Inclusive decays on the lattice

Shoji Hashimoto（KEK, SOKENDAI）

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Based on the collaborations of

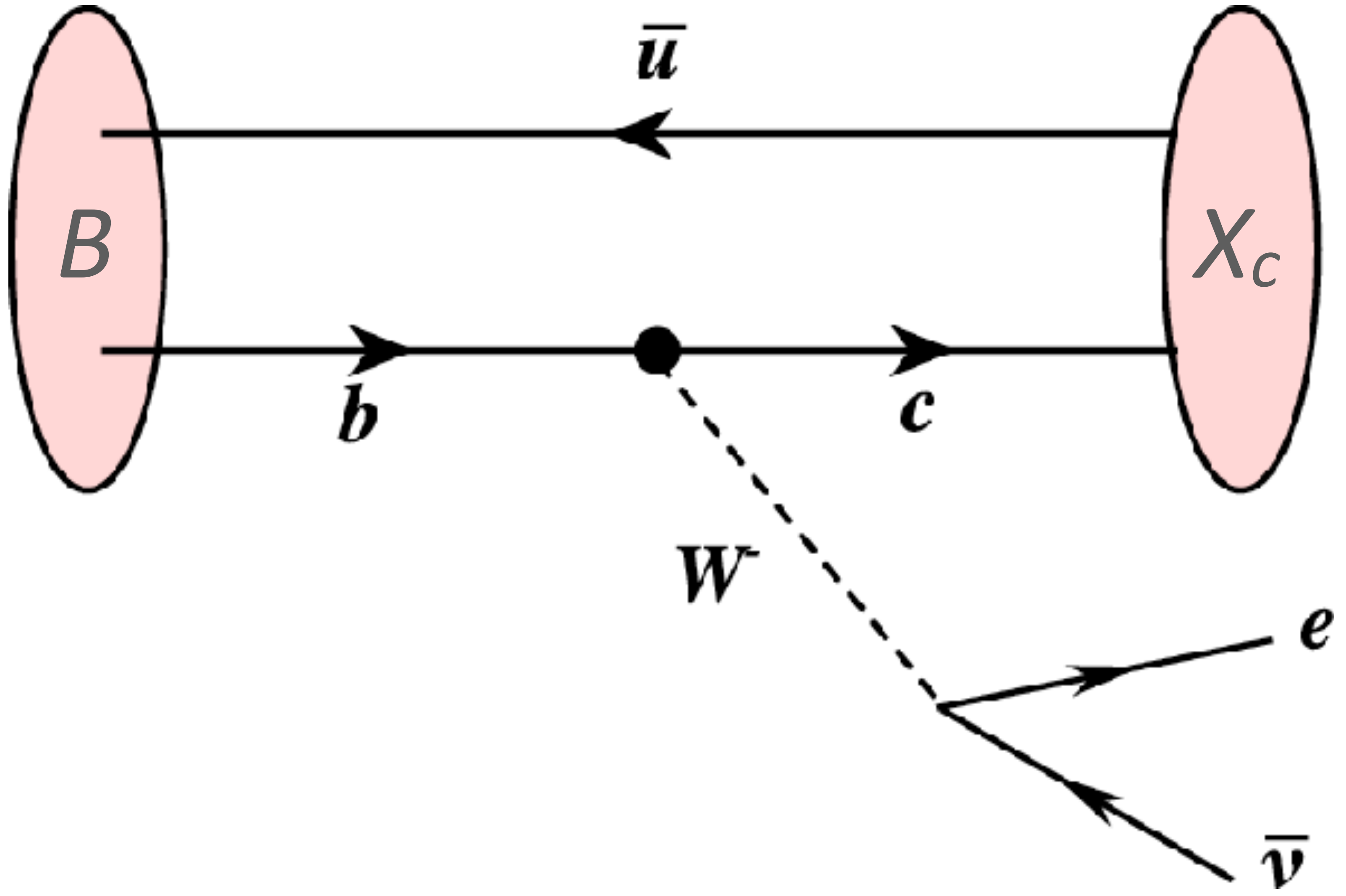
- Gambino, SH, PRL 125 (2020) 032001; 2005.13730
- Gambino, SH, Machler. Panero, Sanfilippo, Simula, Smecca, Tantalo, 2203.11762
- Barone (Southampton), Kellerman (SOKENDAI), SH, Kaneko (KEK), Juttner (CERN), work in progress

but the talk is full of my own prejudice/opinions

SOKENDAI

April 21, 2022

Semi-leptonic B decay



exclusive particular final states (D, D*, ...)

inclusive sum over final states
so far computed by perturbation theory (or OPE)

Gambino and SH, arXiv:2005.13730

Inclusive rate can be evaluated from two-current inserted matrix element.

$$\langle B(\mathbf{0})|\tilde{J}^{\dagger}_{\mu}(-\boldsymbol{q};t) \,\vdots\, \tilde{J}_{\nu}(\boldsymbol{q};0)|B(\mathbf{0})\rangle$$

all possible final states

Basic idea

Lattice calculation of Euclidean matrix elements (like those for form factors)

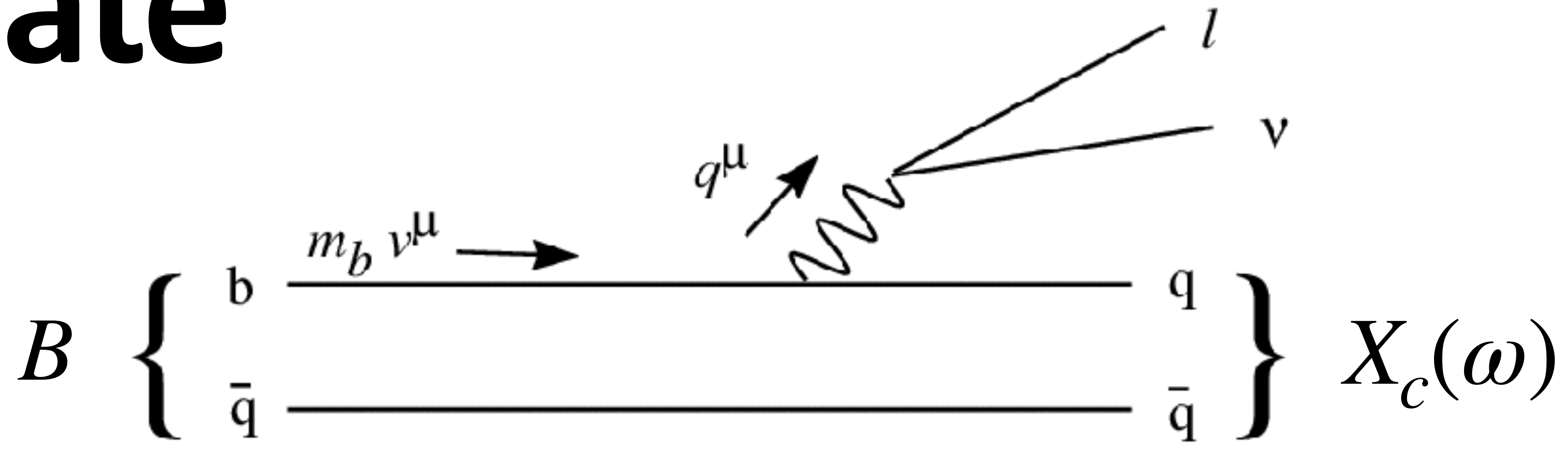
$$\langle B(\mathbf{0})|\tilde{J}_\mu^\dagger(-\boldsymbol{q};t)\ \ \tilde{J}_\nu(\boldsymbol{q};0)|B(\mathbf{0})\rangle \ \sim\ A_0 e^{-E_0 t} + A_1 e^{-E_1 t} + A_2 e^{-E_2 t} + \cdots$$

corresponding to all possible final states

The necessary info were there; but summed with unwanted weights e^{-Et}

To evaluate the inclusive rate, each state has to be summed according to the (semi-leptonic) kinematics.

