$|V_{ub}|$ and LFU ratios in B decays using Lattice QCD and Unitarity

Work in collaboration with G. Martinelli and S. Simula

[PRD '21 (2105.02497), 2202.10285, …]

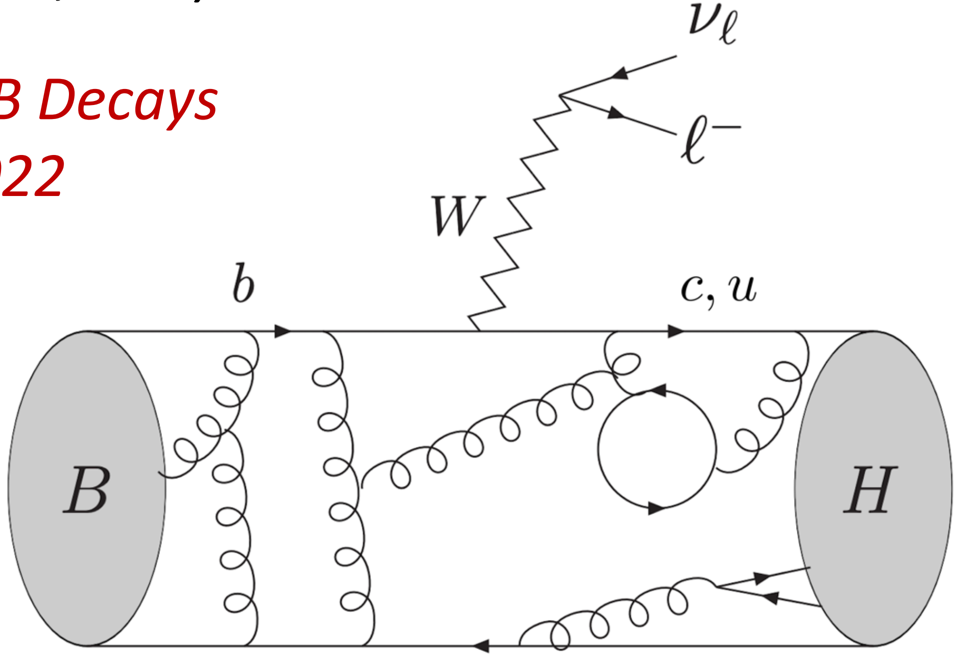
Ludovico Vittorio (SNS & INFN, Pisa)

Challenges in Semileptonic B Decays
Barolo, 19-23 April 2022

SCUOLA NORMALE SUPERIORE

INFN PISA
Istituto Nazionale di Fisica Nucleare
Sezione di Pisa

MINISTERO DELL' ISTRUZIONE, DELL'UNIVERSITÀ E DELLA RICERCA
PRIN "The consequences of flavor"



(from J.Phys.G 46 (2019) 2, 023001)

The Dispersive Matrix (DM) approach to the Form Factors (FFs)

In case of production of a pseudoscalar meson (*i.e.* π, K):

$$\frac{d\Gamma(B_{(s)} \to \pi(K)\ell\nu_\ell)}{dq^2} = \frac{G_F^2|V_{ub}|^2}{24\pi^3}\left(1-\frac{m_\ell^2}{q^2}\right)^2\left[|\vec{p}_{\pi(K)}|^3\left(1+\frac{m_\ell^2}{2q^2}\right)|f_+^{\pi(K)}(q^2)|^2 \right.$$
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- **will be entirely based on first principles (LQCD evaluation of 2- and 3-point Euclidean correlators)**
- **will be independent of any assumption on the functional dependence of the FFs on the momentum transfer**
- **can be applied to theoretical calculations of the FFs, but also to experimental data**
- **keep theoretical calculations and experimental data separated**
- **is universal: it can be applied to any exclusive semileptonic decays of mesons and baryons**

The Dispersive Matrix (DM) approach to the Form Factors (FFs)

See Silvano's talk for the details about the implementation of the DM method!

↑

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Two LQCD inputs have been used for our DM method (**arXiv:2202.10285**):

- **3 RBC/UKQCD synthetic data (points) [PRD '15 (1501.05363)]**
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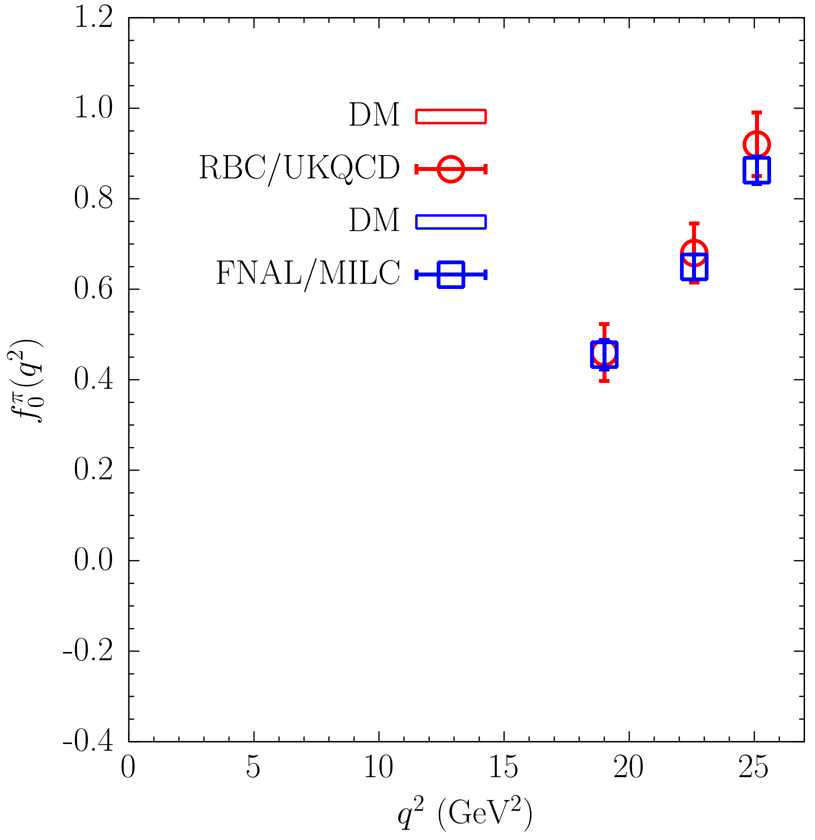
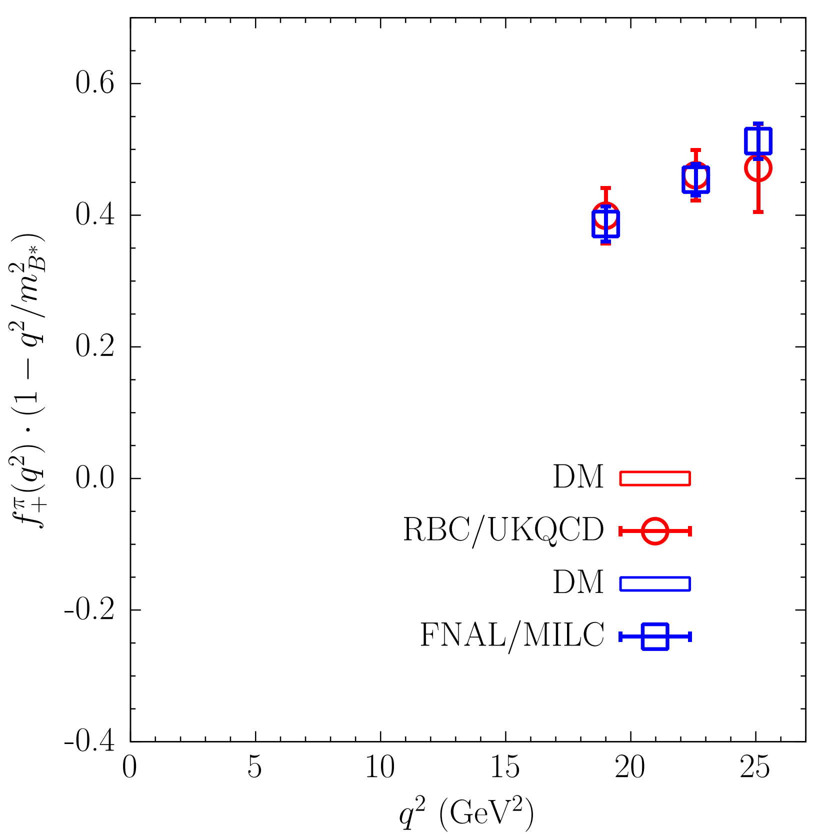
One KC: $f_0(0) = f_+(0)$





