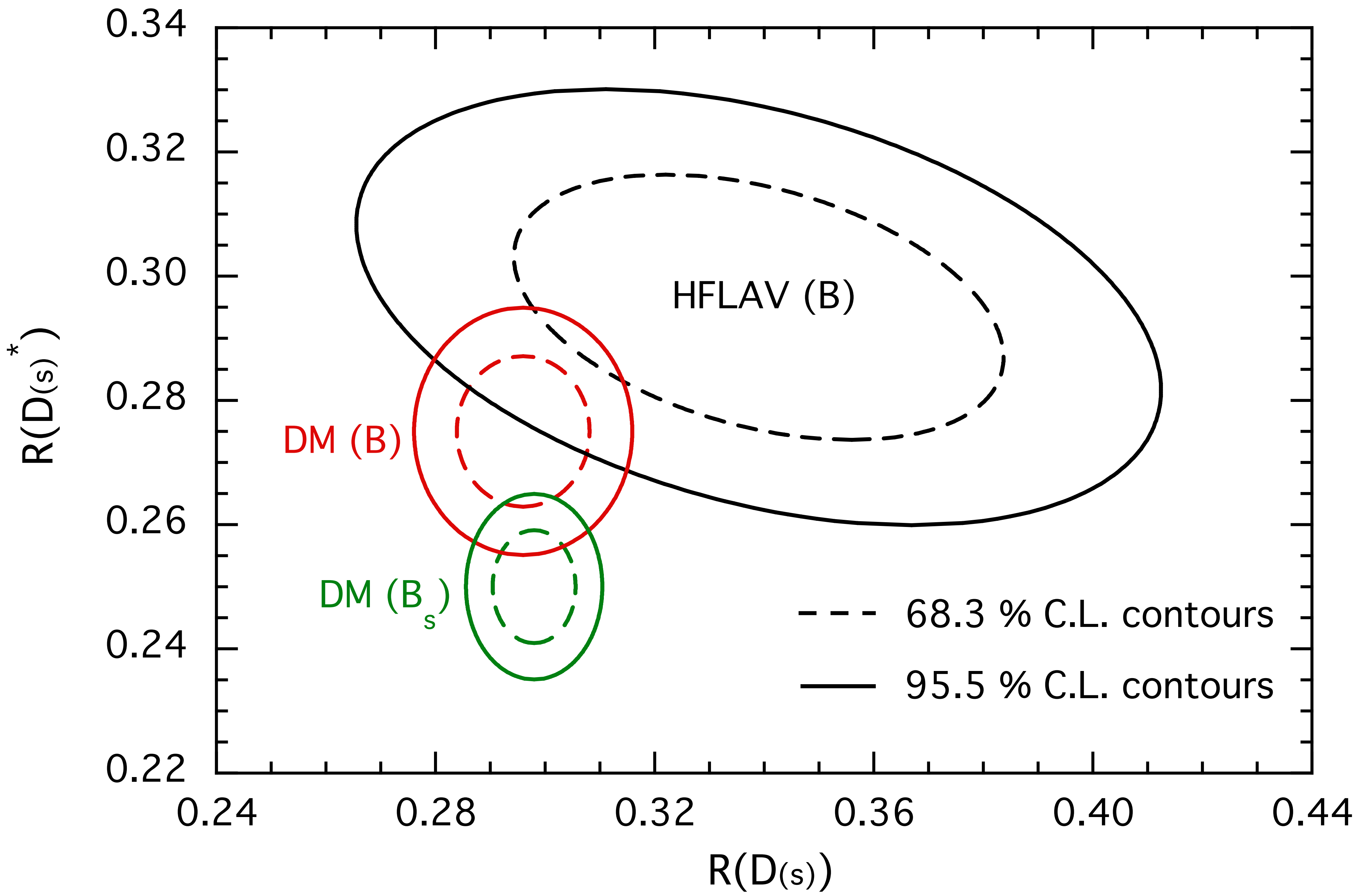
decay	$\vert V_{cb}\vert^{DM} \cdot 10^3$	inclusive [2107.00604]	exclusive [FLAG 21]
$B \to D$	41.0 ± 1.2		
$B \to D^*$	41.3 ± 1.7		
$B_s \to D_s$	42.4 ± 2.0		
$B_s \to D_s^*$	41.4 ± 2.6		
average	41.4 ± 0.8	42.16 ± 0.50	39.36 ± 0.68
difference		$\simeq 0.8\,\sigma$	$\simeq 1.9\,\sigma$

observable	DM	experiment	difference
$R(D)$	0.296 (8)	0.340 (27) (13)	$\simeq 1.4\,\sigma$
$R(D^*)$	0.275 (8)	0.295 (11) (8)	$\simeq 1.3\,\sigma$
$R(D_s)$	0.298 (5)		
$R(D_s^*)$	0.250 (6)		



	decays	DM	FLAG '21	inclusive
$\lvert V_{cb}\rvert \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.4 (8)	39.48 (68)	42.16 (50)
$\lvert V_{ub}\rvert \cdot 10^3$	$B_{(s)} \to \pi, K$	3.85 (27)	3.63 (14)	4.13 (26)