Inclusive semi-leptonic rate



Differential decay rate:

$$d\Gamma \sim |V_{cb}|^2 l^{\mu\nu} W_{\mu\nu}$$

Structure function (or hadronic tensor):

$$W_{\mu\nu} = \sum_X (2\pi)^2 \delta^4(p_B - q - p_X) \frac{1}{2M_B} \langle B(p_B)|J_\mu^\dagger(0)|X\rangle\langle X|J_\nu(0)|B(p_B)\rangle$$

$$\rightarrow \langle B(\mathbf{0})|\tilde{J}_\mu^\dagger(-\boldsymbol{q};t)\ \delta(\omega - \hat{H})\ \tilde{J}_\nu(\boldsymbol{q};0)|B(\mathbf{0})\rangle$$

Total decay rate:

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega - \hat{H})\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

kinematical (phase-space) factor

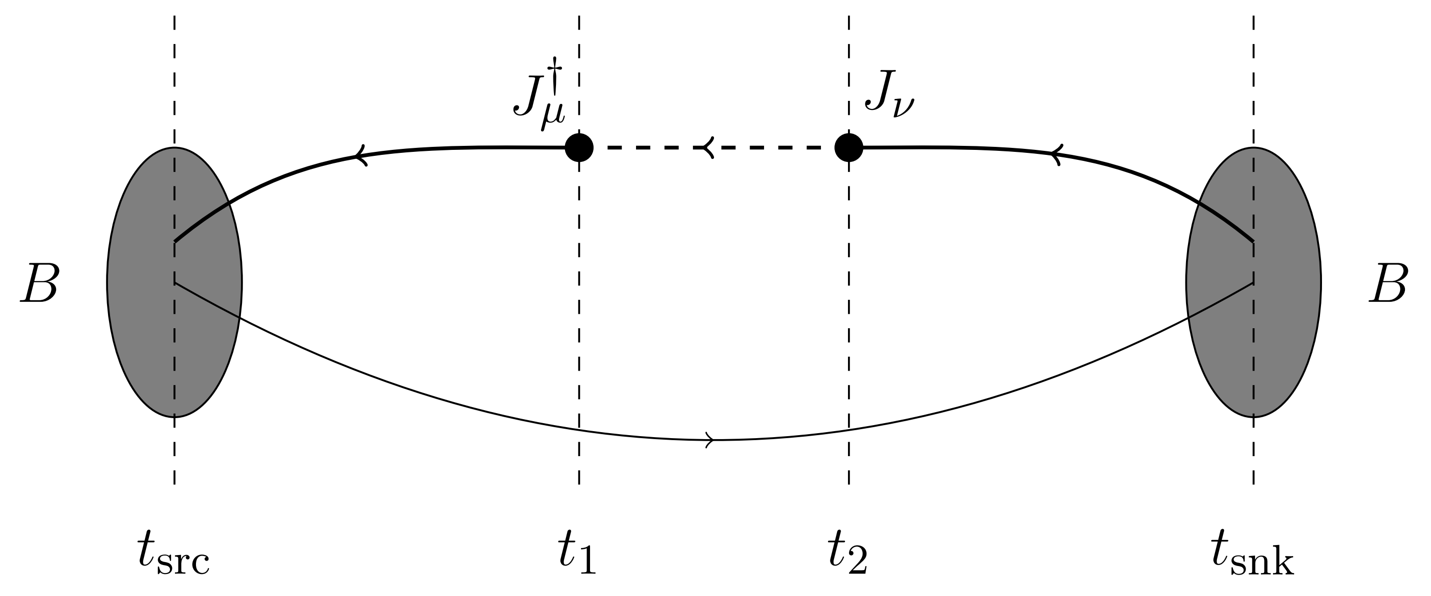
Energy integral to be evaluated:

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

$$= \langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})K(\hat{H};\boldsymbol{q}^2)\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

Compton amplitude obtained on the lattice:

$$\langle B(\mathbf{0})|\tilde{J}^\dagger_\mu(-\boldsymbol{q};t)\ \tilde{J}_\nu(\boldsymbol{q};0)|B(\mathbf{0})\rangle \longrightarrow \langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})e^{-\hat{H}t}\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$



Need an approx :

$$K(\hat{H}) = k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-k_N \hat{H}}$$

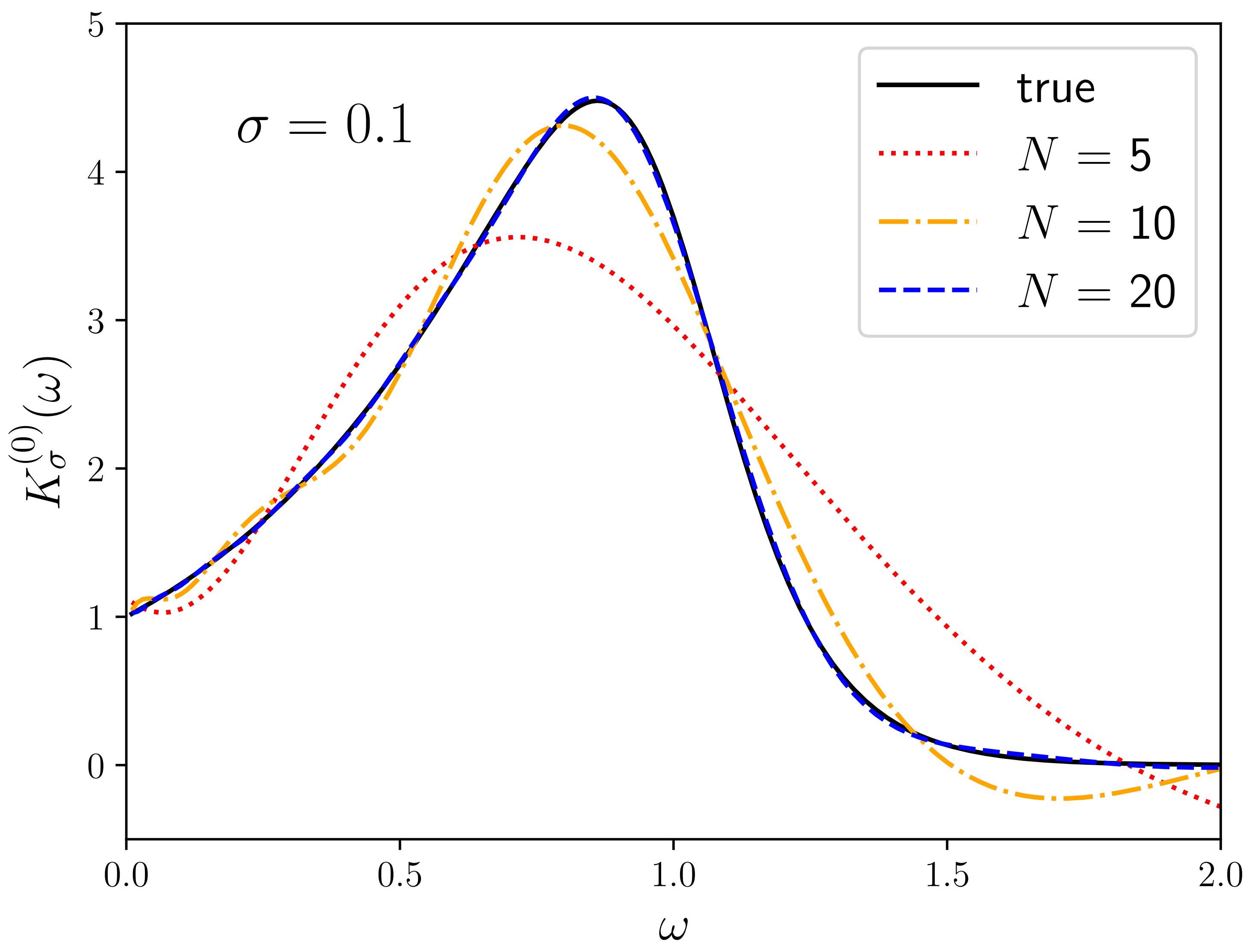
Approximation

- Backus-Gilbert method — Hansen, Lupo, Tantalo (2019)
- <u>Chebyshev polynomial</u> — Bailas, SH, Ishikawa (2000)

