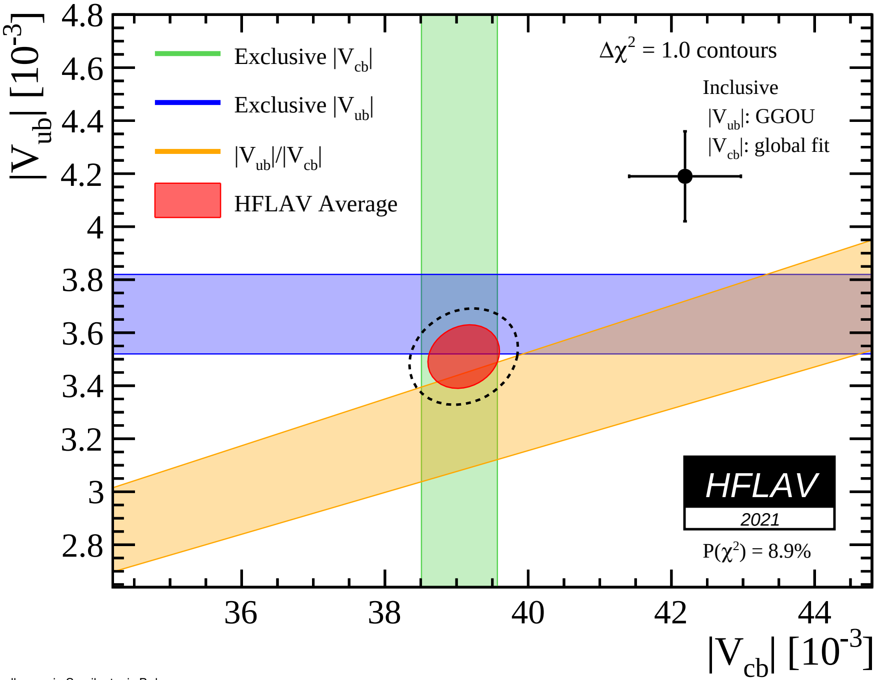
~~Three~~ Five loop calculations for inclusive semileptonic *B* decays

Challenges in Semileptonic *B* decays – Barolo

Matteo Fael | 20 Apr. 2022| with K. Schönwald and M. Steinhauser

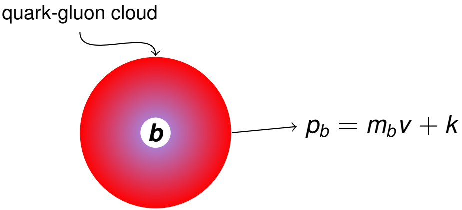


Exclusive $|V_{cb}|$
Exclusive $|V_{ub}|$
$|V_{ub}|/|V_{cb}|$
HFLAV Average
$\Delta\chi^2$ = 1.0 contours
Inclusive
$|V_{ub}|$: GGOU
$|V_{cb}|$: global fit
HFLAV
2021
$P(\chi^2)$ = 8.9%
$|V_{ub}|$ [10^{-3}]
4.8
4.6
4.4
4.2
4
3.8
3.6
3.4
3.2
3
2.8
36
38
40
42
44
$|V_{cb}|$ [10^{-3}]

The Heavy-Quark Expansion

$$\Gamma_{\rm sl} = \Gamma_0 + \Gamma_{\mu_\pi} \frac{\mu_\pi^2}{m_b^2} + \Gamma_{\mu_G} \frac{\mu_G^2}{m_b^2} + \Gamma_{\rho_D} \frac{\rho_D^3}{m_b^3} + \Gamma_{\rho_{LS}} \frac{\rho_{LS}^3}{m_b^3} + \dots$$

Reviews:
Benson, Bigi, Mannel, Uraltsev, Nucl.Phys. B665 (2003) 367;
Dingfelder, Mannel, Rev.Mod.Phys. 88 (2016) 035008.

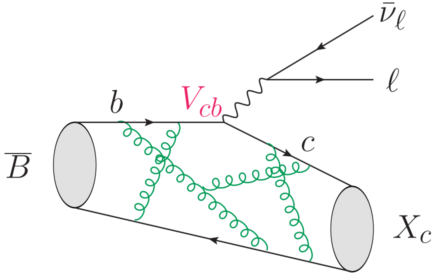


- Γ_i are computed in **perturbative QCD**.
- The HQE parameters: $\mu_\pi, \mu_G, \rho_D, \rho_{LS} \sim \langle B | \mathcal{O}_i^{\bar{b}b} | B \rangle$
- HQE parameters are **extracted from kinematic moments**.

  $\longrightarrow$ see talk M. Bordone, R. van Tonder, M. Rotondo
- Ongoing pilot studies also on the lattice.