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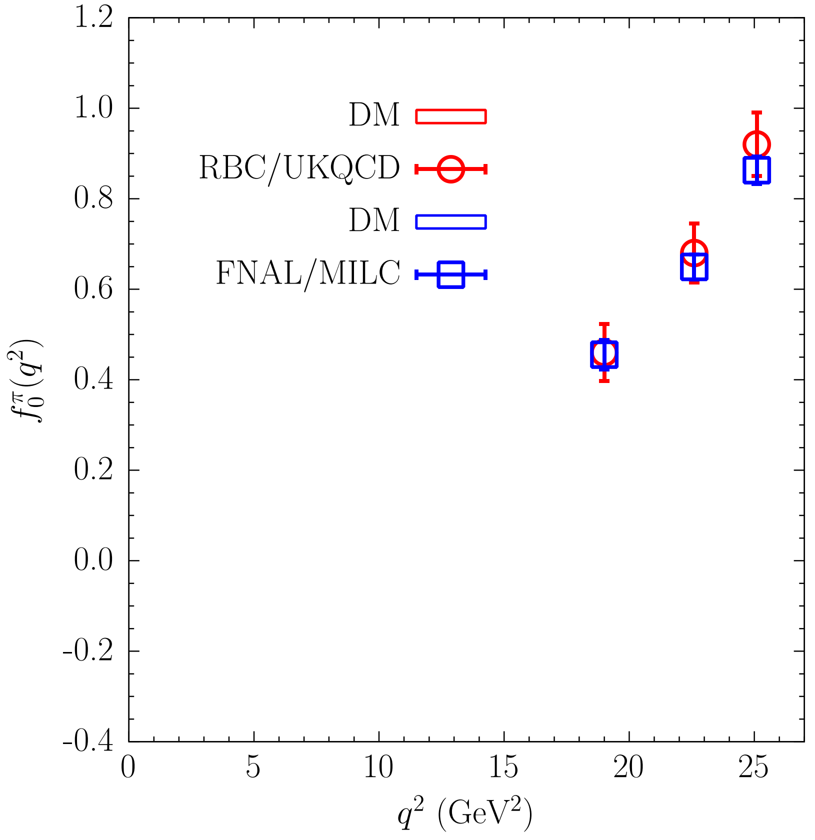
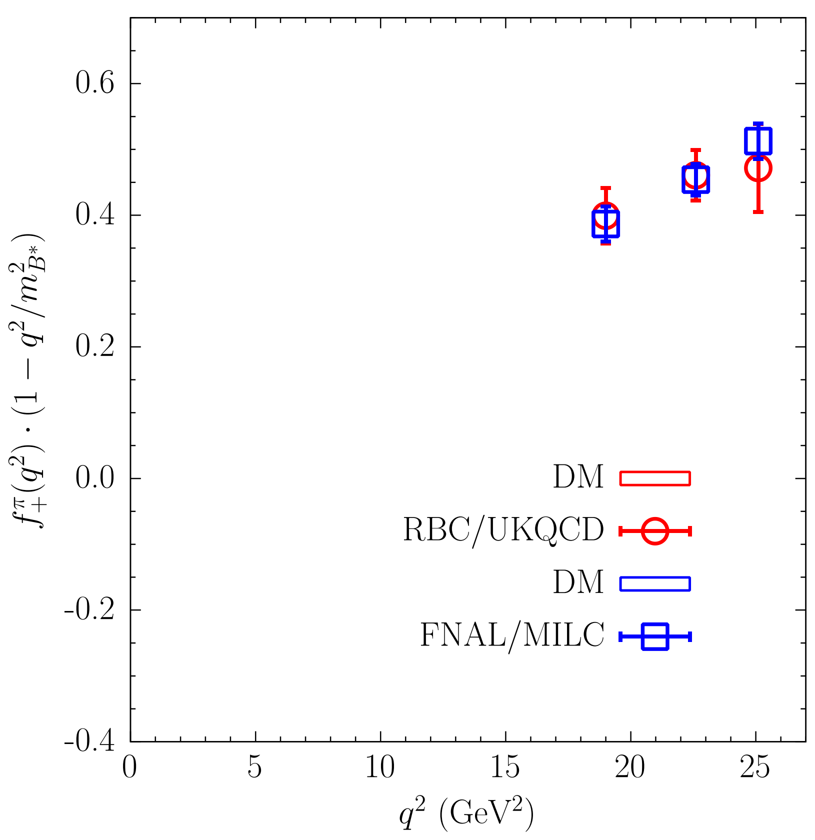
$$f^\pi(q^2=0)|_{\rm FNAL/MILC} = -0.01 \pm 0.16$$

$$f^\pi(q^2=0)|_{\rm combined} = -0.04 \pm 0.22$$

Peculiarity of B → π decays: LONG extrapolation in q^2





It seems that the mean value and the uncertainty are not stable under variation of the truncation order of a series expansion of the FFs…

The DM approach is independent of this issue!!!

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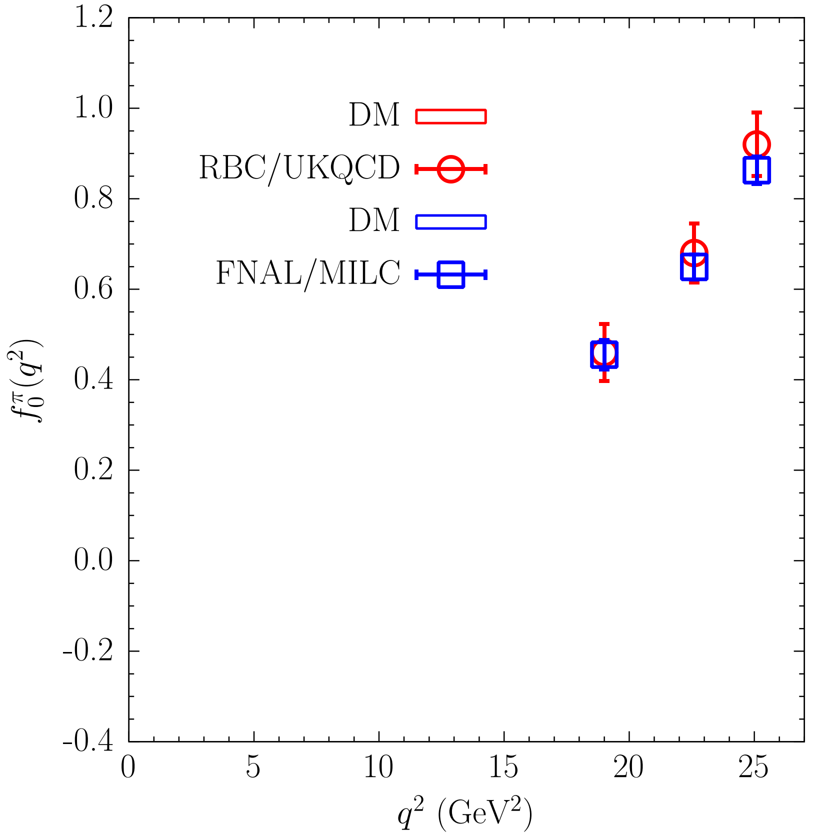