see Ludovico's slides in the discussion session

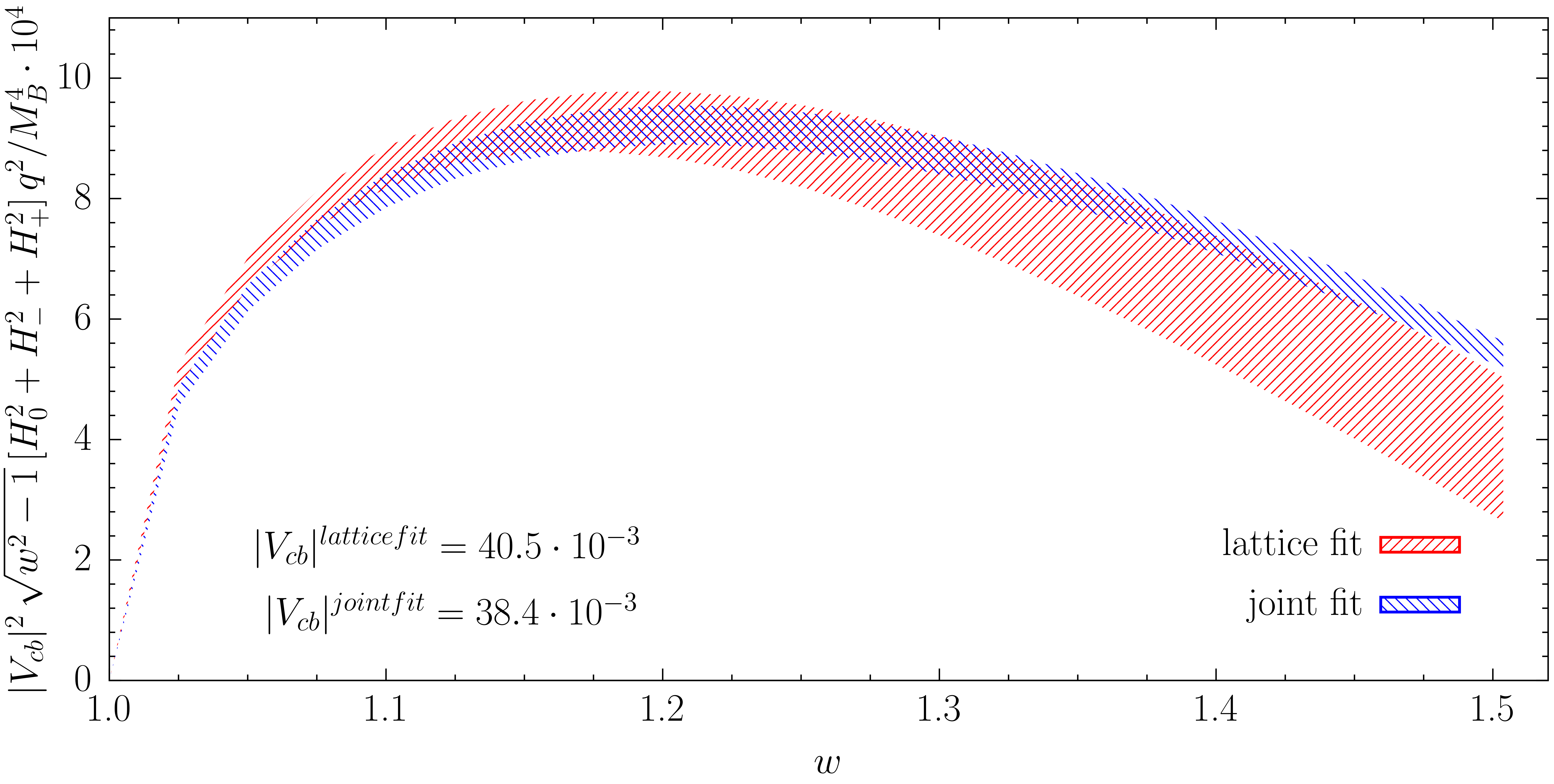


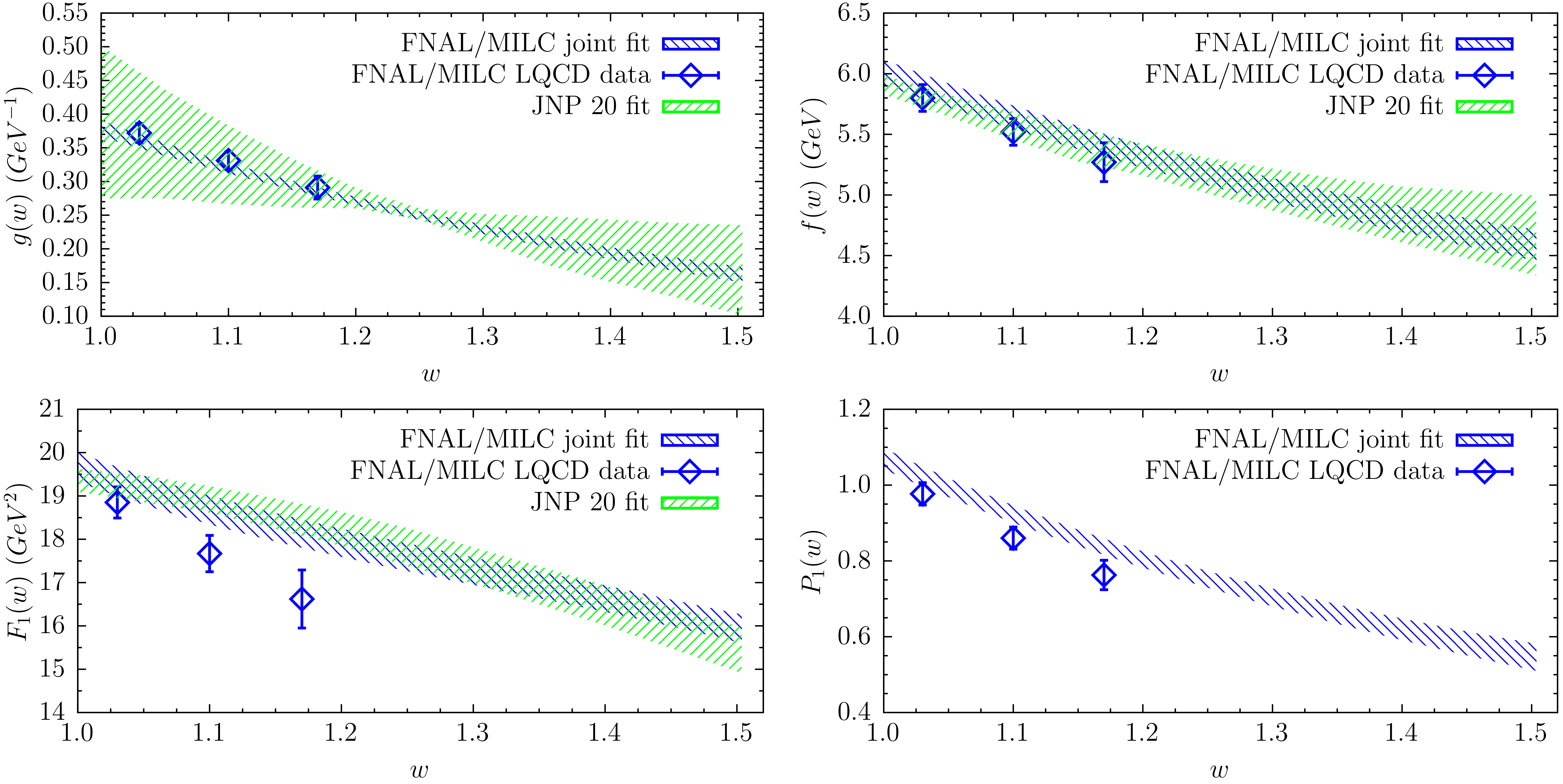
	DM	HFLAV '21
R(D)	0.296 (8)	0.339 (26) (14)
R(D*)	0.275 (8)	0.295 (10) (10)
$R(D_s)$	0.298 (5)	
$R(D_s^*)$	0.250 (6)	