$$K(\hat{H}) \simeq \sum_{j=0}^{N} c_j T_j(e^{-\hat{H}})$$

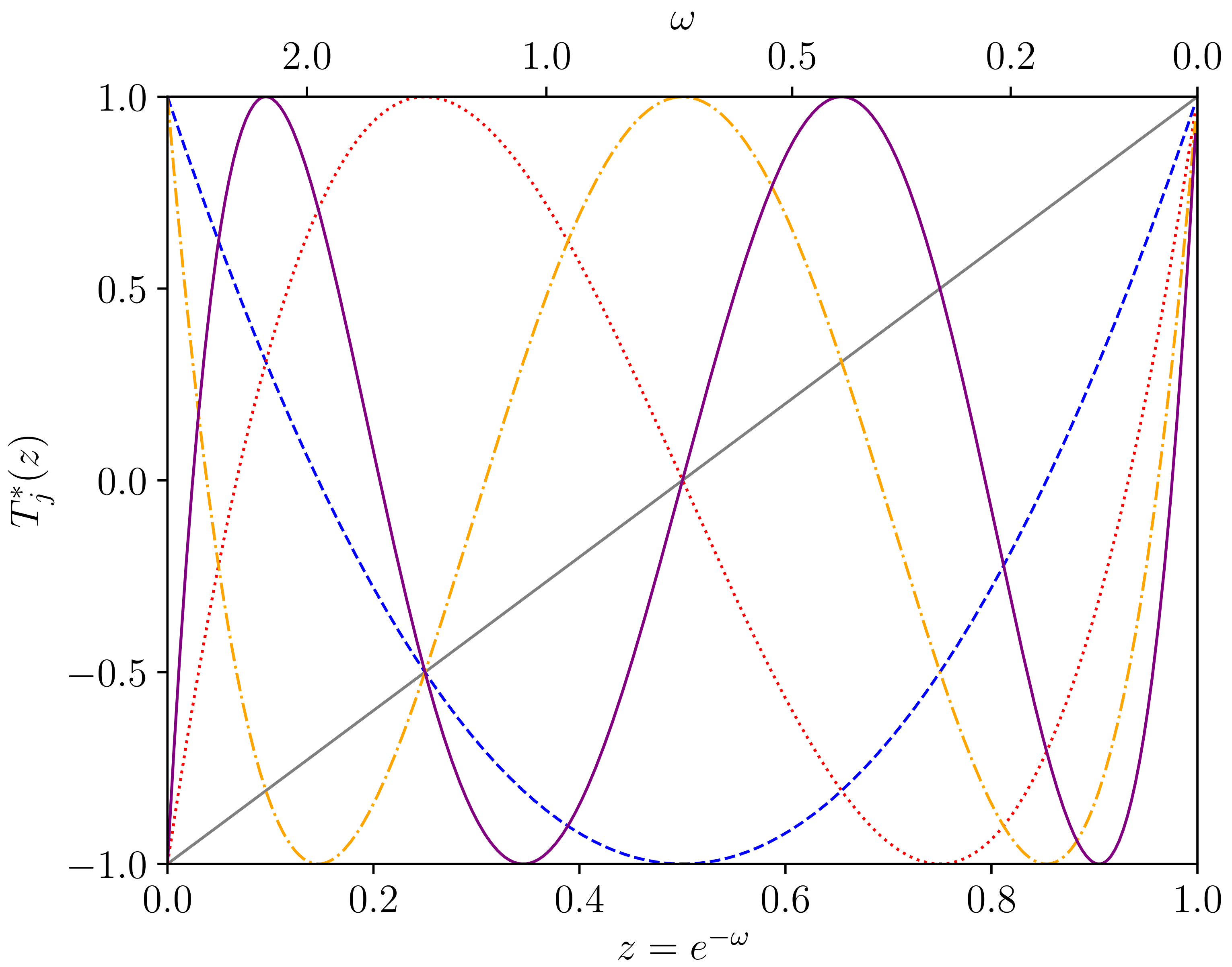
- “best” approx (= maximal deviation is minimal)
- only smooth functions can be approximated.
- constraint $|T_j(z)| < 1$ helps stabilize.

example of the Chebyshev approx:



(shifted) Chebyshev polynomials

$$T_0^*(x) = 1$$
$$T_1^*(x) = 2x - 1$$
$$T_2^*(x) = 8x^2 - 8x + 1$$
$$\vdots$$
$$T_{j+1}^*(x) = 2(2x-1)T_j^*(x) - T_{j-1}^*(x)$$