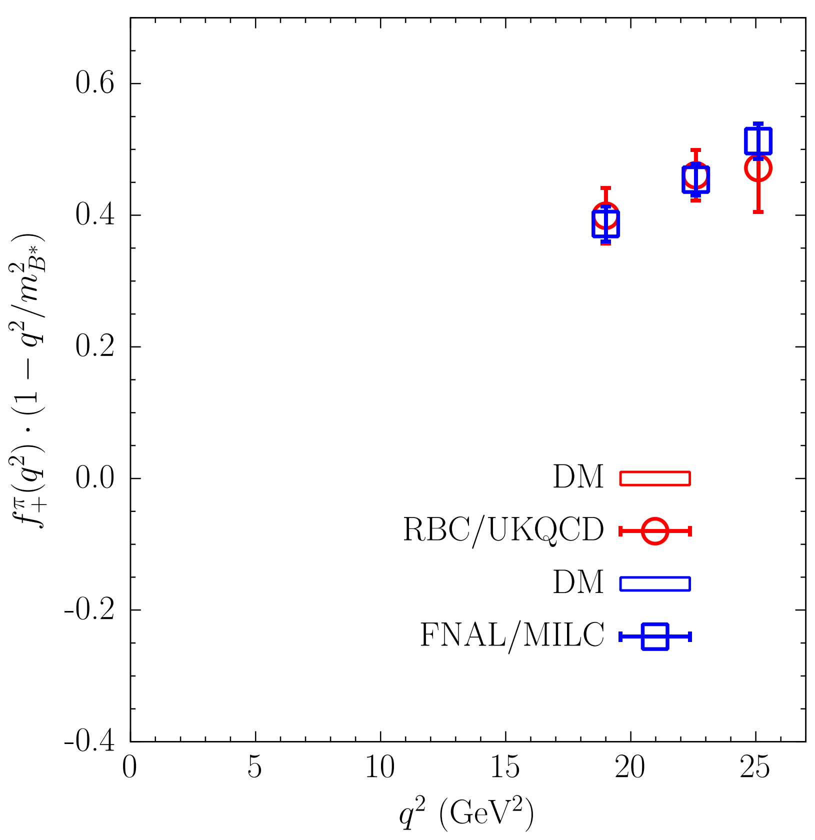

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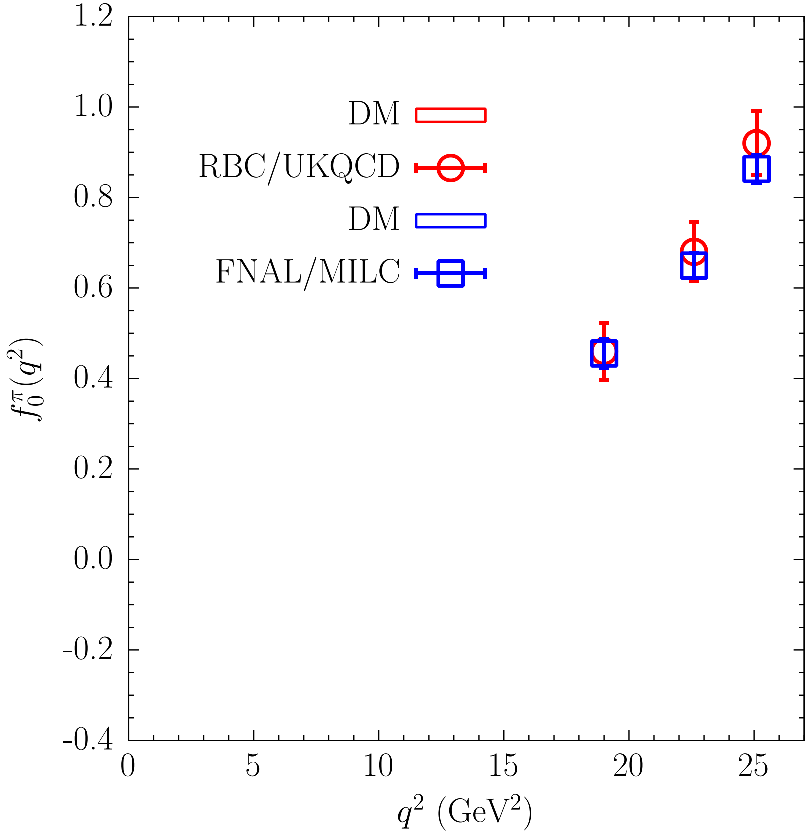
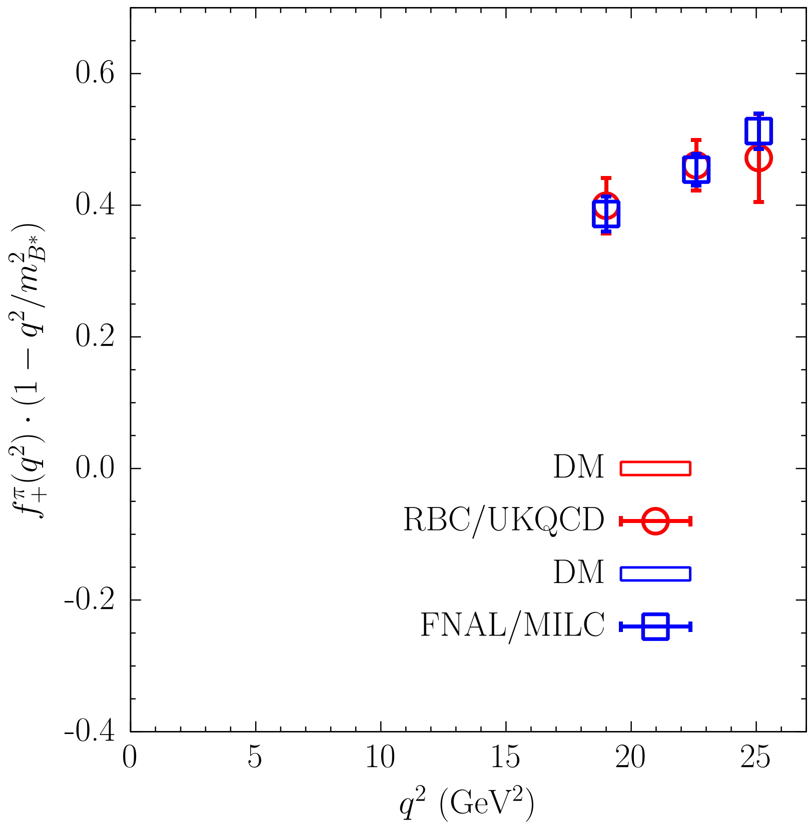
IMPORTANT: new LQCD computations published by JLQCD Collaboration [arXiv:2203.04938]!





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LFU in semileptonic B → π decays

This issue is of capital importance to test LFU:

$$R_\pi^{\tau/\mu} \equiv \frac{\Gamma(B \to \pi\tau\nu_\tau)}{\Gamma(B \to \pi\mu\nu_\mu)}$$

THEORY with DM method

Input	RBC/UKQCD	FNAL/MILC	combined
$R_\pi^{\tau/\mu}$	0.767(145)	0.838(75)	0.793(118)

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For further investigation of possible NP effects in the future, it is fundamental to extrapolate appropriately the FFs behaviour in the whole kinematical range

$|V_{ub}|$ from semileptonic B → π decays

Six sets of data from Belle and BaBar collaborations:
BaBar 2011, 1 channel [PRD '11 (1005.3288)]
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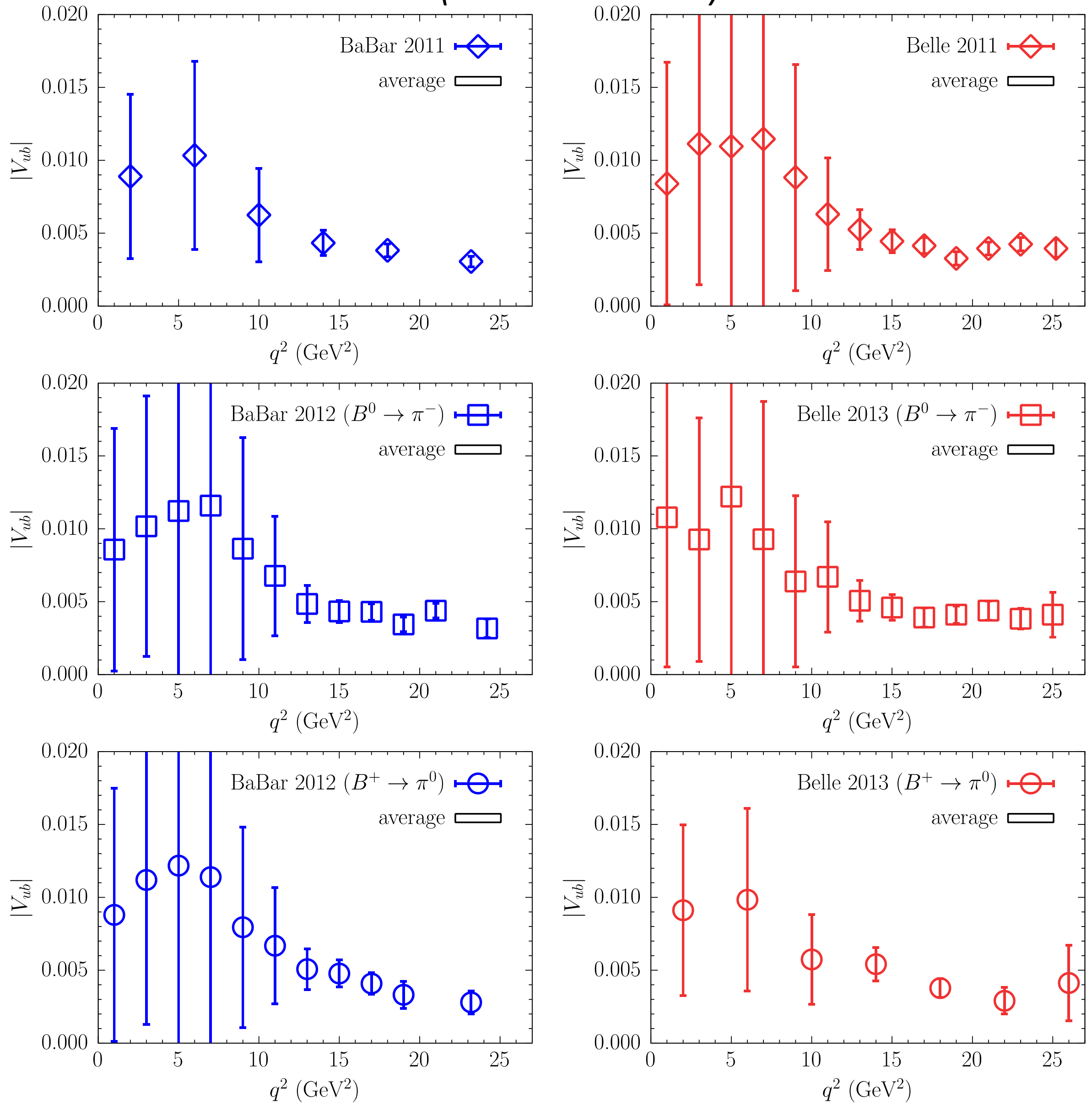
Multiplicity factor — Exper. data

$$|V_{ub}|_i = \sqrt{\frac{C_v}{\tau_{B^v}} \cdot \frac{\Delta\mathcal{B}|_i^{exp}}{\Delta\zeta_i}}$$