reduced tensions in both $|V_{cb}|$, $|V_{ub}|$ and $R(D^*)$
when theory and experiments are not fitted simultaneously

backup slides

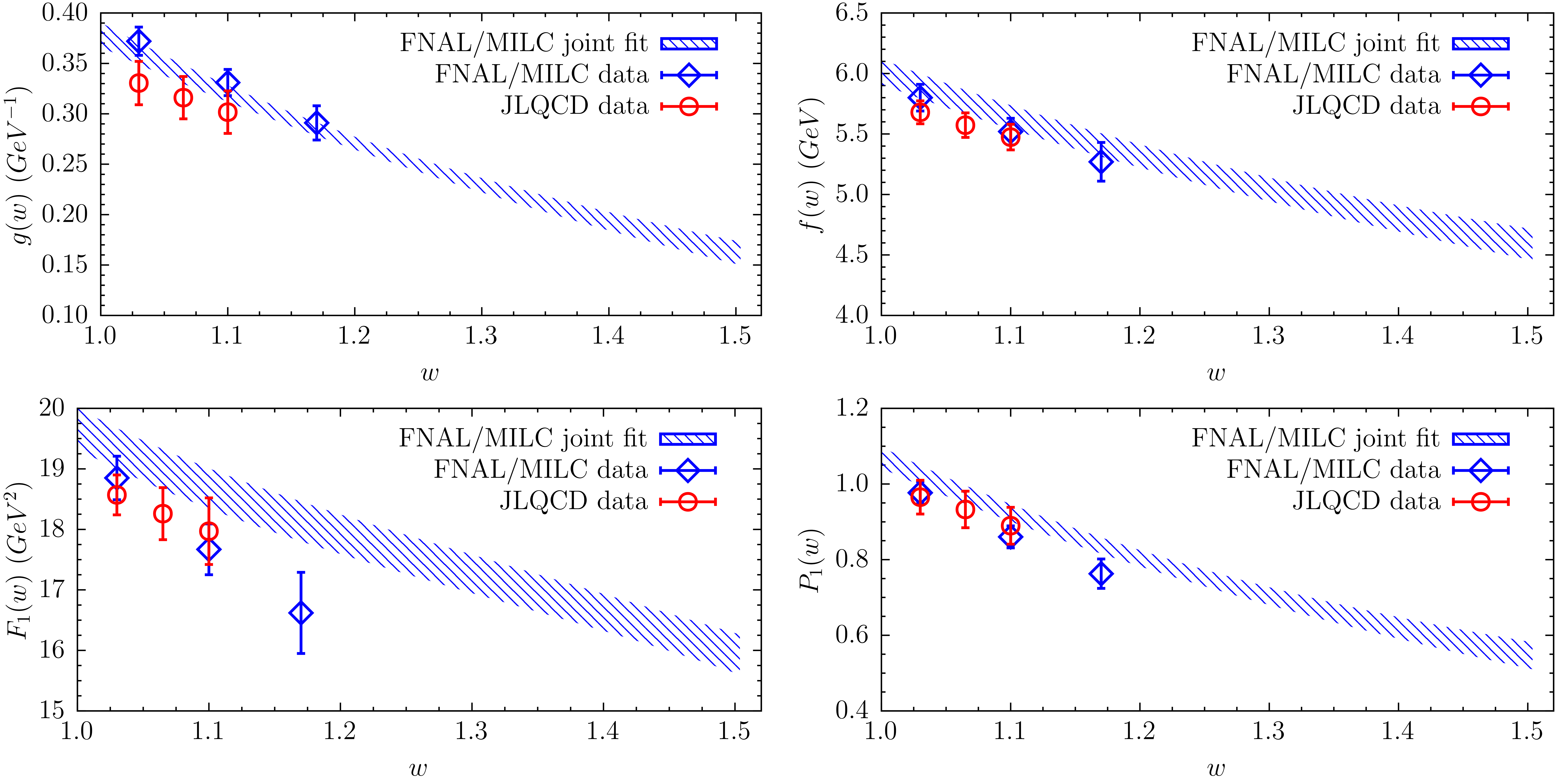
$$\frac{d\Gamma}{dw} \propto |V_{cb}|^2 \sqrt{w^2-1}\,\frac{q^2}{M_B^4}\left[H_0^2(w)+H_-^2(w)+H_+^2(w)\right] = |V_{cb}|^2\sqrt{w^2-1}\left\{\left(\frac{\mathcal{F}_1(w)}{M_B^2}\right)^2 + 2\frac{q^2}{M_B^2}\left[\left(\frac{f(w)}{M_B}\right)^2 + r^2(w^2-1)\,m_B^2\,g^2(w)\right]\right\} \qquad m_\ell = 0$$