  $\longrightarrow$ see talk S. Hashimoto

Hashimoto, Gambino, Phys.Rev.Lett. 125 (2020) 032001 and hep-lat/2203.11762;
Gambino, Melis, Simula, Phys.Rev.D 96 (2017) 014511;
Hansen, Meyer, Robaina, Phys.Rev.D 96 (2017) 9, 094513;



		tree	α_s	α_s^2	α_s^3	
1		✓	✓	✓*	✓*	Jezabek, Kuhn, NPB 314 (1989) 1; Gambino et al., NPB 719 (2005) 77; Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015. MF, Schönwald, Steinhauser, PRD 104 (2021) 016003;
$1/m_b^2$	μ_π	✓	✓			Becher, Boos, Lunghi, JHEP 0712 (2007) 062; Alberti, Ewerth, Gambino, Nandi, NPB 870 (2013) 16
	μ_G	✓	✓			Alberti, Gambino, Nandi, JHEP 1401 (2014) 147; Mannel, Pivovarov, Rosenthal, PRD 92 (2015) 054025.
$1/m_b^3$	ρ_D	✓	✓*			Mannel, Pivovarov, PRD100 (2019) 093001; Mannel, Moreno, Pivovarov, PRD 105 (2022) 054033.
	ρ_{LS}	✓	✓*			
$1/m_b^{4,5}$		✓				Dassinger, Mannel, Turczyk, JHEP 0703 (2007) 087, JHEP 1011 (2010) 109 MF, Mannel, Vos, JHEP 02 (2019) 177, JHEP 12 (2019) 067.
$\overline{m}_b - m_b^{\rm kin}$			✓	✓	✓	Bigi et al, PRD 56 (1997) 4017; Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189. MF, Steinhauser, Schönwald, PRL 125 (2020) 5, PRD 103 (2021) 014005.

✓ = known fully differential ✓* = known for selected observables

		tree	α_s	α_s^2	α_s^3	
1		✓	✓	✔	✔	Jezabek, Kuhn, NPB 314 (1989) 1; Gambino et al., NPB 719 (2005) 77; Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015. MF, Schönwald, Steinhauser, PRD 104 (2021) 016003;
$1/m_b^2$	μ_π	✓	✓			Becher, Boos, Lunghi, JHEP 0712 (2007) 062; Alberti, Ewerth, Gambino, Nandi, NPB 870 (2013) 16
	μ_G	✓	✓			Alberti, Gambino, Nandi, JHEP 1401 (2014) 147; Mannel, Pivovarov, Rosenthal, PRD 92 (2015) 054025.
$1/m_b^3$	ρ_D	✓	✔			Mannel, Pivovarov, PRD100 (2019) 093001; Mannel, Moreno, Pivovarov, PRD 105 (2022) 054033.
	ρ_{LS}	✓	✔			
$1/m_b^{4,5}$		✓				Dassinger, Mannel, Turczyk, JHEP 0703 (2007) 087, JHEP 1011 (2010) 109 MF, Mannel, Vos, JHEP 02 (2019) 177, JHEP 12 (2019) 067.
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✓ = known fully differential ✔ = known for selected observables

see talk by K. Vos

Third order corrections for $B \to X_c \ell \nu_\ell$

- Total semileptonic rate

  MF, Schönwald, Steinhauser, PRD 104 (2021) 016003, JHEP 10 (2020) 087

- Relation between $\overline{\text{MS}}$ mass and the kinetic mass

  MF, Schönwald, Steinhauser, PRL 125 (2020) 052003, PRD 103 (2021) 014005

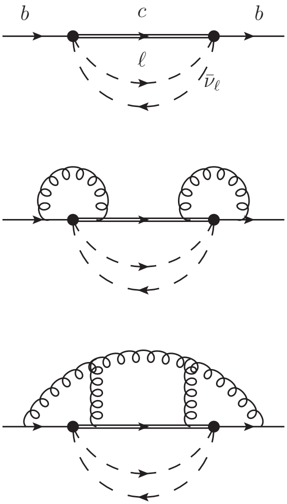
- First glance to the spectral moments.

  MF, Schönwald, Steinhauser, in preparation

α_s^3 corrections to $b \to X_c \ell \bar{\nu}_\ell$ width

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[X_0(\rho) + C_F \sum_{n \geq 1} \left(\frac{\alpha_s}{\pi} \right)^n X_n(\rho) \right]$$

with $\rho = m_c / m_b$



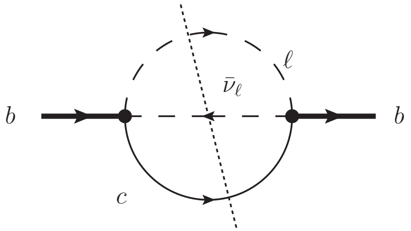
Possible strategies

- Exact analytic result with m_b and m_c only at $O(\alpha_s)$.
  Nir, Phys.Lett.B 221 (1989) 184
- Numerical approach
  Melnikov, PLB 666 (2008) 336
- **Approximation exploiting $m_c < m_b$**
  Czarnecki, Pak, PRD 78 (2008) 114015

Computational Method

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2\left[1 - 8\rho^2 - 12\rho^4\log(\rho^2) + 8\rho^6 - \rho^8\right]$$

where $\rho = m_c/m_b$.