B0/B+ meson lifetime — Theor. decay width

(COMBINED case)



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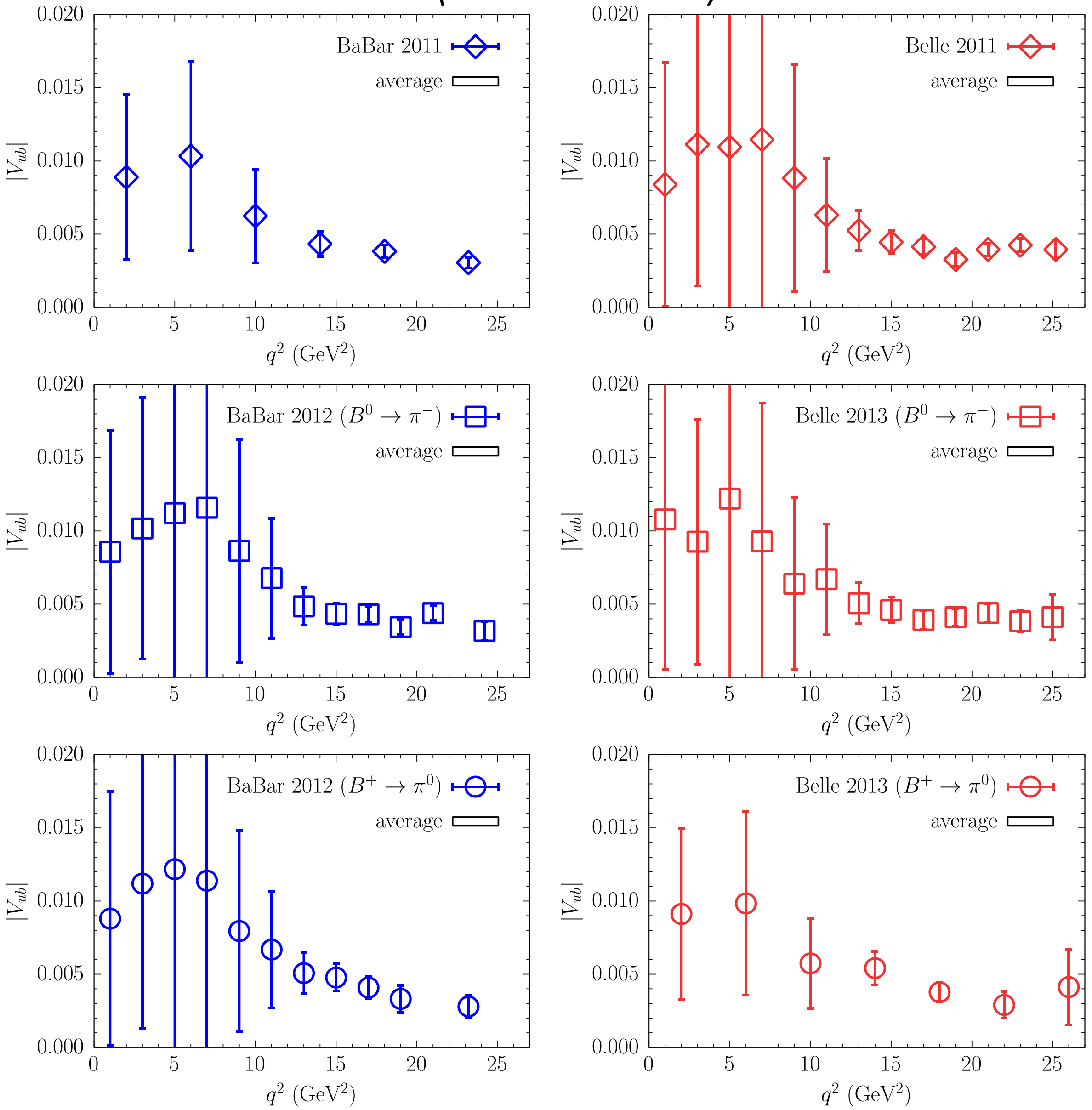

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Input	$\lvert V_{ub}\rvert \times 10^3$
RBC/UKQCD	**3.52(49)**
FNAL/MILC	**3.76(41)**
Combined	**3.62(47)**
PDG exclusive [PTEP 2020, 083C01]	3.70(16)
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DM results (RBC/UKQCD, FNAL/MILC, Combined)

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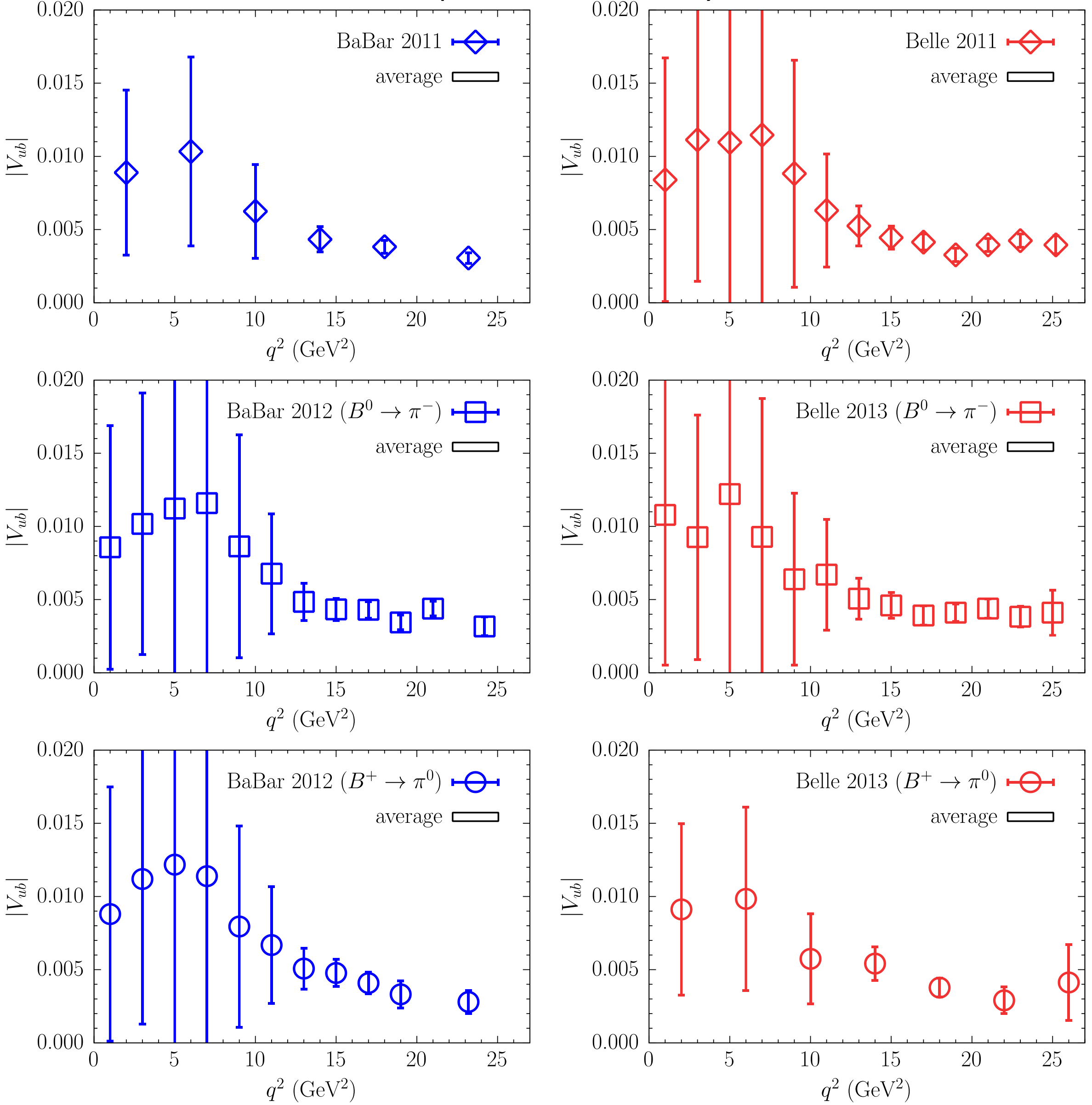