FNAL/MILC joint fit (arXiv:2105.14019) uses Belle+BaBar data and new FNAL/MILC LQCD points

JNP 20 fit (Jaiswal et al. JHEP '20) uses Belle data + old FNAL/MILC LQCD point $h_{A_1}(1)$



FNAL/MILC data from arXiv:2105.14019

JLQCD data from slides of T. Kaneko at CKM '21

FNAL/MILC fit to lattice points (arXiv:2105.14019)

TABLE XI. Results of linear, quadratic, and unitarity-constrained cubic z expansions using only lattice-QCD data.

	Linear	Quadratic	Cubic
a_0	0.0330(12)	0.0330(12)	0.0330(12)
a_1	−0.157(52)	−0.155(55)	−0.155(55)
a_2		−0.12(98)	−0.12(98)
a_3			−0.004(1.000)
b_0	0.01229(23)	0.01229(24)	0.01229(23)
b_1	−0.002(10)	−0.003(12)	−0.003(12)
b_2		0.07(53)	0.05(55)
b_3			−0.01(1.00)
c_1	−0.0057(22)	−0.0058(25)	−0.0057(25)
c_2		−0.013(91)	−0.02(10)
c_3			0.10(95)
d_0	0.0508(15)	0.0509(15)	0.0509(15)
d_1	−0.317(59)	−0.327(67)	−0.327(67)
d_2		−0.03(96)	−0.02(96)
d_3			−0.0006(1.0000)
χ^2/dof	0.83/5	0.64/3	0.64/3
$\sum_i^N a_i^2$	0.026(16)	0.04(24)	0.04(24)
$\sum_i^N (b_i^2 + c_i^2)$	0.000193(69)	0.005(70)	0.01(18)
$\sum_i^N d_i^2$	0.103(37)	0.110(61)	0.110(52)

quadratic fit

$$\sum_{i=1}^{2} a_i^2 = 0.04 \pm 0.24 \quad ???$$

indeed: $a_2 = -0.12 \pm 0.98$

with 1σ one has $|a_2| > 1$!!!

what's going on ?

linearization of the error

$$(a_2 + \delta a_2)^2 - a_2^2 \to 2|a_2|\delta a_2 \approx 0.24$$

wrong when $\delta a_2 >> |a_2|$

- start from a set of input data $\{f_i\}$ with a given covariance matrix C_{ij} and a (eventually correlated) susceptibility χ
- generate a multivariate distribution of N_{boot} events
- for each event $k = 1,2,\ldots,N_{boot}$ evaluate the lower $f^k_{lo}(t)$ and upper $f^k_{up}(t)$ values of the form factor at a given t