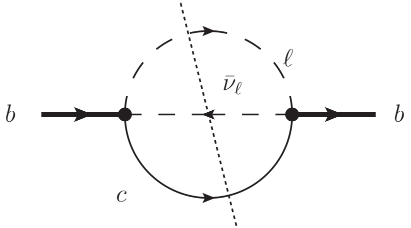


- Optical theorem.
- Multi-loop diagrams with **two scales m_c and m_b.**
- Use **method of regions**.
  Beneke, Smirnov, NPB 522 (1998) 321; Smirnov, Springer Tracts Mod.Phys. 177
- Expansion in $\rho = m_c/m_b$: too difficult to extend at $O(\alpha_s^3)$.
- Expansion in $\delta = 1 - m_c/m_b$: Crucial factorization of loop integrals.

$\Gamma_{\rm sl}$ at tree level

- Each loop momenta can scale as **hard** or **soft**.

$$\Gamma_\mu = \Gamma^{(\rm hh)} + \Gamma^{(\rm ss)} + \Gamma^{(\rm hs)} + \Gamma^{(\rm sh)}$$
$$= \Gamma_0 \left[1 - 8\rho^2 - 12\rho^4 \log(\rho^2) + 8\rho^6 - \rho^8 \right]$$

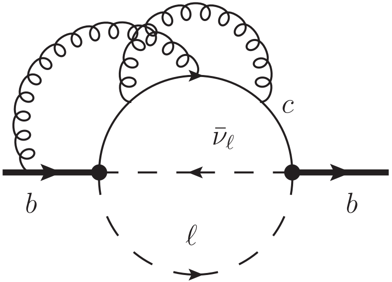


where $\rho = m_c/m_b$ and $\Gamma_0 = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2$

$$\Gamma^{(\rm hh)} \sim 1 - 8\rho^2 - \frac{\rho^4}{12\varepsilon} - 24\rho^4 \log\left(\frac{\mu^2}{m_b^2}\right) - 24\rho^4 + 16\rho^6 - 2\rho^8$$
$$\Gamma^{(\rm hs)} \sim + \frac{\rho^4}{12\varepsilon} + 24\rho^4 \log\left(\frac{\mu^2}{m_b^2}\right) - 12\rho^4 \log(\rho^2) + 24\rho^4 - 8\rho^6 + \rho^8$$
$$\Gamma^{(\rm ss)} = \Gamma^{(\rm sh)} = 0$$

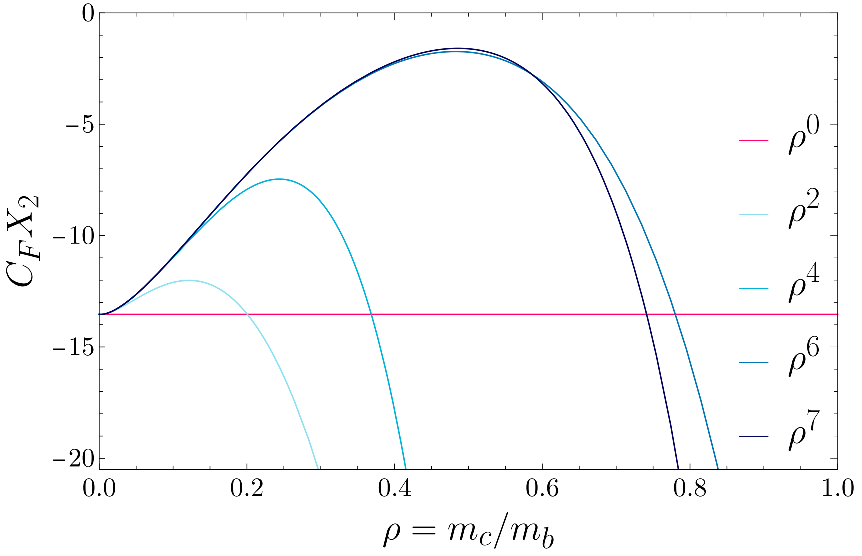
Second order corrections

$$\Gamma_{\mathrm{sl}} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \left[X_0 + C_F \frac{\alpha_s}{\pi} X_1 + C_F \left(\frac{\alpha_s}{\pi}\right)^2 X_2 + \ldots \right]$$



Czarnecki, Pak, PRD 78 (2008) 114015; PRL 100 (2008) 241807.

- Four-loop diagrams.
- Loop momenta **hard** (m_b) or **soft** (m_c).
- **11 different regions**.
- **33 four-loop master integrals**.
- Expansion depth: $O(\rho^7)$ ($\rho = m_c/m_b$).



0
−5
−10
−15
−20
$C_F X_2$
0.0
0.2
0.4
0.6
0.8
1.0
$\rho = m_c/m_b$
ρ^0
ρ^2
ρ^4
ρ^6
ρ^7

Towards the third order corrections

	α_s^2		α_s^3
n. diagrams	62	$\to$	1450
n. loops	4	$\to$	5
regions	11	$\to$	O(20)
expansion depth	7	$\to$	?
master integrals	33	$\to$	?