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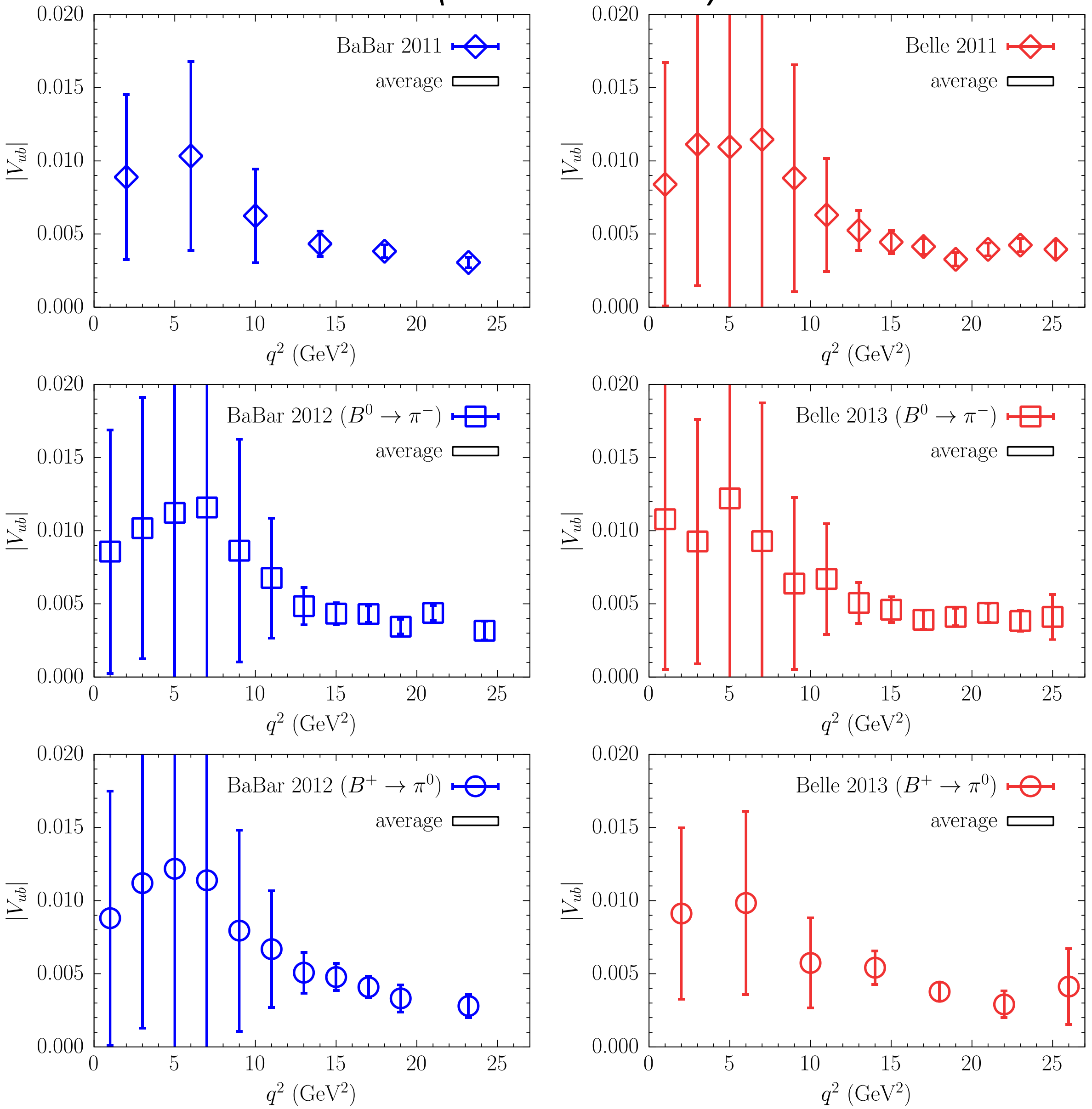

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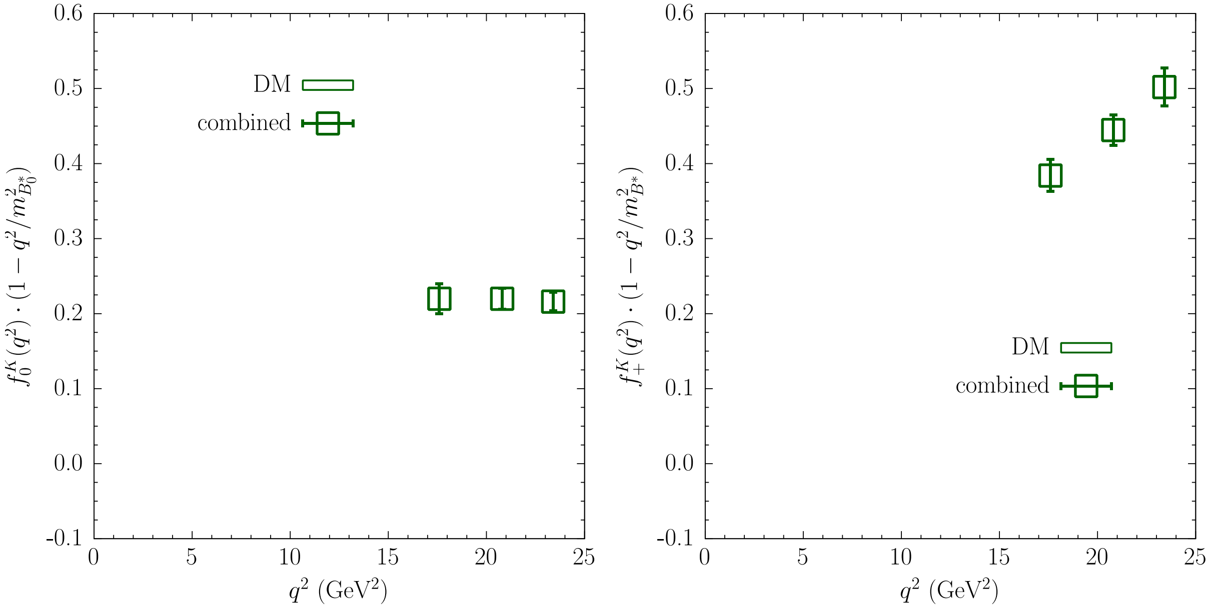

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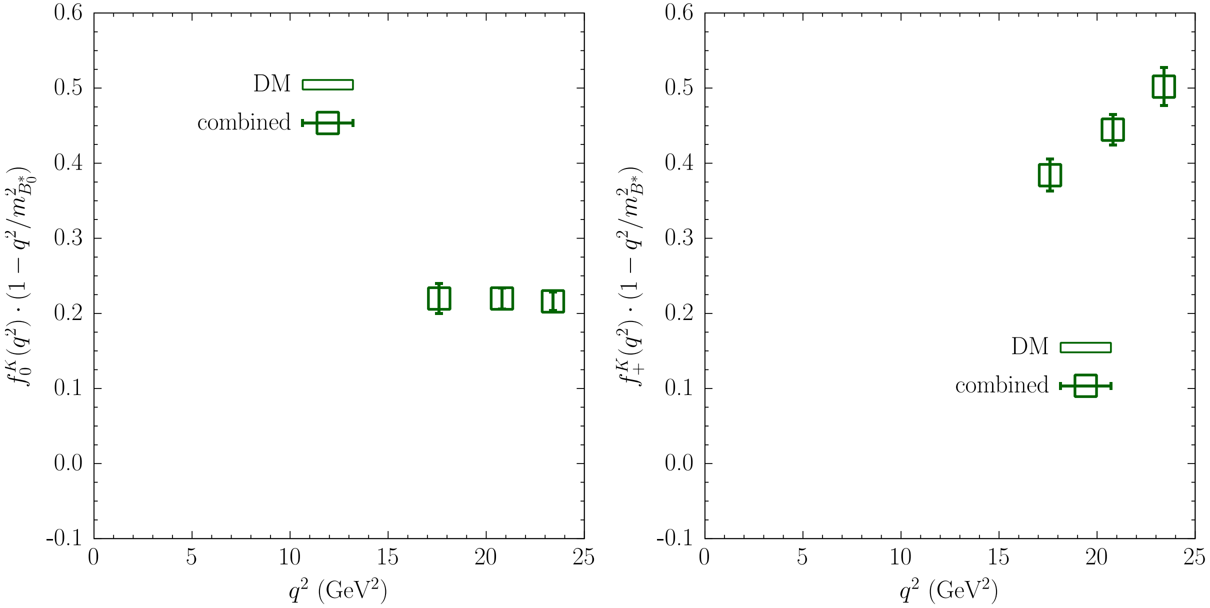

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$$R_{BF} \equiv \frac{\mathcal{B}(B_s^0 \to K^- \mu^+ \nu_\mu)}{\mathcal{B}(B_s^0 \to D_s^- \mu^+ \nu_\mu)}$$

Low-q^2: $q^2 \leq 7\,\mathrm{GeV}^2$

High-q^2: $q^2 \geq 7\,\mathrm{GeV}^2$

LHCb Collaboration, PRL '21 [2012.05143]