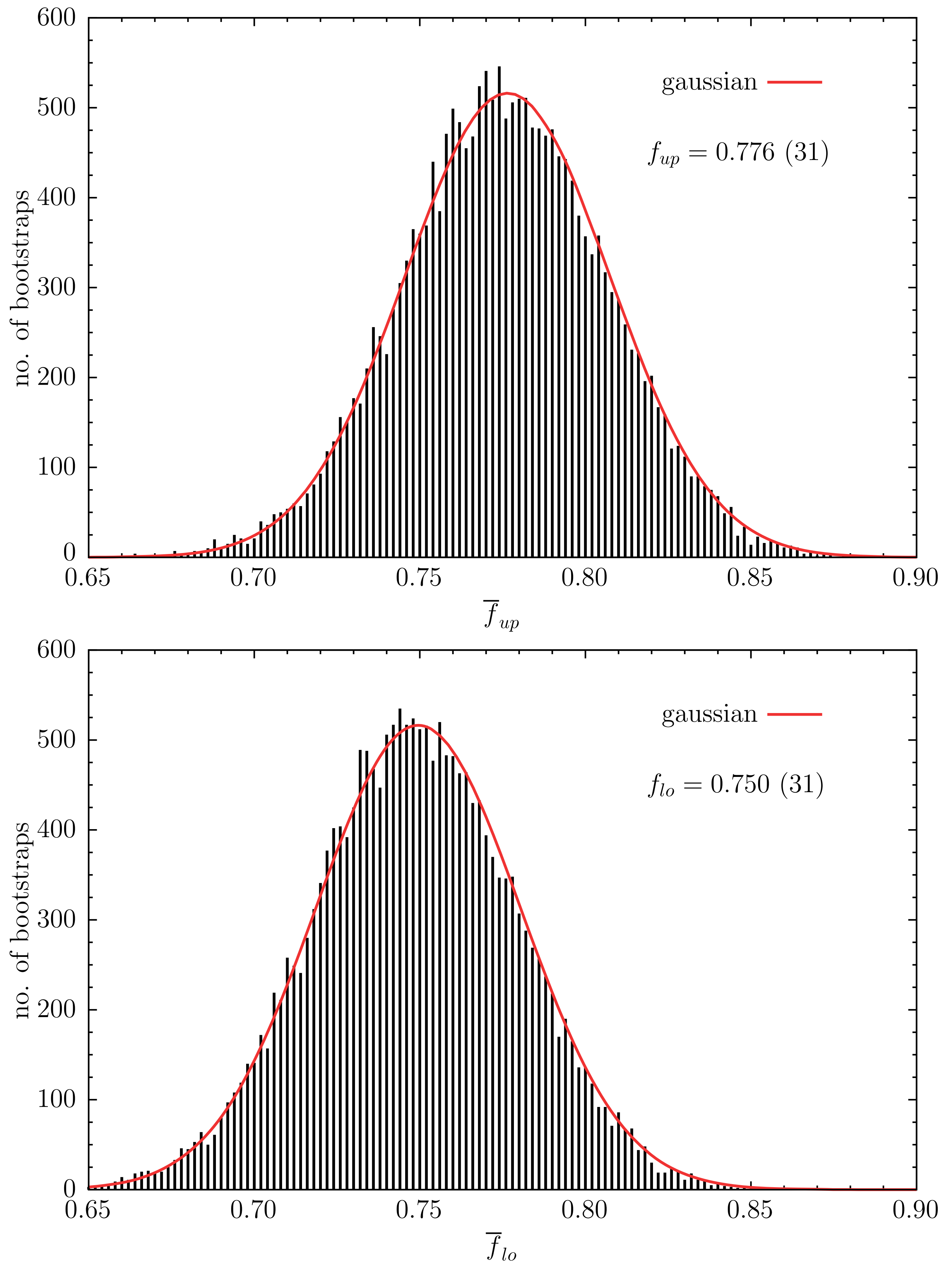


FIG. 1. Histograms of the values of $\bar{f}_{\rm up}$ (upper panel) and $\bar{f}_{\rm lo}$ (lower panel) for the bootstrap events that pass the unitarity filter in the case of the vector form factor $f_+(t = 0\ {\rm GeV}^2)$ of the $D \to K$ transition.

averages: $$\bar{f}_{lo(up)}(t) = \frac{1}{N_{boot}} \sum_{k=1}^{N_{boot}} f^k_{lo(up)}(t)$$

covariance: $$C_{L(U),L(U)} \equiv \frac{1}{N_{boot}-1} \sum_{k=1}^{N_{boot}} \left[f^k_{lo(up)}(t) - \bar{f}_{lo(up)}(t)\right] \left[f^k_{lo(up)}(t) - \bar{f}_{lo(up)}(t)\right]$$

correlated bivariate: $$P_{LU}(f_L, f_U) = \frac{\sqrt{\det(C^{-1})}}{2\pi} e^{-\frac{1}{2}\left[C^{-1}_{LL}(f_L-\bar{f}_{lo})^2 + 2C^{-1}_{LU}(f_L-\bar{f}_{lo})(f_U-\bar{f}_{up}) + C^{-1}_{UU}(f_U-\bar{f}_{up})^2\right]}$$

uniform distribution: $$P(f) = \frac{1}{f_U - f_L}\theta(f - f_L)\theta(f_U - f)$$

final average: $$f(t) \equiv \frac{\bar{f}_{lo}(t) + \bar{f}_{up}(t)}{2}$$

final variance: $$\sigma^2_f(t) \equiv \frac{1}{12}\left[\bar{f}_{lo}(t) - \bar{f}_{up}(t)\right]^2 + \frac{1}{3}\left[C_{LL}(t) + C_{UU}(t) + C_{LU}(t)\right]$$

* kinematical constraint: $f_+(0) = f_0(0)$

for each event $k = 1,2,\ldots,N_{boot}$: $f(0)|_{lo} \leq f(0) \leq f(0)|_{up}$

$$f_0(0)|_{lo} = \max(f_0(0)|_{lo}, f_+(0)|_{lo})$$
$$f_0(0)|_{up} = \min(f_0(0)|_{up}, f_+(0)|_{up})$$

addition of one (common) point at $q^2 = 0$ in the dispersion matrices of f_0 and f_+ uniformely distributed in $\left[f(0)|_{lo}, f(0)|_{up}\right]$

* when the percentage of events satisfying the unitarity and/or kinematical constraints is too low, the reliability of the DM bands may become questionable and we apply a procedure to recover a larger percentage of events passing the filters

<u>skeptical procedure</u> (from D'Agostini, arXiv: 2001.03466)

1. modify the standard deviations σ_i of the input data by a factor r_i while keeping fixed the averages f_i (a common value r is typically enough)
2. enlarge the number of bootstraps by extracting Nr values of r distributed according to a exponential distribution
3. select the the events passing the filters and compute their average value r*
4. select the event with r closest to r*

<u>iterative procedure</u> [arXiv:2109.15248]

1. recalculate the mean values and the covariance matrix of the subset of inpout data passing the filters
2. generate a new multivariate distribution
3. check unitarity and kinematical constraints
4. repeat steps 1-3 until convergence of the percentage of events passing the filters is reached