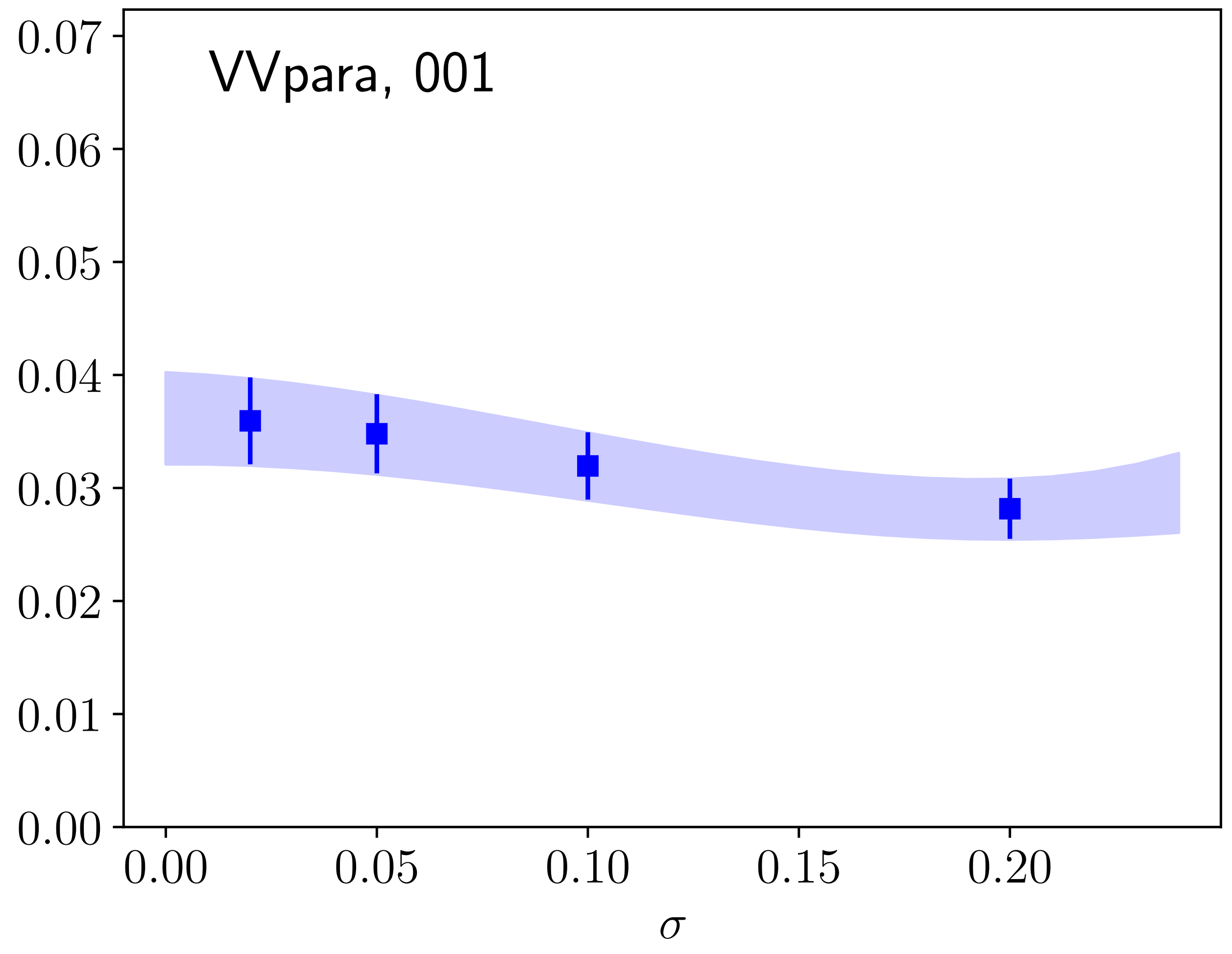
Phase-space factor as a kernel

upper limit

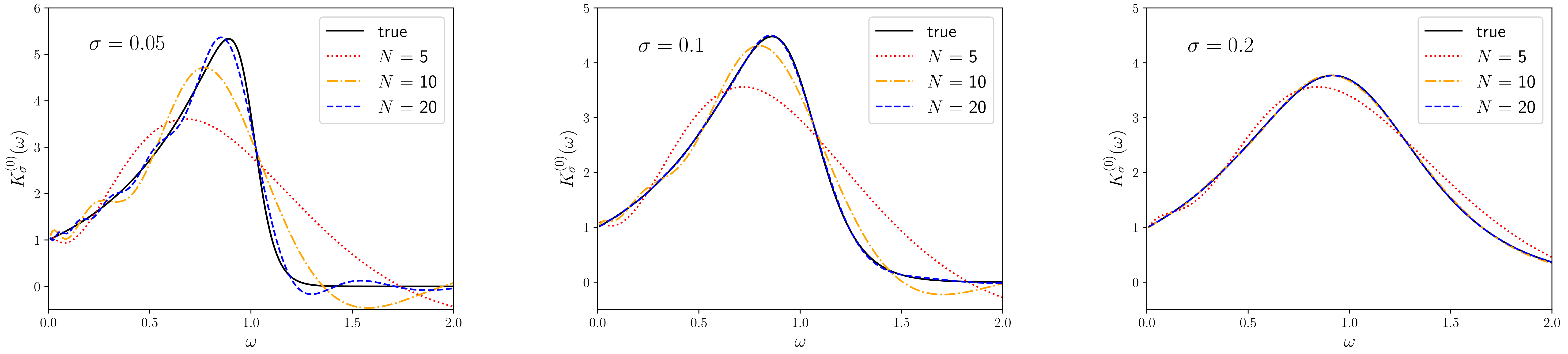
$$K(\omega) \sim e^{2\omega t_0} (m_B - \omega)^l \theta(m_B - |\mathbf{q}| - \omega)$$

kinematical

smear by sigmoid with a width σ;
Need to take the σ→ 0 limit



narrower smearing wider

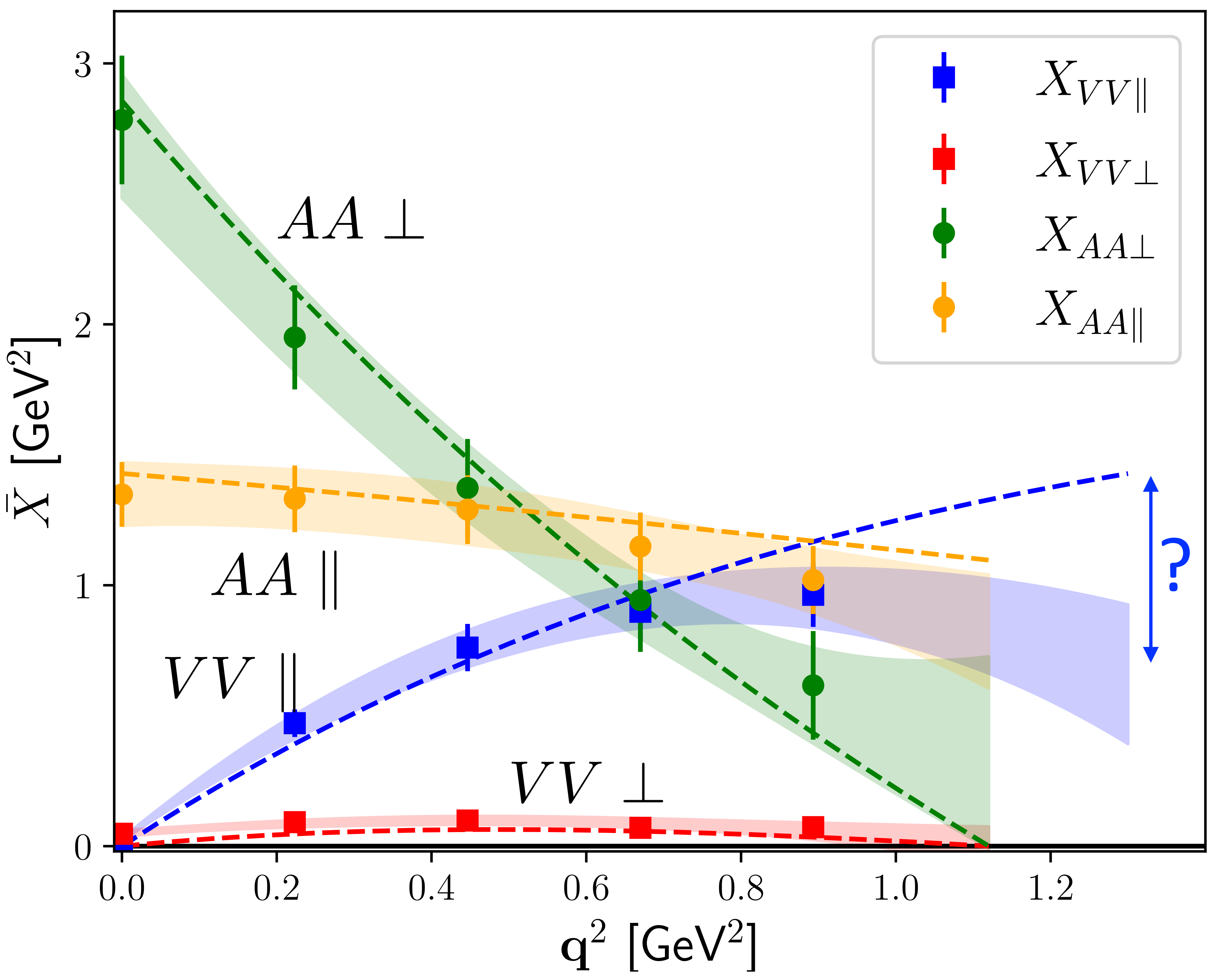


Inclusive decay rate

- Prototype lattice calculation
  - $B_s \to Xc$
  - the b quark is lighter than physical.