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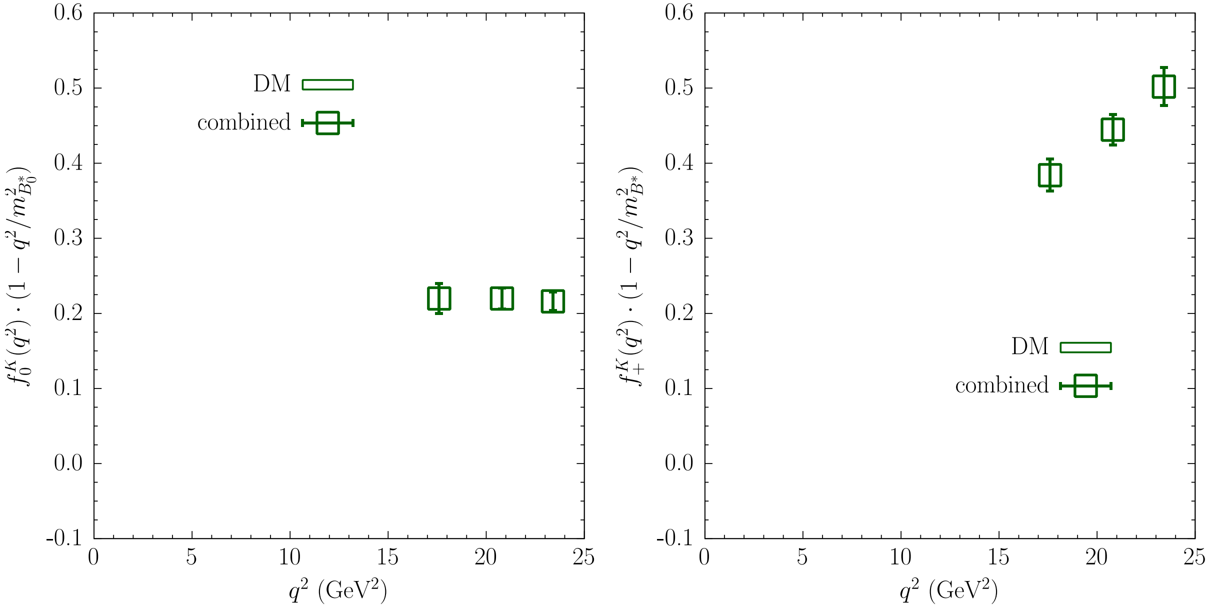

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by using the exp. value of the BR @ denominator

q^2-bin	RBC/UKQCD	FNAL/MILC	HPQCD	combined
low	6.70 ± 3.26	6.43 ± 2.03	3.57 ± 1.94	5.31 ± 3.02
high	4.20 ± 0.56	4.10 ± 0.38	3.54 ± 0.43	3.94 ± 0.59
average	3.93 ± 0.46	3.93 ± 0.35	3.54 ± 0.35	3.77 ± 0.48

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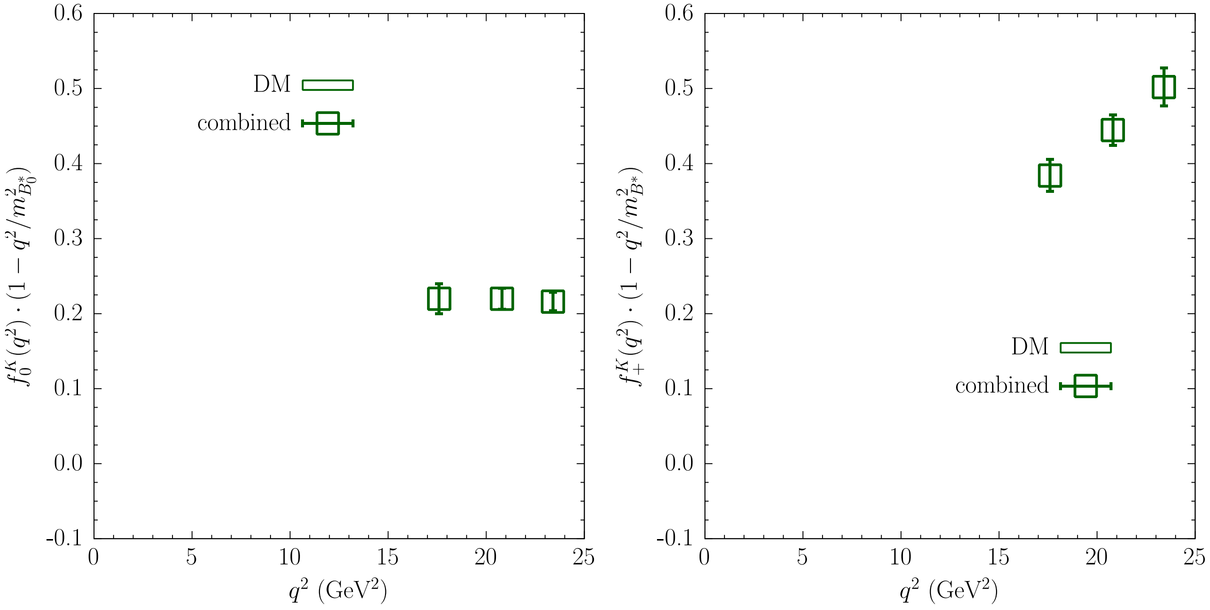

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DM Vub value: $|V_{ub}|\cdot 10^3 = 3.69 \pm 0.34$

when averaged with the $B \to \pi$ result

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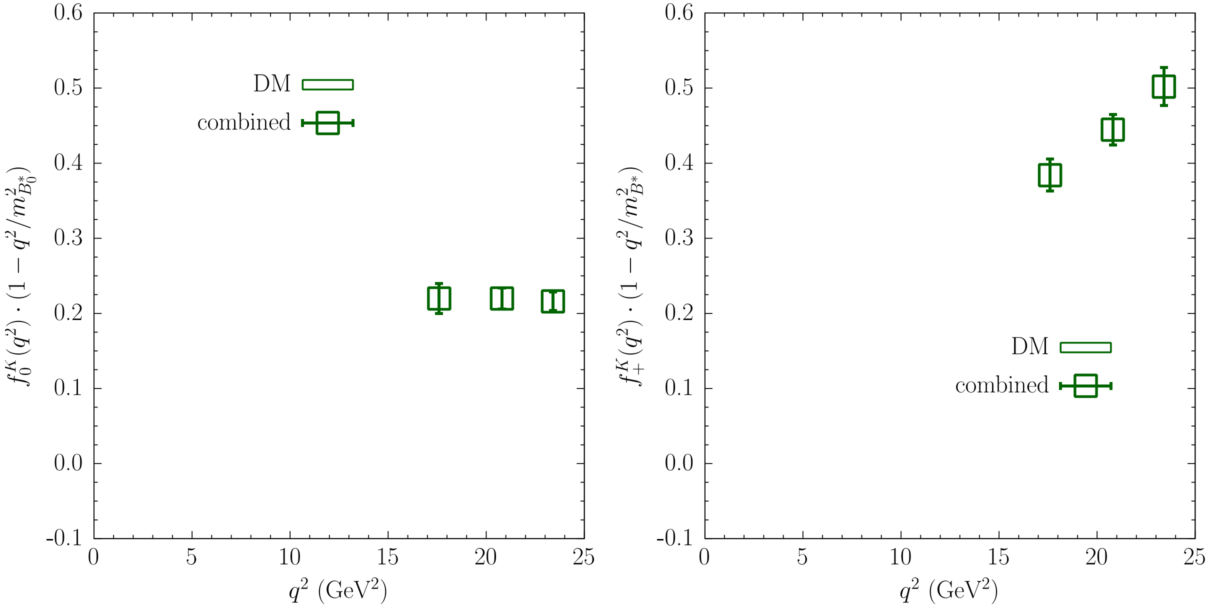

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Low-q²: $q^2 \leq 7\,\mathrm{GeV}^2$

High-q²: $q^2 \geq 7\,\mathrm{GeV}^2$

LHCb Collaboration, PRL '21 [2012.05143]

*We can also investigate **LFU**:* $R_K^{\tau/\mu} = 0.755 \pm 0.138$

by using the exp. value of the BR @ denominator

q^2-bin	RBC/UKQCD	FNAL/MILC	HPQCD	combined
low	6.70 ± 3.26	6.43 ± 2.03	3.57 ± 1.94	5.31 ± 3.02
high	4.20 ± 0.56	4.10 ± 0.38	3.54 ± 0.43	3.94 ± 0.59
average	3.93 ± 0.46	3.93 ± 0.35	3.54 ± 0.35	3.77 ± 0.48