⇓

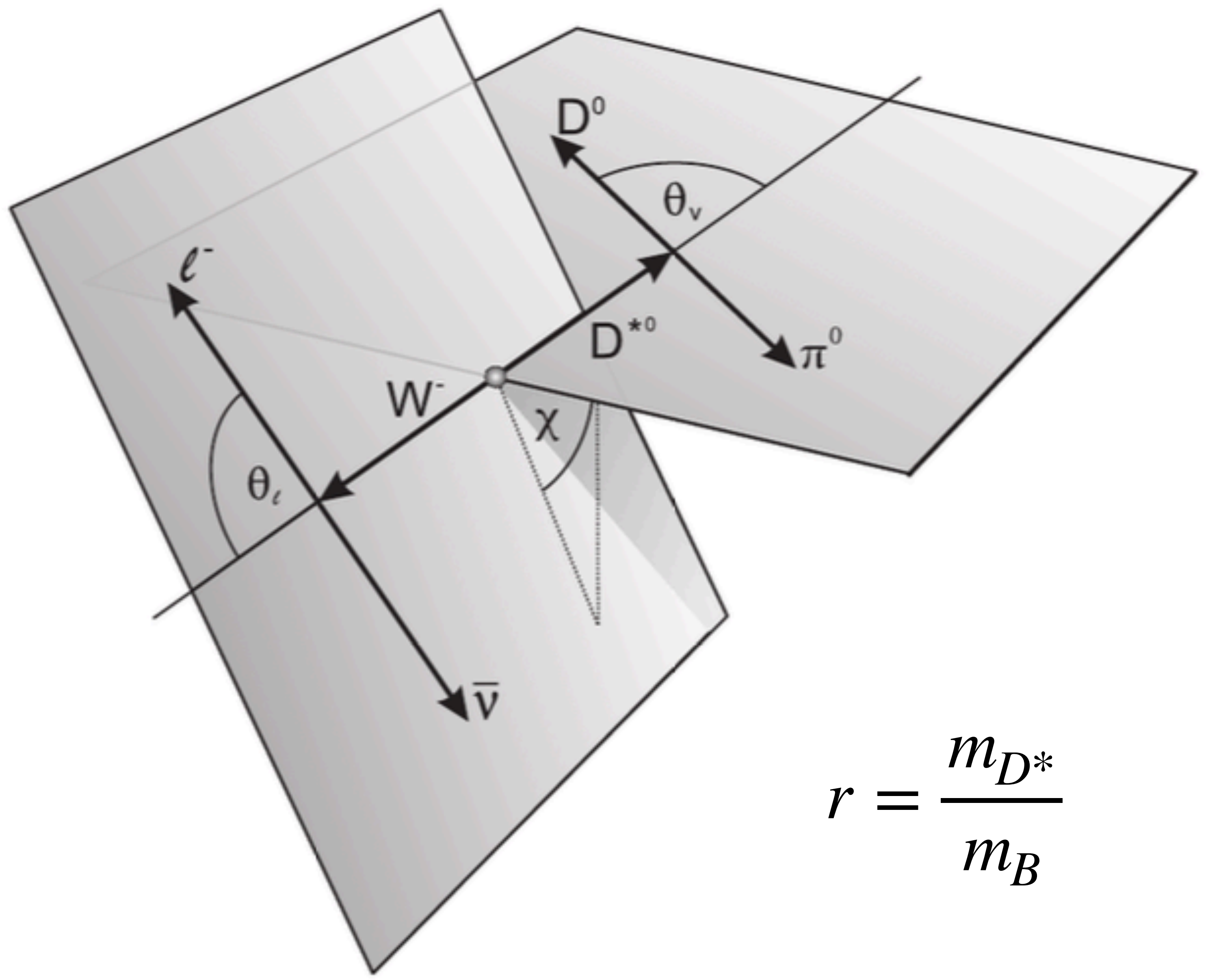
simpler and more effective procedure

hadronic form factors in semileptonic $B \to D^{(*)}\ell\nu_\ell$ decays

$$\frac{d\Gamma(B \to D\ell\nu_\ell)}{dq^2} = \frac{G_F^2}{24\pi^3}\eta_{EW}^2 |V_{cb}|^2 p_D^3 \, f_+^2(q^2)$$

for massless leptons

$$\frac{d^4\Gamma(B \to D^*\ell\nu_\ell)}{dw \; d\cos\theta_v \; d\cos\theta_\ell \; d\chi} = \frac{3}{4}\frac{G_F^2}{(4\pi)^4}\eta_{EW}^2 |V_{cb}|^2 m_B^3 \, r^2 \sqrt{w^2-1}(1+r^2-2rw)$$
$$\cdot \Big\{ H_+^2(w) \sin^2\theta_v \,(1-\cos\theta_\ell)^2 + H_-^2(w) \sin^2\theta_v \,(1+\cos\theta_\ell)^2$$
$$+ \; 4\, H_0^2(w) \cos^2\theta_v \sin^2\theta_\ell - 2\, H_-(w)H_+(w) \sin^2\theta_v \sin^2\theta_\ell \cos 2\chi$$
$$- \; 2\, H_+(w)H_0(w) \cos 2\theta_v \sin\theta_\ell \,(1-\cos\theta_\ell) \cos\chi$$
$$+ \; 2\, H_-(w)H_0(w) \cos 2\theta_v \sin\theta_\ell \,(1+\cos\theta_\ell) \cos\chi \Big\}$$



$$r = \frac{m_{D^*}}{m_B}$$

$$g(w) = \frac{1}{r\sqrt{w^2-1}}\frac{H_+(w) - H_-(w)}{2m_B^2}$$

$$f(w) = \frac{H_+(w) + H_-(w)}{2}$$

$$F_1(w) = m_B\sqrt{1-2rw+r^2}\,H_0(w)$$

for massive leptons one should add $f_0(q^2)$ for $B \to D$ and $P_1(w)$ for $B \to D^*$

experimental data for $B \to D^* \ell \nu_\ell$ decays

- two sets of data from Belle collaboration arXiv:1702.01521 and arXiv:1809.03290
- four different differential decay rates $d\Gamma/dx$ where $x = \{w, \cos\theta_v, \cos\theta_\ell, \chi\}$: 10 bins for each variable

total of 80 data points

*** we do not mix theoretical calculations with experimental data to describe the shape of the FFs ***

$$|V_{cb}|_i \equiv \sqrt{\frac{(d\Gamma/dx)_i^{exp}}{(d\Gamma/dx)_i^{th}}} \qquad i = 1, \ldots, N_{bins}$$