The heavy daughter limit

Dowling, Piclum, Czarnecki, PRD 78 (2008) 074024

- Is the most natural expansion parameter also the best one?

$$\frac{m_c}{m_b} \sim 0.3 \qquad 1 - \frac{m_c}{m_b} \sim 0.7$$

- Heavy daughter limit $m_c \sim m_b$:

$$\delta = 1 - \rho = 1 - \frac{m_c}{m_b} \ll 1$$

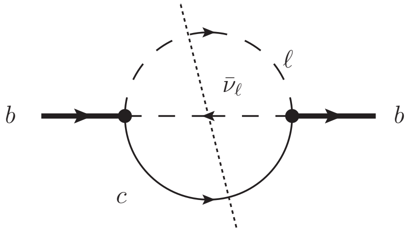
- Leading power in δ:

$$\Gamma_{\mathrm{sl}} \overset{m_c \to m_b}{\simeq} \frac{G_F^2}{192\pi^3}(m_b - m_c)^5 = \frac{G_F^2 m_b^5}{192\pi^3}\delta^5$$

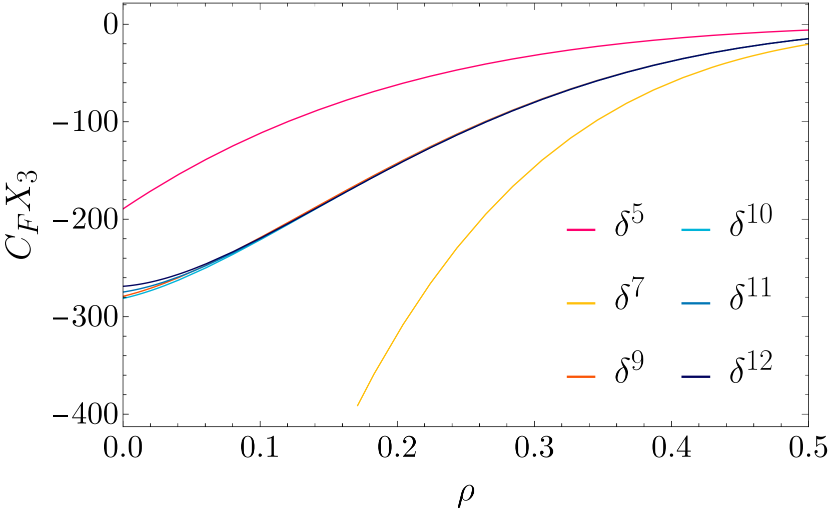
Γ_{sl} Reloaded

$$\Gamma_{\mathrm{sl}} = \Gamma^{(\mathrm{hs})}$$
$$= \Gamma_0 \left[\frac{64}{5}\delta^5 - \frac{96}{5}\delta^6 + \frac{288}{35}\delta^7 + \ldots \right]$$

where $\delta = 1 - m_c/m_b$ and $\Gamma_0 = \frac{G_F^2 m_b^5}{192\pi^3}|V_{cb}|^2$



- **Crucial factorisation** in the heavy daughter limit: **5 loops $\longrightarrow$ 3 loops**!
- At least one electron's propagator must scale soft to generate an imaginary part $\log(-\delta)$.
- **Less regions to consider**, e.g. at leading order $\Gamma^{(\mathrm{hh})} = \Gamma^{(\mathrm{sh})} = \Gamma^{(\mathrm{hs})} = 0$



$$C_F X_3(\rho = 0.28) = -91.2 \pm 0.4 \quad (0.4\%)$$

MF, Schönwald, Steinhauser, Phys.Rev.D 104 (2021) 1, 016003

The kinetic scheme

$$\Gamma_{\mathrm{sl}} = \frac{G_F^2 |V_{cb}|^2 (m_b^{\mathrm{OS}})^5}{192\pi^3} f(0.28) \left[1 - 1.72 \left(\frac{\alpha_s}{\pi}\right) - 13.1 \left(\frac{\alpha_s}{\pi}\right)^2 - 163.3 \left(\frac{\alpha_s}{\pi}\right)^3 \right] + O\left(\frac{1}{m_b^2}\right)$$

See: Bigi, Shifman, Uraltsev, Vainshtein PRD 50 (1994) 2234; Beneke, Braun, NPB 426 (1994) 301; Ball, Beneke, Braun, PRD 52 (1995) 3929; Melnikov, van Ritbergen, PLB 482 (2000) 99.

Meson-quark mass relation

$$m_b = M_B - \overline{\Lambda} - \frac{\mu_\pi^2}{2m_b} + \ldots$$

- $\overline{\Lambda}$: the B-meson binding energy.
- μ_π: the kinetic energy induced by the residual motion of the heavy quark.