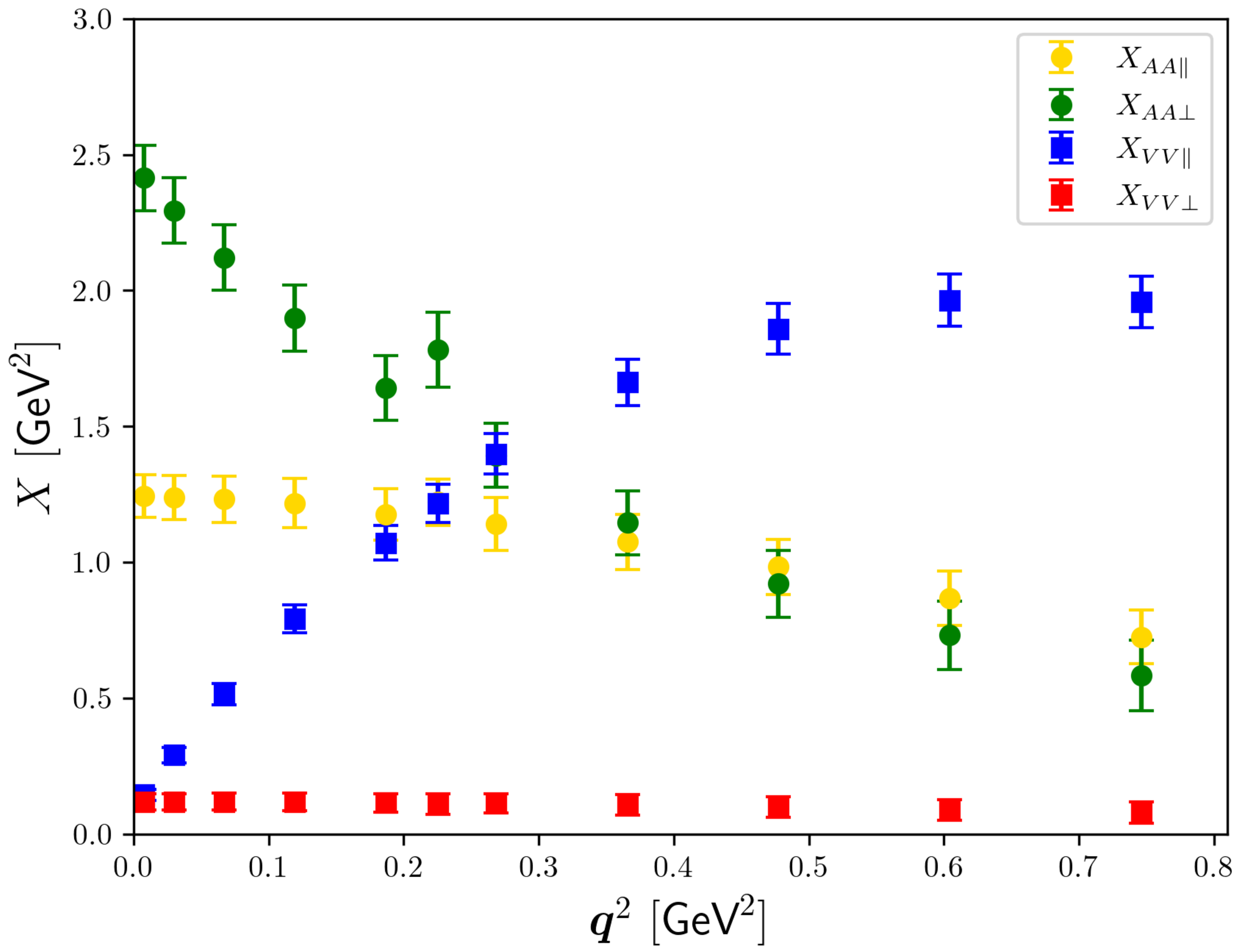
- Decay rate in each channel
  - VV and AA
  - parallel or perpendicular to the recoil momentum
  - compared to “exclusive” (dashed lines)
    - $VV_{||}$ is dominated by B→D
    - Others are by B→D*

differential rate / |**q**|



JLQCD data from
Gambino et al., 2203.11762

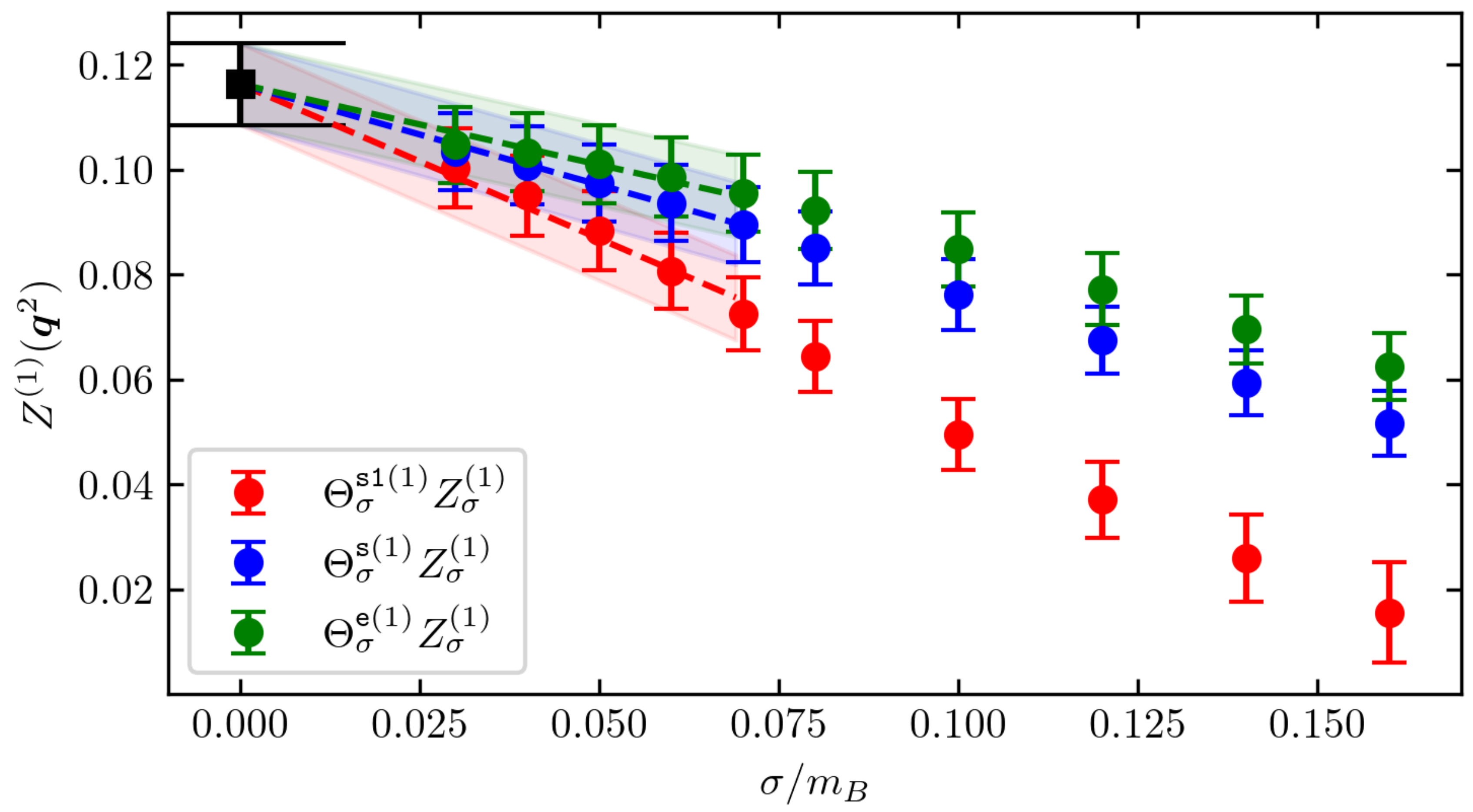
Inclusive decay rate



ETMC data from
Gambino et al., 2203.11762

From 2203.11762
Analysis with Backus-Gilbert (by Smecca et al)

- Backus-Gilbert works equally well
- σ→0 limit is taken (with different smearings)



- calculated at many **q**^{2} points
- lighter b quark

Sum over states: dangerous game?

Sum over states with a kernel $K(s)$: $\int_0^\infty ds\, K(s)\rho(s)$

Crucially depends on our ability to approximate the energy integral.

- Possible to treat any $K(s)$?
- Probably not, because $K(s) = \delta(s)$ leads us back to the ill-posed problem (reconstruction of full spectral function from lattice data!)