* issue with the covariance matrix C_{ij}^{exp} of the Belle data: $\Gamma^{exp} \equiv \sum_{i=1}^{10} \left(\frac{d\Gamma}{dx}\right)_i^{exp}$ should be the same for all the variables x

- we recover the above property by evaluating the correlation matrix of the experimental ratios

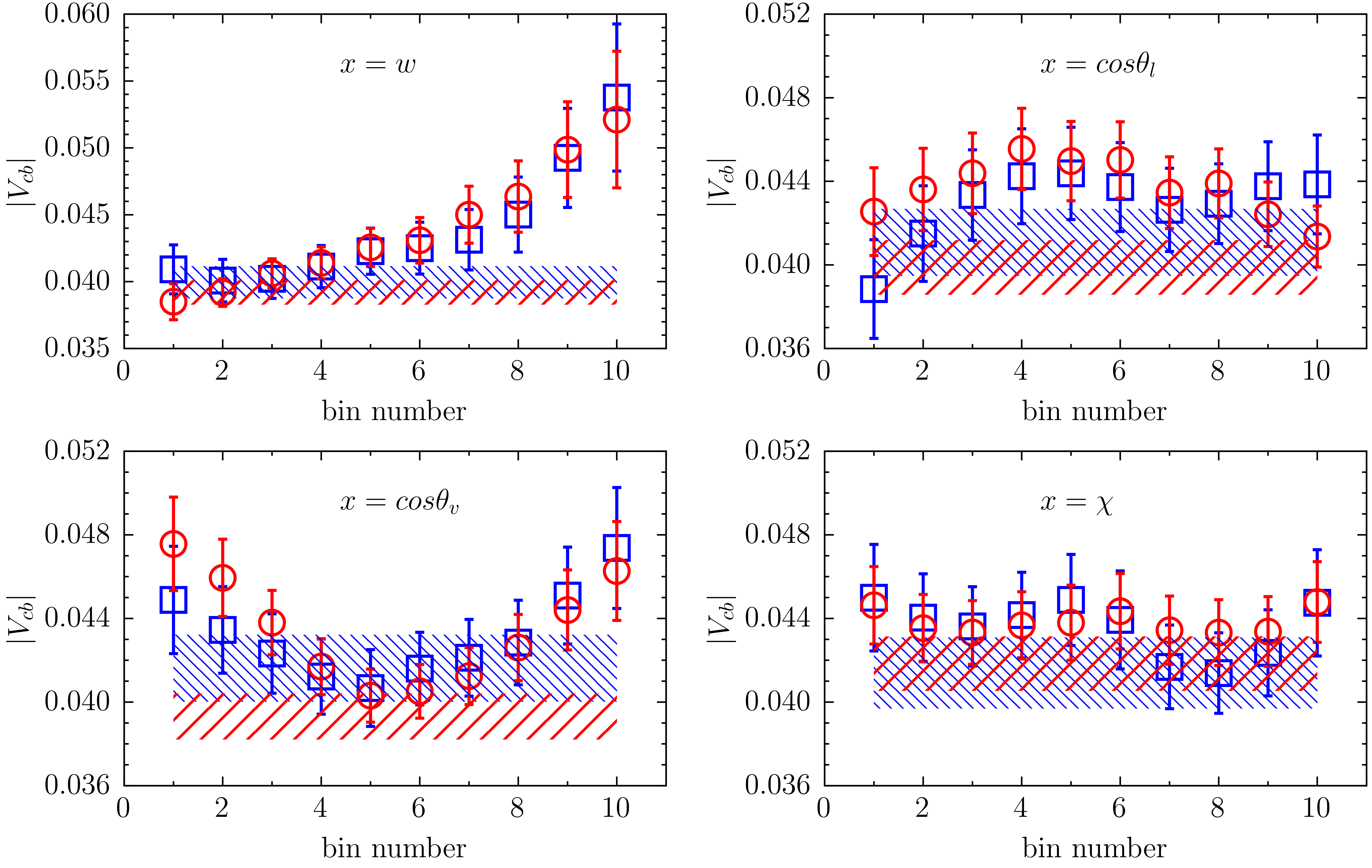
$$\frac{1}{\Gamma^{exp}} \left(\frac{d\Gamma}{dx}\right)_i^{exp}$$

and by considering the new covariance matrix of the experimental data given by (see arXiv:2105.08674)

$$\widetilde{C}_{ij}^{exp} \to \rho_{ij}^{ratios} \sqrt{C_{ii}^{exp.} C_{jj}^{exp}}$$

extraction of $|V_{cb}|$ from $B \to D^* \ell \nu_\ell$ decays

original covariance matrix of Belle data



blue data: Belle 1702.01521

red data: Belle 1809.03290

bands are (correlated) weighted averages

$$|V_{cb}| = \frac{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij} |V_{cb}|_j}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} ,$$

$$\sigma^2_{|V_{cb}|} = \frac{1}{\sum_{i,j=1}^{10} (\mathbf{C}^{-1})_{ij}} ,$$

$$|V_{cb}| \cdot 10^3 = 40.5 \pm 1.7$$

BGL approach

(Boyd, Grinstein and Lebed '95-'97)

* the hadronic form factors corresponding to definite spin-parity can be represented as an expansion, originating from unitarity, analyticity and crossing symmetry, in terms of the conformal variable z ($|z| \leq 1$)

$$f_+(q^2) = \sqrt{\frac{n_I}{\chi_{1^-}(q_0^2)}} \frac{1}{\phi_+(z(q^2), q_0^2)\; P_+(z(q^2))} \sum_{n=0}^{\infty} a_n\, z^n(q^2)$$

$$z(t) \equiv \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}$$

$$t_0 \to t_-$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

$\phi_+(z(q^2), q_0^2)$ = kinematical function (q_0^2 = auxiliary quantity)