The relevant parameter in Γ_{sl} is $\boldsymbol{m_b^5}$**, not** $\boldsymbol{M_B^5}$:

$$\Gamma_{\mathrm{sl}} \simeq \frac{G_F^2 |V_{cb}|^5}{192\pi^3} (M_B - \overline{\Lambda})^5$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017

The kinetic mass

$$m_b^{\mathrm{kin}}(\mu) = m_b^{\mathrm{OS}} - [\overline{\Lambda}(\mu)]_{\mathrm{pert}} - \frac{[\mu_\pi^2(\mu)]_{\mathrm{pert}}}{2 m_b^{\mathrm{kin}}(\mu)} - \dots$$

Bigi, Shifman, Uraltsev, Vainshtein, PRD 56 (1997) 4017.
see also: Czarnecki, Melnikov, Uraltsev, PRL 80 (1998) 3189;
Gambino, JHEP 09 (2011) 055;

- In pQCD, we can make a short-distance mass definition by identifying:

$$m_b(\mu) \to m_b^{\mathrm{kin}}(\mu) \qquad \overline{M}_B \to m_b^{\mathrm{OS}}$$

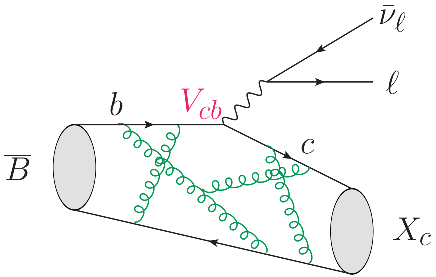
$$\overline{\Lambda}(\mu) \to [\overline{\Lambda}(\mu)]_{\mathrm{pert}} \qquad [\mu_\pi^2(\mu)] \to [\mu_\pi^2(\mu)]_{\mathrm{pert}}$$

The Small Velocity Sum Rules

- How to give an operative definition of $\overline{\Lambda}$ and μ_π^2?
- Moments of the excitation energy:

$$I_n(\vec{q}^2) = \int \mathrm{d}\omega\, \omega^n \frac{d\Gamma}{d\omega d\vec{q}^2}$$

with $\omega = E_{X_c} - M_D$ and $q = p_\ell + p_\nu$



The Small Velocity Sum Rules

Take the limit where the X_c's velocity is small: $|\vec{v}| = |\vec{q}/m_c| \ll 1$:

$$I_0(\vec{q}^{\,2}) = |\vec{q}\,| \frac{G_F^2 |V_{cb}|^2}{8\pi^3} (m_b - m_c)^2 + O\left(|\vec{v}|^2, \frac{\Lambda_{\mathrm{QCD}}}{m_b} \right)$$

$$I_1(\vec{q}^{\,2}) = I_0 \frac{\vec{v}^{\,2}}{2} \overline{\Lambda} + O\left(|\vec{v}|^3, \frac{\Lambda_{\mathrm{QCD}}^2}{m_b^2} \right)$$

$$I_2(\vec{q}\,) = I_0 \frac{\vec{v}^{\,2}}{3} \mu_\pi^2 + O\left(|\vec{v}|^3, \frac{\Lambda_{\mathrm{QCD}}^3}{m_b^3} \right)$$

Let’s include radiative corrections . . .

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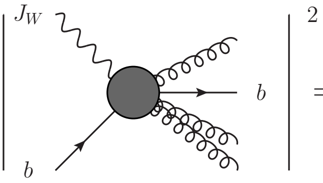
$$I_2(\vec{q}^2) = \underbrace{\int dq_0\, \omega^2 \frac{d\Gamma_{\rm tree}}{d\omega d\vec{q}^2} + \int_0^{\mu} d\omega\, \omega^2 \frac{d\Gamma_{\alpha_s}}{d\omega d\vec{q}^2}}_{\text{use this to define } \mu_\pi^2(\mu)} + \int_{\mu}^{\omega^{\rm max}} d\omega\, \omega^2 \frac{d\Gamma_{\alpha_s}}{d\omega d\vec{q}^2}$$

$$\mu_\pi^2(0) = \mu_\pi^2(\mu) - [\mu_\pi^2]_{\rm pert}$$
$$\rho_D^3(0) = \rho_D^3(\mu) - [\rho_D^3]_{\rm pert}$$



$$= W(\omega, \vec{v})$$

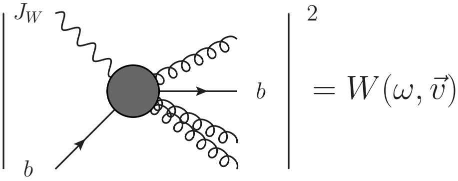
The Small Velocity Sum Rules

- The OPE for the structure function $W(\omega, \vec{v})$ tells us:

$$[\overline{\Lambda}(\mu)]_{\rm pert} = \lim_{\vec{v}\to 0} \lim_{m_b\to\infty} \frac{2}{\vec{v}^{\,2}} \frac{\int_0^{\mu} d\omega\, \omega\, W(\omega, \vec{v})}{\int_0^{\mu} d\omega\, W(\omega, \vec{v})}$$

$$[\mu_\pi^2(\mu)]_{\rm pert} = \lim_{\vec{v}\to 0} \lim_{m_b\to\infty} \frac{3}{\vec{v}^{\,2}} \frac{\int_0^{\mu} d\omega\, \omega^2\, W(\omega, \vec{v})}{\int_0^{\mu} d\omega\, W(\omega, \vec{v})}$$