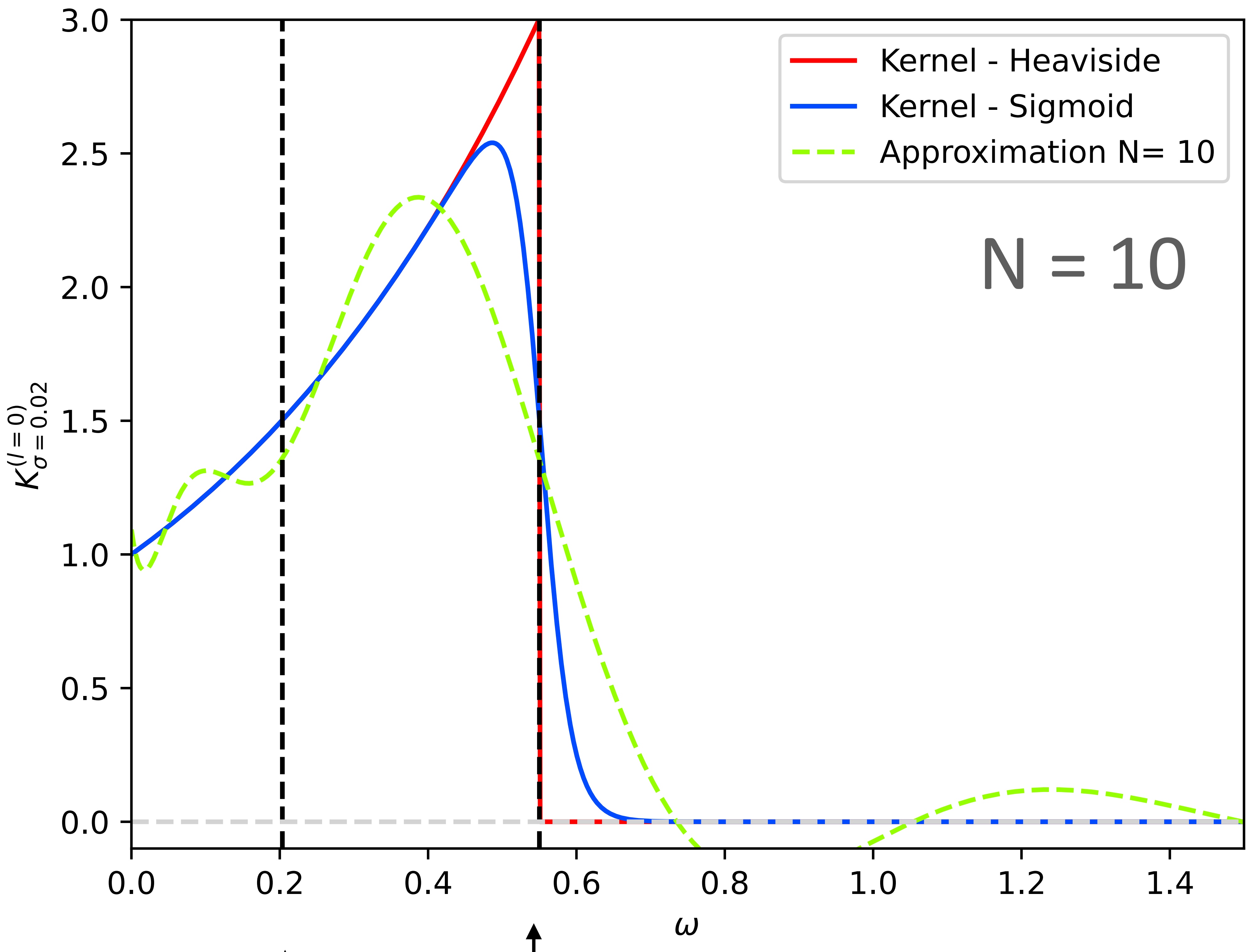
- Then, what is the limitation?

Kernel approximation: an example

$$K(\omega) \sim e^{2\omega t_0}(m_B - \omega)^l \theta(m_B - |\mathbf{q}| - \omega)$$

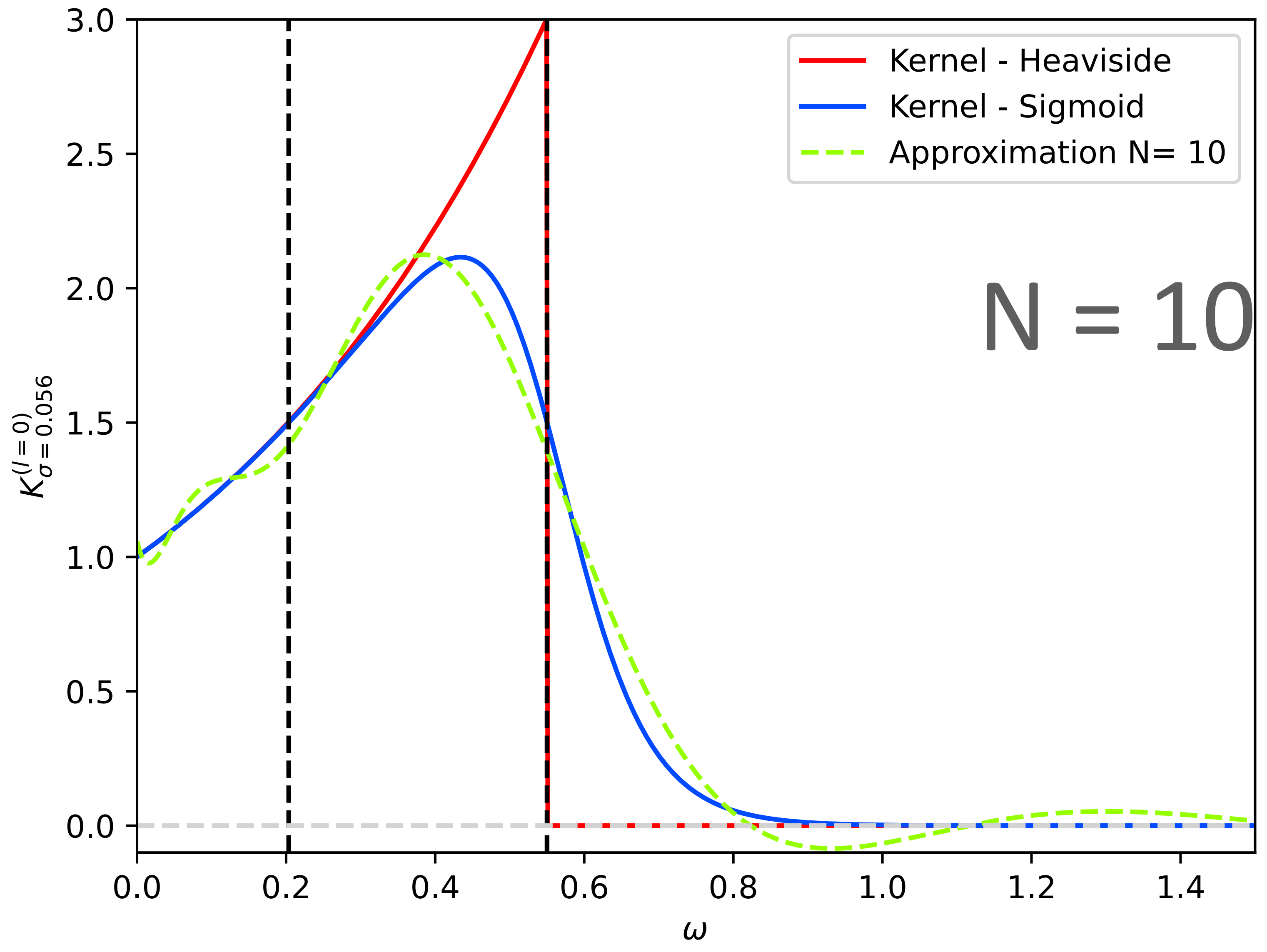
narrow smearing (σ = 0.02)

medium (σ = 0.056)