$P_+(z(q^2))$ = Blaschke factor including resonances below the pair-production threshold t_+

n_I = factor counting the number of spectator quarks

$$\chi_{1^-}(q_0^2) = \text{transverse vector susceptibility} \equiv \frac{1}{2} \frac{\partial^2}{\partial (q_0^2)^2} \left[q_0^2\, \Pi_{1^-}(q_0^2) \right] = \frac{1}{\pi} \int_0^\infty ds \frac{s\, \text{Im}\Pi_{1^-}(s)}{(s - q_0^2)^3}$$

calculable nonperturbatively from appropriate 2-point lattice correlators (see arXiv:2105.07851)

unitarity constraint: $\sum_{n=0}^{\infty} a_n^2 \leq 1$

Γ_i : mean values $\overline{\Gamma}_i$ and covariance matrix C_{ij} $\qquad i, j = 1, \ldots, N$

$$\Gamma = \sum_{i=1}^{N} \Gamma_i$$

$$\text{mean value } \overline{\Gamma} = \sum_{i=1}^{N} \overline{\Gamma}_i \text{ and variance } \sigma_\Gamma^2 = \sum_{i,j=1}^{N} C_{ij}$$

$$\sum_{i,j=1}^{N} C_{ij} = \sum_{i,j=1}^{N} \langle (\Gamma_i - \overline{\Gamma}_i)(\Gamma_j - \overline{\Gamma}_j) \rangle = \left\langle \left[\sum_{i=1}^{N} (\Gamma_i - \overline{\Gamma}_i) \right]^2 \right\rangle = \langle (\Gamma - \overline{\Gamma})^2 \rangle \equiv \sigma_\Gamma^2$$

$$\Gamma_i = r_i \cdot \Gamma \qquad i = 1,\ldots,N$$

r_i : mean values $\bar{r}_i$ and covariance matrix R_{ij}

Γ : mean value $\overline{\Gamma}$ and variance σ_Γ^2 uncorrelated with all the $r_i's$

Γ_i : mean values $\overline{\Gamma}_i = \bar{r}_i \cdot \overline{\Gamma}$ and covariance matrix $C_{ij} = R_{ij} \cdot \left[\overline{\Gamma}^2 + \sigma_\Gamma^2\right] + \bar{r}_i \bar{r}_j \sigma_\Gamma^2$

$$r_i = \bar{r}_i + \sqrt{R_{ii}} \sum_{k=1}^{N} U_{ik}^T \sqrt{\lambda_k} \cdot \xi_k \qquad R_{ij} = \sqrt{R_{ii} R_{jj}} \sum_{k=1}^{N} U_{ik}^T \lambda_k U_{kj}$$

ξ_k : uncorrelated variables

$< \xi_k > = 0$ and $< \xi_k \xi_{k'} > = \delta_{kk'}$

$$\Gamma = \overline{\Gamma} + \sigma_\Gamma \cdot \xi_\Gamma$$

ξ_Γ : uncorrelated variable with all the ξ_k variables

$< \xi_\Gamma > = 0$ and $< \xi_\Gamma^2 > = 1, < \xi_\Gamma \xi_k > = 0$

$$C_{ij} = < (r_i \cdot \Gamma - \bar{r}_i \cdot \overline{\Gamma})(r_j \cdot \Gamma - \bar{r}_j \cdot \overline{\Gamma} > = \sqrt{R_{ii} R_{jj}} \sum_{k=1}^{N} U_{ik}^T \lambda_k U_{jk}^T \left[\overline{\Gamma}^2 + \sigma_\Gamma^2\right] + \bar{r}_i \bar{r}_j \sigma_\Gamma^2 = R_{ij} \left[\overline{\Gamma}^2 + \sigma_\Gamma^2\right] + \bar{r}_i \bar{r}_j \sigma_\Gamma^2$$

* LHCb ratios from arXiv:2003.08453

$$\Delta r_j = \frac{\Delta\Gamma_j(B_s \to D_s^* \mu \nu_\mu)}{\Gamma(B_s \to D_s^* \mu \nu_\mu)} \qquad j = 1,\ldots,7$$

- constrain $\sum_{j=1}^{7} \Delta r_j = 1$ ⇨ 1) one null eigenvalue of the covariance matrix R_{ij}^{LHCb} (six independent ratios)

  2) $\sum_{i,j} R_{ij}^{\mathrm{LHCb}} = 0$ (null variance for the sum)

- experimental covariance matrix R_{ij}^{LHCb}:

  eigenvalues $\lambda_j = \{0.072,\ 0.21,\ 0.33,\ 0.53,\ 0.73,\ 1.03,\ 2.33\} \cdot 10^{-4}$ and $\sum_{i,j} R_{ij}^{\mathrm{LHCb}} = 1.45 \cdot 10^{-3}$

  → ~ 3.8% to be added (in quadrature) to the error of the total decay rate

- modified covariance matrix $\widetilde{R}_{ij}^{\mathrm{LHCb}}$: $\widetilde{\Delta r}_j = \Delta r_j / \sum_{k=1}^{7} \Delta r_k$

  eigenvalues $\widetilde{\lambda}_j = \{0.0,\ 0.073,\ 0.22,\ 0.34,\ 0.53,\ 0.74,\ 1.14\} \cdot 10^{-4}$ and $\sum_{i,j} \widetilde{R}_{ij}^{\mathrm{LHCb}} = 0$