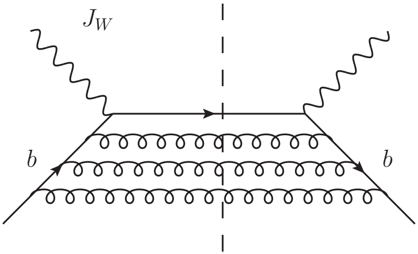
- Scattering of a heavy quark on a current J_W in the SV kinematic.



The Kinetic Mass as a Threshold Mass

- Excite the heavy quark, but just a bit . . .

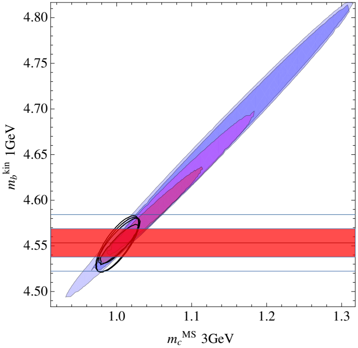
$$y = s - m_b^2 \simeq 2 m_b \omega \ll m_b^2$$



- SV Limit corresponds to **one-particle Threshold limit**.
- Factorization can be understood in terms of Eikonal factorization

$$W(\omega, \vec{v}) \simeq H \cdot U(\omega, \vec{v})$$

Precise $\overline{m}_b - m_b^{\text{kin}}$ conversion up to $O(\alpha_s^3)$



Gambino, Schwanda, PRD 89 (2014) 014022
Horizontal error bands superimposed by MF

- Mass relation implemented in `(C)RunDec` and `REvolver`
  Herren, Steinhauser, Comput.Phys.Commun.224, 333 (2018)
  Hoang, Lepenik, Mateu, Comput.Phys.Commun. 270 108145 (2022)
- Input from FLAG19:
  - $\overline{m}_b(\overline{m}_b) = 4.198(12)$ GeV
  - $\overline{m}_c(3\text{ GeV}) = 0.988(7)$ GeV

$$m_b^{\text{kin}}(1\text{ GeV}) = 4.198 + 0.261 + 0.079 + 0.027 = 4.564\text{ GeV}$$

- Conversion-uncertainty $\delta m_b = 15$ MeV (half $O(\alpha_s^3)$ correction)
- Uncertainty at $O(\alpha_s^2)$ was $\delta m_b = 40$ MeV

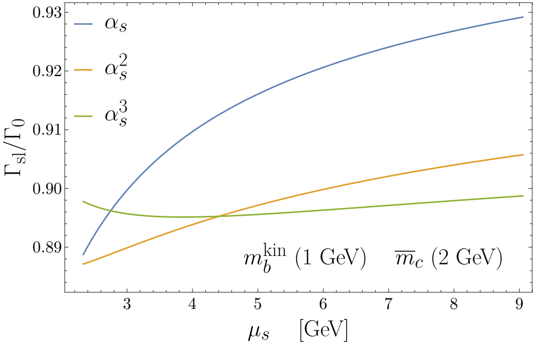
$$m_b^{\text{kin}}(1\text{ GeV}) = 4.565\,(15)_{\text{th}}(13)_{\text{lat}}\text{ GeV} = 4.565\,(20)\text{ GeV}$$

Implications for $\overline{B} \to X_c \ell \bar{\nu}_\ell$

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} f(\rho) \left[1 + \sum_n Y_n \left(\frac{\alpha_s}{\pi}\right)^n \right]$$

with $\alpha_s \equiv \alpha_s^{(4)}(m_b)$.

n=1 Jezabek, Kühn, Jezabek, Kuhn, NPB 314 (1989) 1
n=2 Melnikov, PLB 666 (2008) 336; Pak, Czarnecki, PRD 78 (2008) 114015.
n=3 Fael, Schönwald, Steinhauser, hep-ph/2011.13654

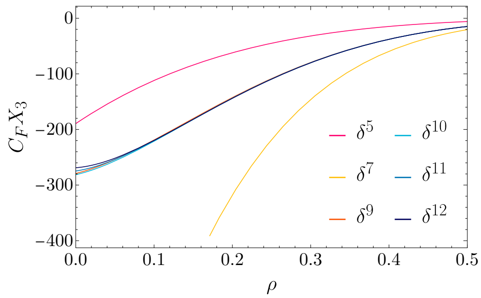


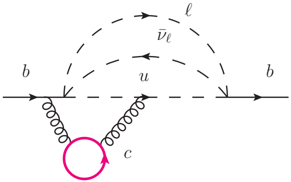
$$m_b^{\rm OS} : m_c^{\rm OS} \quad 1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 13.1\left(\frac{\alpha_s}{\pi}\right)^2 - 163.3\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm kin}(1\ \text{GeV}) : \overline{m}_c(2\ \text{GeV}) \quad 1 - 1.24\left(\frac{\alpha_s}{\pi}\right) - 3.65\left(\frac{\alpha_s}{\pi}\right)^2 - 1.0\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\rm 1S} : m_c \text{ via HQET} \quad 1 - 1.38\left(\frac{\alpha_s}{\pi}\right) - 6.32\left(\frac{\alpha_s}{\pi}\right)^2 - 33.1\left(\frac{\alpha_s}{\pi}\right)^3$$

A new puzzle in V_{ub}?