upper limit

lowest energy state



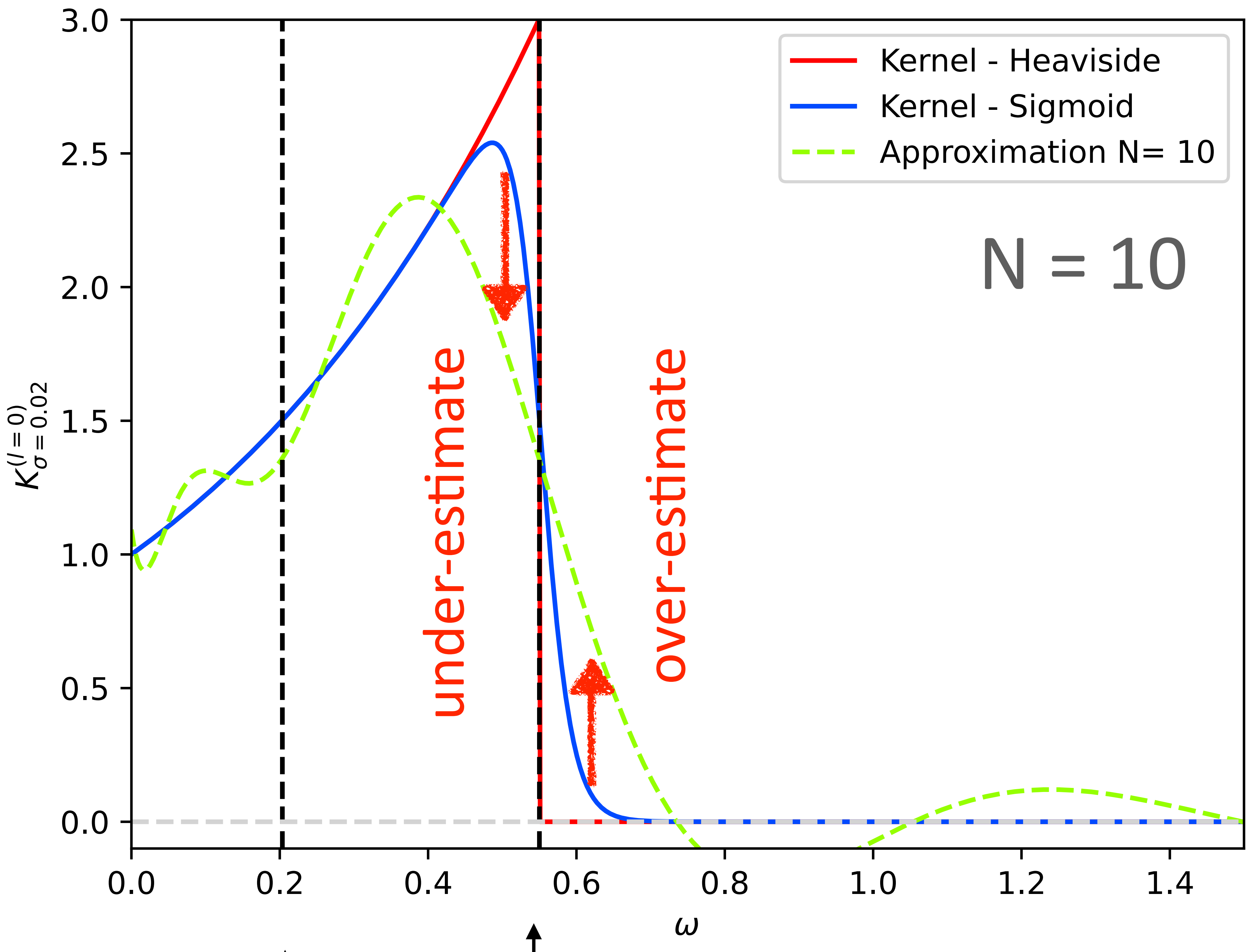
Smearing:

- Too wide = away from the true func
- Too narrow = bad approx

Kernel approximation: an example

$$K(\omega) \sim e^{2\omega t_0}(m_B - \omega)^l \theta(m_B - |\mathbf{q}| - \omega)$$

narrow smearing (σ = 0.02)



upper limit

lowest energy state

Good news:

- Error cancels due to the oscillating approximation (Chebyshev polynomial) when the states distribute evenly.

Bad news:

- Physical spectrum may not be flat. (A large gap between ground and excited states, for instance.)
- The integral range gets narrower for larger $\mathbf{q}^2$. The problem gets harder. (But we can keep the ground state only, there.)

Prospects

"The devil is in the details."

- Still in the early stage. Concerning the errors, I am optimistic, but more studies are necessary for various kinematical setups.

- Real calculation of $B \to X_c$, X_u at physical masses still to be done.