- $\Gamma_{\rm sl}(B \to X_u \ell \nu_\ell)$ from $\rho \to 0$ limit: $C_F X_3^u = -269 \pm 26$.

$$\Gamma_{b\to u}(m_b^{\rm kin}) = \Gamma_0 \left[1 - 0.020|_{\alpha_s} + 0.014|_{\alpha_s^2} + 0.031|_{\alpha_s^3} \right]$$

$$\Gamma_{b\to u}^{\rm no\ charm}(m_b^{\rm kin}) = \Gamma_0 \left[1 - 0.020|_{\alpha_s} + 0.012|_{\alpha_s^2} + 0.016|_{\alpha_s^3} \right]$$

$$\Gamma_{b\to u}(m_b^{\rm 1S}) = \Gamma_0 \left[1 - 0.116\epsilon - 0.032\epsilon^2 + 0.002\epsilon^3 \right]$$

- Relevant for $|V_{ub}|$ in GGOU scheme

Gambino, Giordano, Ossola, Uraltsev, JHEP 10 (2007) 058

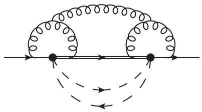
- Large m_c effects at $O(\alpha_s^3)$?
- Kinetic mass not good for $b \to u$?

Moments of kinematic distributions

$$M^n[w] = \int_{\text{veto}} \mathrm{d}\Phi \, w^n(p_\ell, q, v_B) \frac{d\Gamma}{dq_0 \, dq^2 \, dE_\ell}$$

Observable	$w(p_\ell, q, v_B)$
Semileptonic rate	1
Electron energy moments	$p_\ell \cdot v_B$
Hadronic invariant mass	$(M_B v_B - q)^2$
Leptonic invariant mass	q^2

- Extend computation strategy of Γ_{sl} to **moments without cuts**.

$w^n(p_\ell, q, v) \times$ 

First glance to α_s^3 corrections for the moments

- Centralized moments (no cuts):

$$\left\langle M^n[w] \right\rangle = \frac{M^n[w]}{M^0[w]} \quad \rightarrow \quad \left\langle (M[w] - \langle M[w] \rangle)^n \right\rangle$$

- QCD corrections up to $O(\alpha_s^3)$ at leading order in the HQE.
- Tree level contribution to $O(1/m_b^2)$ and $O(1/m_b^3)$.
- Results in the kinetic scheme: m_b^{kin} and $\overline{m}_c(3\ \mathrm{GeV})$.
- We quote:
  - Higher QCD corrections flagged by "α_s^n".
  - Power correction up to $1/m_b^3$ flagged by "*pw*"
  - Uncertainties in α_s^n from finite δ expansion.
  - Uncertainties from HQE parameters.

  Bordone, Capevila, Gambino, PLB 822 (2021) 136679

q^2 moments: $q_1 = \langle q^2 \rangle$, $\quad q_{n\geq 2} = \langle (q^2 - \langle q^2 \rangle)^n \rangle$

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MF, Schönwald, Steinhauser, in preparation

$$\hat{q}_1 = 0.232947\Big[1 - 0.0106345_{\alpha_s} - 0.008736(15)_{\alpha_s^2} - 0.00505(13)_{\alpha_s^3} - 0.0875(97)_{\mathrm{pw}}\Big],$$

$$\hat{q}_2 = 0.0235256\Big[1 - 0.035937_{\alpha_s} - 0.0217035(20)_{\alpha_s^2} - 0.01118(17)_{\alpha_s^3} - 0.237(27)_{\mathrm{pw}}\Big],$$

$$\hat{q}_3 = 0.0014511\Big[1 - 0.0700381_{\alpha_s} - 0.035693(73)_{\alpha_s^2} - 0.01909(12)_{\alpha_s^3} - 0.726(94)_{\mathrm{pw}}\Big],$$

$$\hat{q}_4 = 0.00120161\Big[1 - 0.0585199_{\alpha_s} - 0.042276(11)_{\alpha_s^2} - 0.02411(20)_{\alpha_s^3} - 0.631(77)_{\mathrm{pw}}\Big].$$

$$q_1(q^2 > 3\ \mathrm{GeV}^2) = 6.23\,(8)\ \mathrm{GeV}^2 \qquad (1.3\%)$$

$$q_2(q^2 > 3\ \mathrm{GeV}^2) = 4.44\,(15)\ \mathrm{GeV}^4 \qquad (3.1\%)$$

$$q_3(q^2 > 3\ \mathrm{GeV}^2) = 4.13\,(68)\ \mathrm{GeV}^6 \qquad (16\%)$$

$$q_3(q^2 > 3\ \mathrm{GeV}^2) = 46.6\,(5.6)\ \mathrm{GeV}^8 \qquad (12\%)$$

Belle, PRD 104 (2021) 112011

Electron energy: $\ell_1 = \langle E_\ell \rangle$, $\quad \ell_{n\geq 2} = \langle (E_\ell - \langle E_\ell \rangle)^n \rangle$

MF, Schönwald, Steinhauser, in preparation

$$
\begin{aligned}
\hat{\ell}_1 &= 0.315615\Big[1 - 0.0101064_{\alpha_s} - 0.005082(17)_{\alpha_s^2} - 0.00227(13)_{\alpha_s^3} - 0.0192(31)_{\rm pw}\Big],\\
\hat{\ell}_2 &= 0.00900585\Big[1 - 0.01992_{\alpha_s} - 0.006152(41)_{\alpha_s^2} + 0.0002(21)_{\alpha_s^3} + 0.017(11)_{\rm pw}\Big],\\
\hat{\ell}_3 &= -0.000464269\Big[1 - 0.0639319_{\alpha_s} - 0.035673(10)_{\alpha_s^2} - 0.0142(46)_{\alpha_s^3} - 0.175(22)_{\rm pw}\Big],\\
\hat{\ell}_4 &= 0.00020743\Big[1 - 0.028854_{\alpha_s} - 0.00717(23)_{\alpha_s^2} - 0.(0.25)_{\alpha_s^3} + 0.(0.021)_{\rm pw}\Big].
\end{aligned}
$$

$$
\begin{aligned}
\ell_1(E_\ell > 0.4\ \mathrm{GeV}) &= 1393.92(6.73)(3.02)\,\mathrm{MeV} && (0.5\%)\\
\ell_2(E_\ell > 0.4\ \mathrm{GeV}) &= 168.77(3.68)(1.53) \times 10^{-3}\ \mathrm{GeV}^2 && (2.3\%)\\
\ell_3(E_\ell > 0.4\ \mathrm{GeV}) &= -21.04(1.93)(0.66) \times 10^{-3}\ \mathrm{GeV}^3 && (9.6\%)\\
\ell_4(E_\ell > 0.4\ \mathrm{GeV}) &= 64.153(1.813)(0.935) \times 10^{-3}\ \mathrm{GeV}^4 && (3.2\%)
\end{aligned}
$$

Belle, PRD 75 (2007) 032001

Hadronic mass: $h_1 = \langle M_X^2 \rangle, \quad h_{n\geq 2} = \langle (M_X^2 - \langle M_X^2 \rangle)^n \rangle$

MF, Schönwald, Steinhauser, in preparation

$$\hat{h}_1 = 0.00899843\Big[+23.4975 \quad +1+0.4223(15)_{\alpha_s^2} \quad +0.147(11)_{\alpha_s^3} \quad +0.04(20)_{\rm pw}\Big],$$

$$\hat{h}_2 = 0.000745468\Big[+0.87352 \quad +1+0.4505(74)_{\alpha_s^2} \quad +0.34(43)_{\alpha_s^3} \quad +3.33(59)_{\rm pw}\Big],$$

$$\hat{h}_3 = 0.0000915954\Big[-0.0729568 \quad +1+0.165(62)_{\alpha_s^2} \quad +2.29(55)_{\alpha_s^3} \quad +7.3(1.1)_{\rm pw}\Big],$$

$$\hat{h}_4 = 0.000091207\Big[+0.0100938 \quad +1+0.51(17)_{\alpha_s^2} \quad +1(145)_{\alpha_s^3} \quad +0.380(52)_{\rm pw}\Big].$$

$$h_1 = 4.541\,(101)\ \text{GeV}^2 \qquad (2\%)$$

$$h_2 = 1.56\,(0.18)\,(0.16)\ \text{GeV}^4 \qquad (15\%)$$

$$h_3 = 4.05\,(0.74)\,(0.32)\ \text{GeV}^6 \qquad (20\%)$$

$$h_4 = 21.1\,(4.5)\,(2.1)\ \text{GeV}^8 \qquad (23\%)$$

DELPHI, EPJ C 45 (2006) 35

Outlook & Conclusions

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What is still needed?

- Short term:
  - NNLO corrections to q^2 moments with cuts (also A_{FB} asymmetries?).
  - Fully differential NLO corrections to ρ_D (Gambino, Nandi et al.)
  - Kinetic scheme at higher order in $1/m_b$ (and $O(\alpha_s^4)$?).
- Long term:
  - N^3LO and NNLO $\times 1/m_b^n$ corrections with cuts for selected observables might be doable.
  - Improve prediction for $b \to u \ell \nu$. Charm mass effects.
  - Applicability of kinetic scheme to $b \to u \ell \nu$?