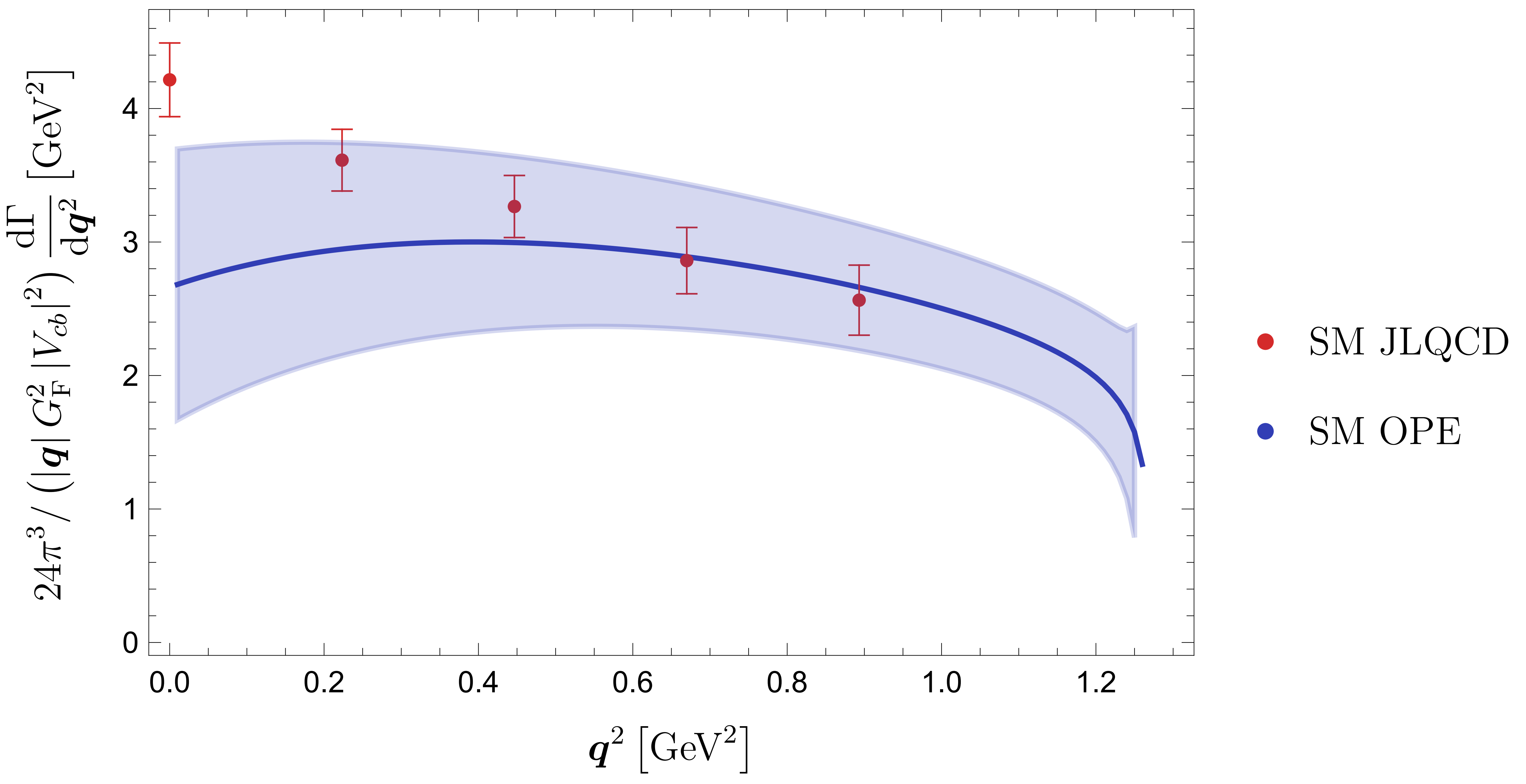
- Many potential applications

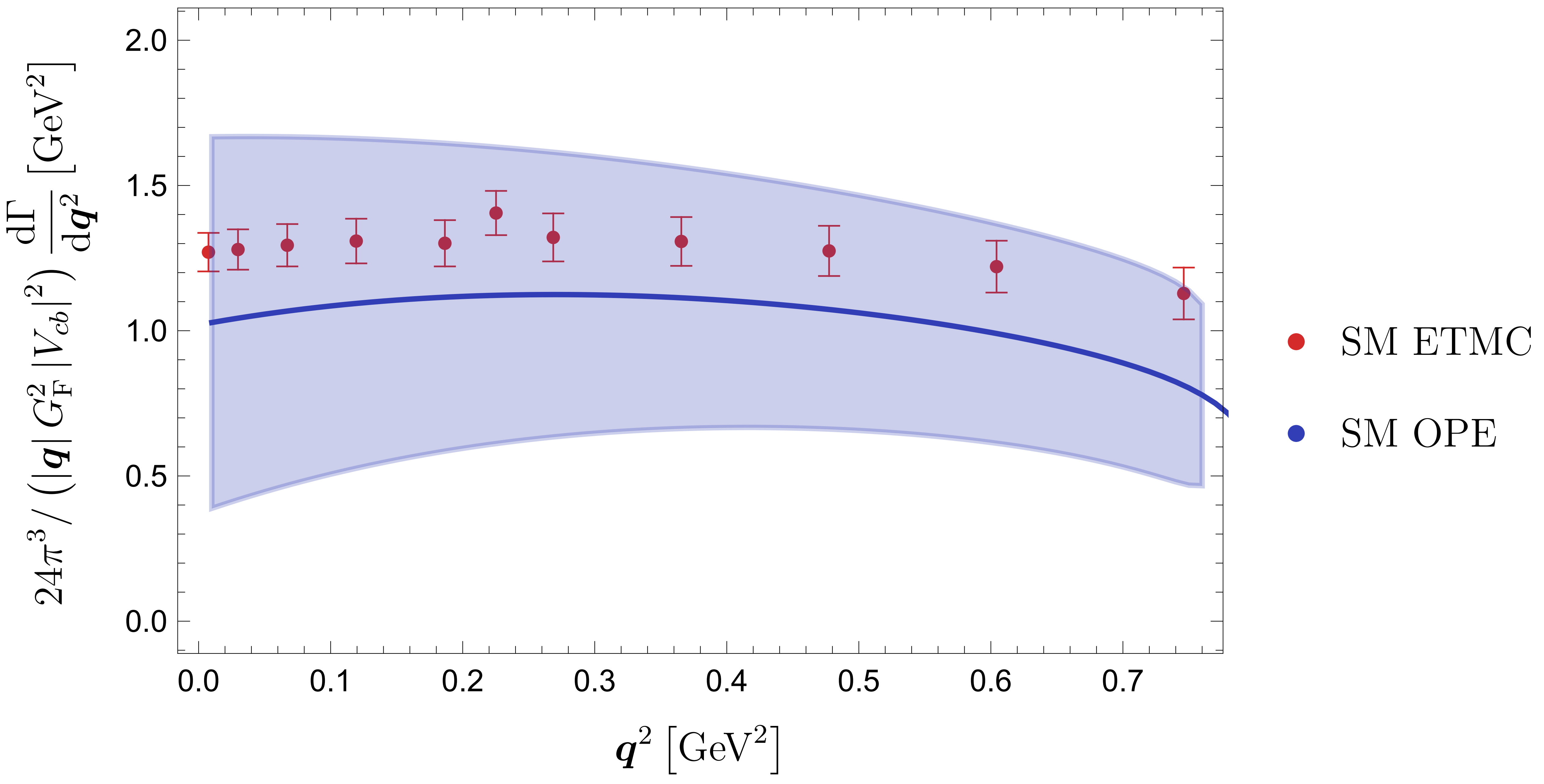
  - D and B

  - Not just total rate, e.g. semi-leptonic moments $<M_X^2>$, $<E_l>$

  - Comparison with OPE, then to determine MEs

see 2203.11762





From 2203.11762

OPE calculation by Gambino and Machler

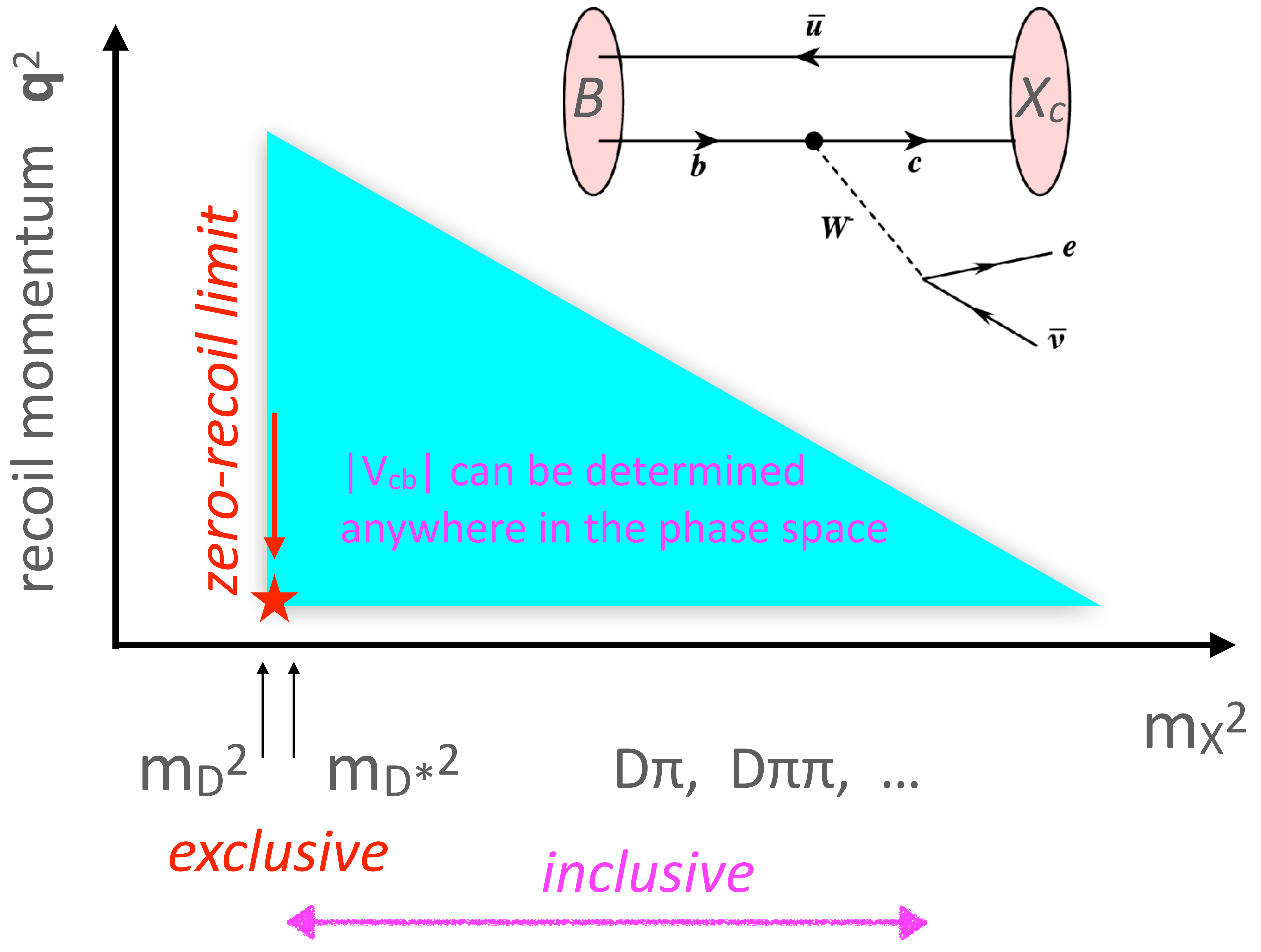
- PT including $O(\alpha_s)$, OPE up to $O(1/m^3)$
- Hadronic parameters μ_π^2 etc are taken from the phono analysis.
- b quark mass is adjusted to match the lattice calculations.
- OPE breaks down near the $\mathbf{q}^2$ endpoint.

✓ Good agreement.

✓ Error of OPE is from the hadronic parameters. Large because of small m_b.

✓ Better for moments $\langle M_X^2 \rangle$, $\langle E_l \rangle$, …

Better use of the phase space?



Two measurement strategies:

- Exclusive, with lattice FF
- Inclusive, with OPE (or lattice)

But, anything in between and more?

Possible?

Minimize the uncertainty by choosing a weight function: like the moments, but there is more freedom

- Smooth weight functions preferred for lattice inclusive
- XXXX preferred for exp't… ?