DM Vub value: $|V_{ub}| \cdot 10^3 = 3.69 \pm 0.34$

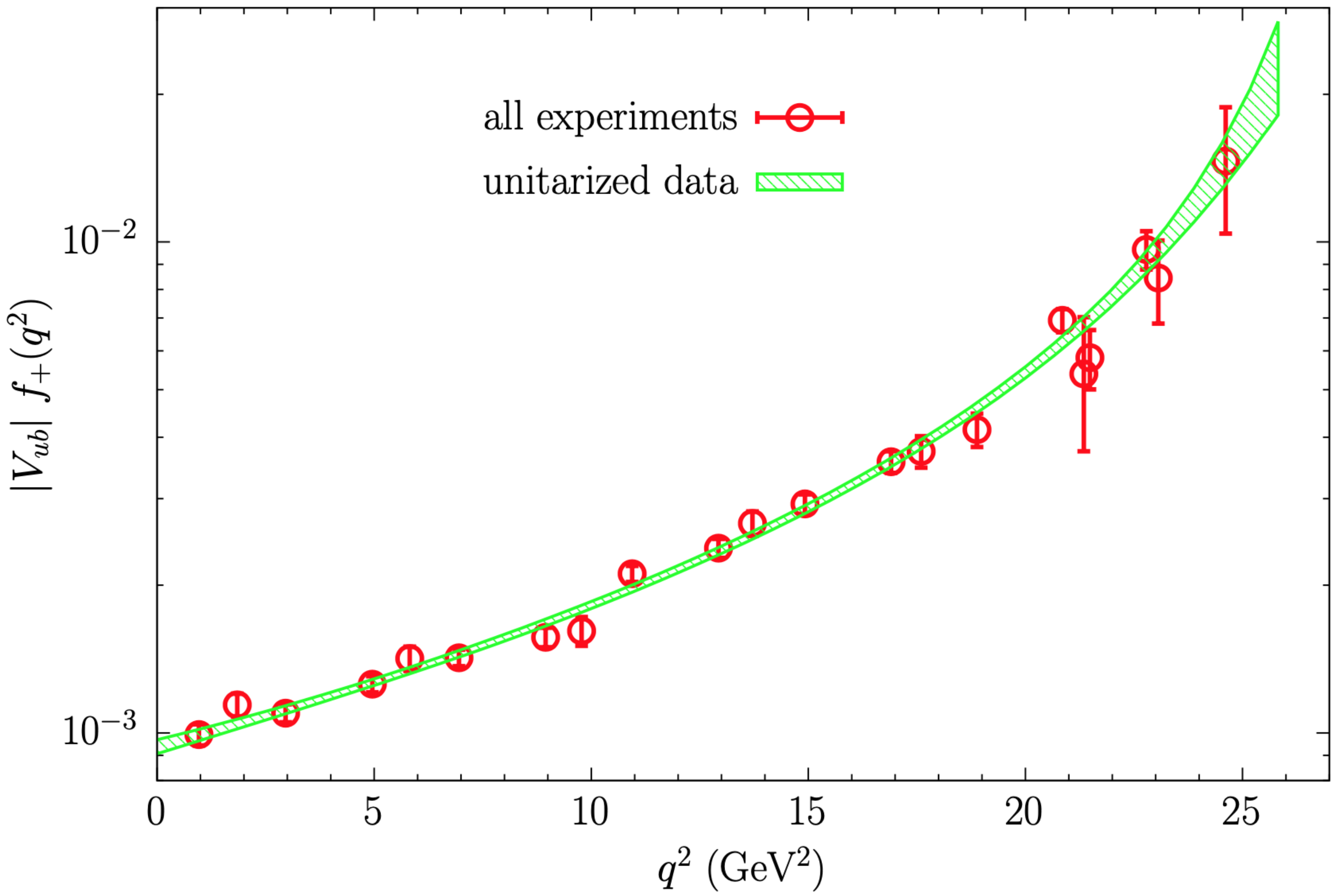
when averaged with the $B \to \pi$ result

$|V_{ub}| \cdot 10^3 = 3.62 \pm 0.47$

Improved determination of $|V_{ub}|$ from semileptonic $B \to \pi$ decays

$$|V_{ub} f_+(q_i^2)| = \sqrt{\frac{d\Gamma}{dq_i^2}\frac{1}{z_i}}$$

z_i = kinematical coefficient in the i-th bin



Unitarity bound $|V_{ub}|\chi_{1-}$ with an initial guess $|V_{ub}|$

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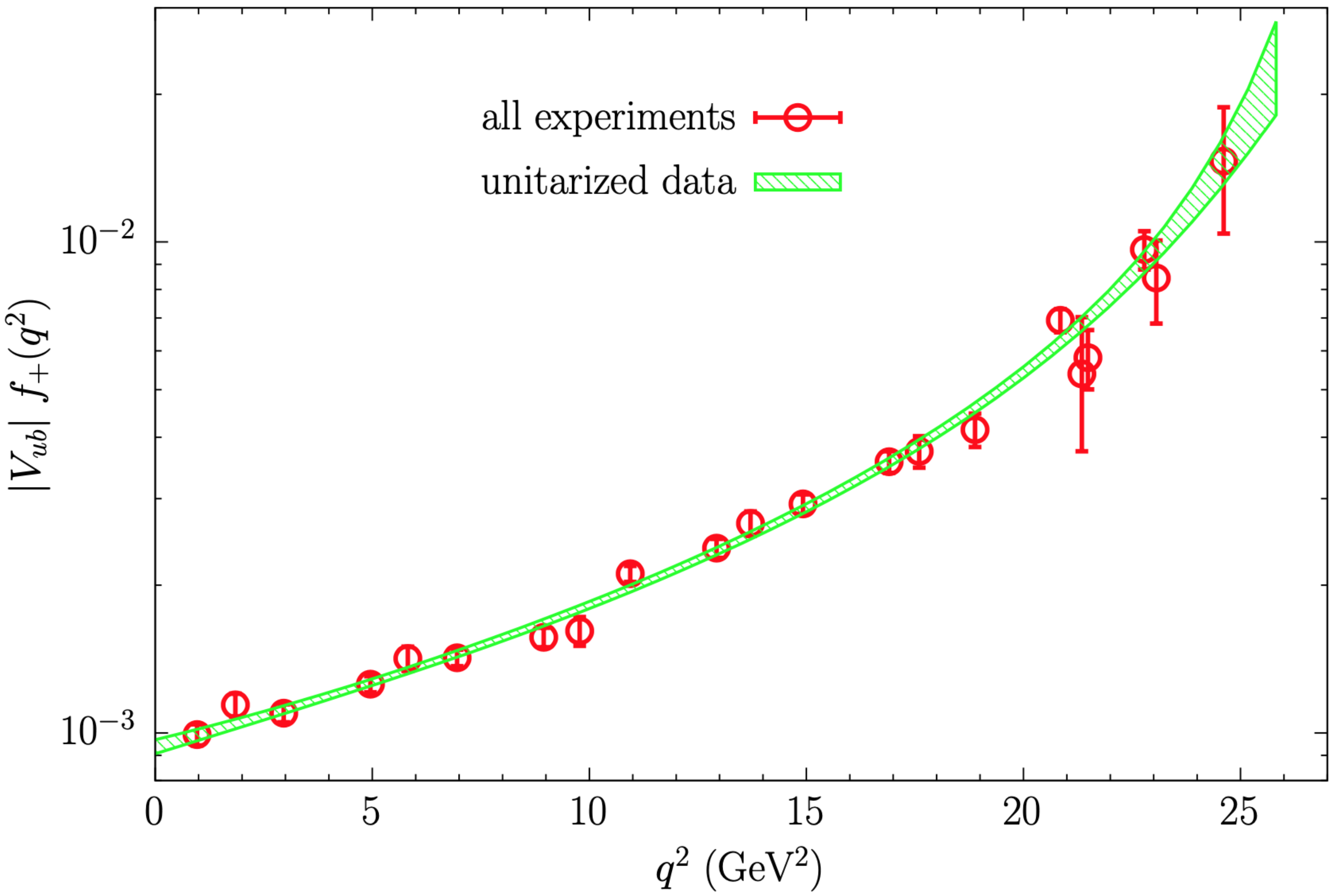

Unitarity bound $|V_{ub}|\chi_{1^-}$ with an initial guess $|V_{ub}|$

$|Vub|$ is then determined by using the theoretical unitary bands for $f_+(q^2)$ and by iterating the procedure until consistency for $|Vub|$ is reached:

$$|V_{ub}|^{\text{impr}}_{\text{B}\pi} \times 10^3 = 3.88 \pm 0.32$$

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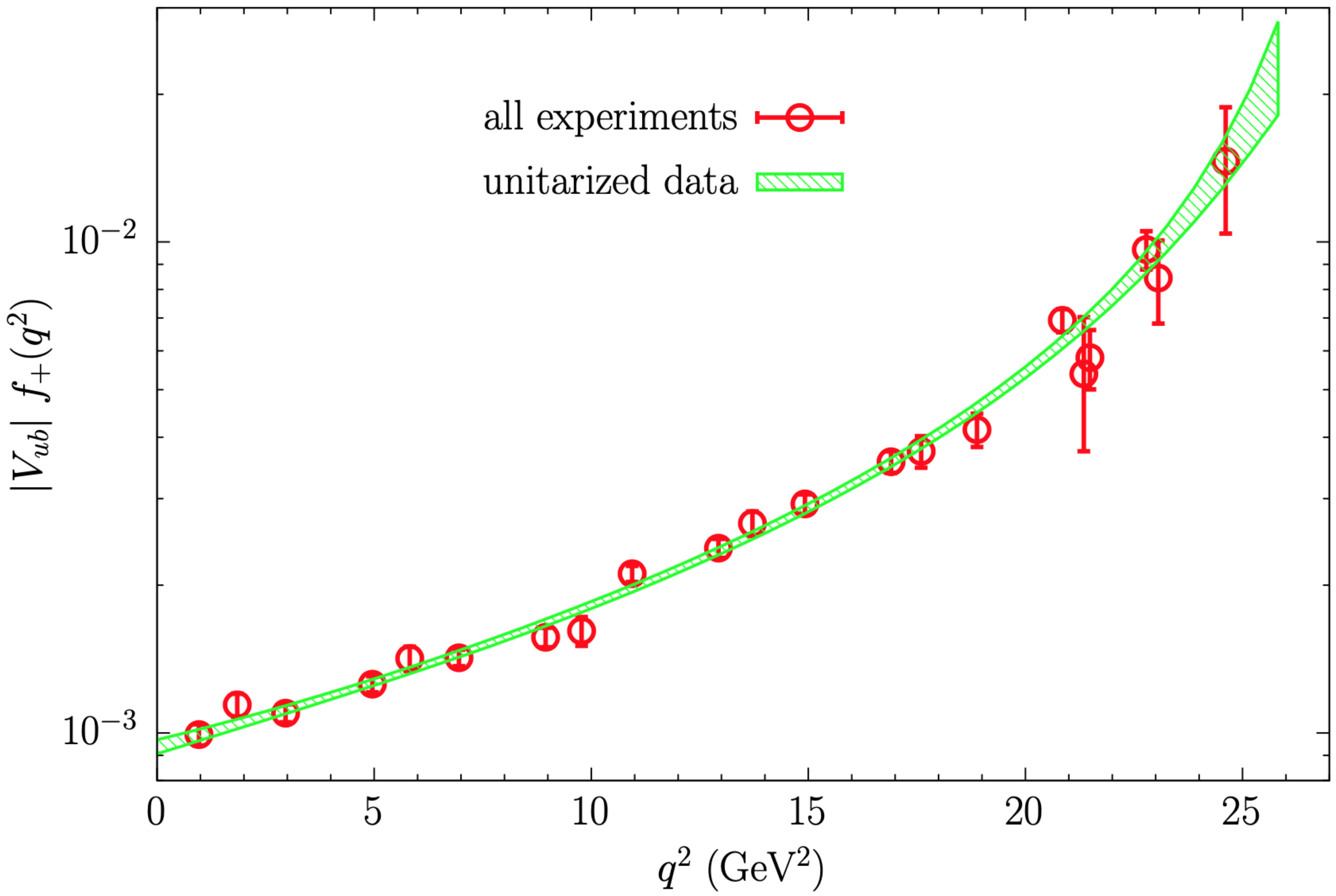

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Final DM Vub value:

$$|V_{ub}|^{\rm final}_{\rm DM} \times 10^3 = 3.85 \pm 0.27$$

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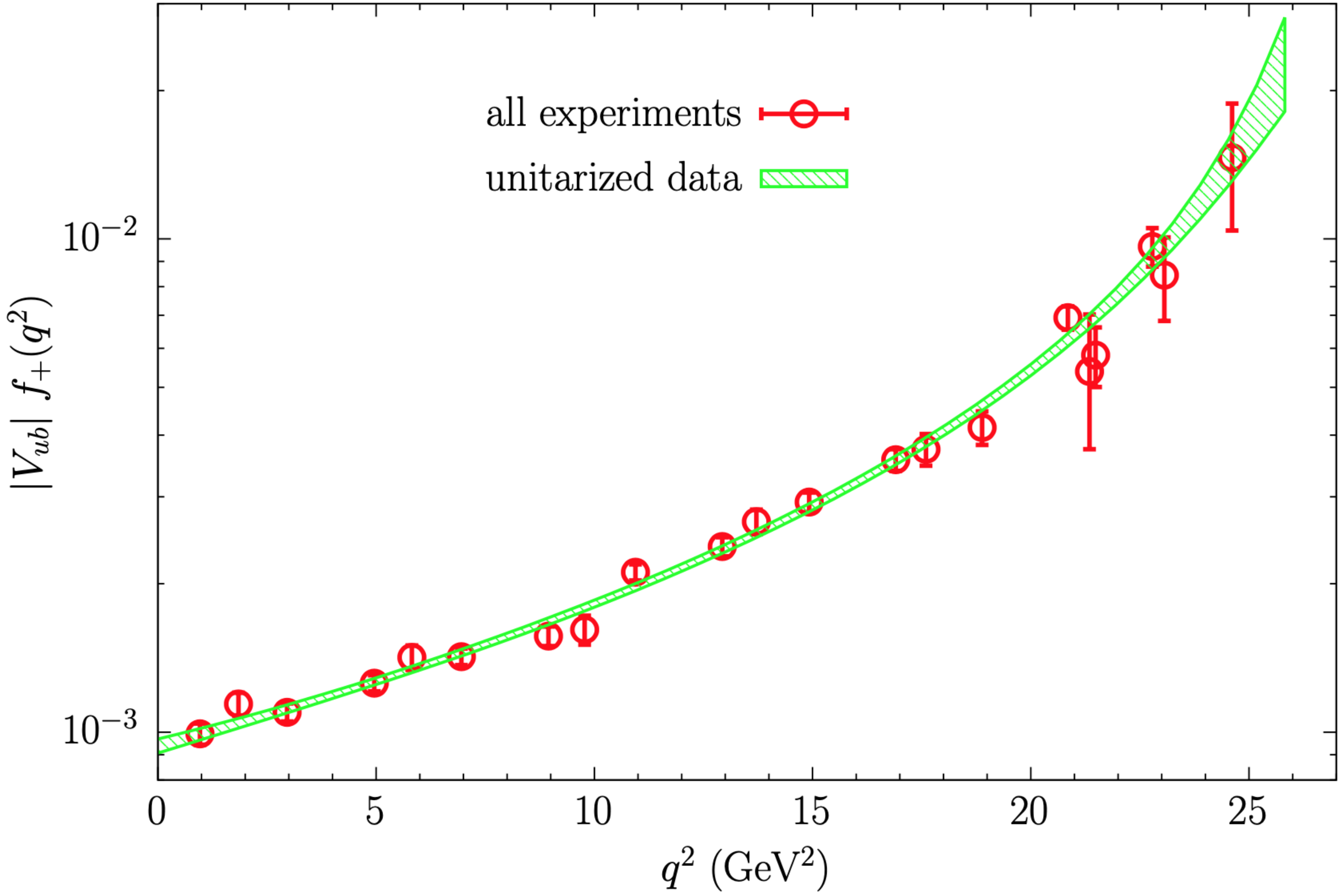

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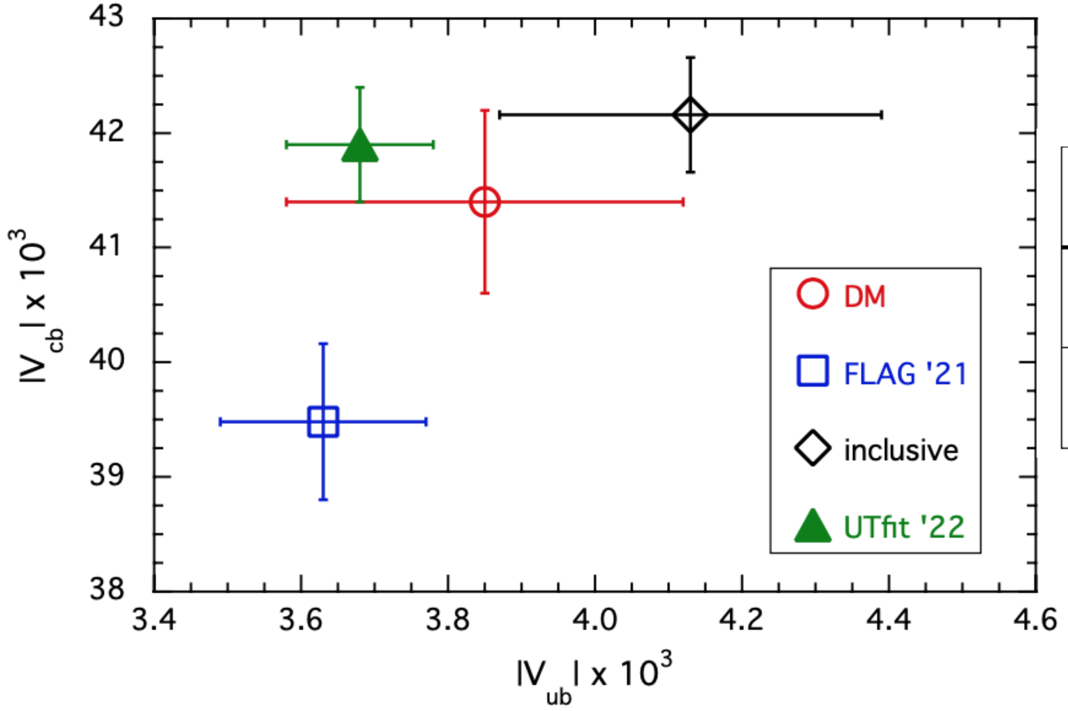
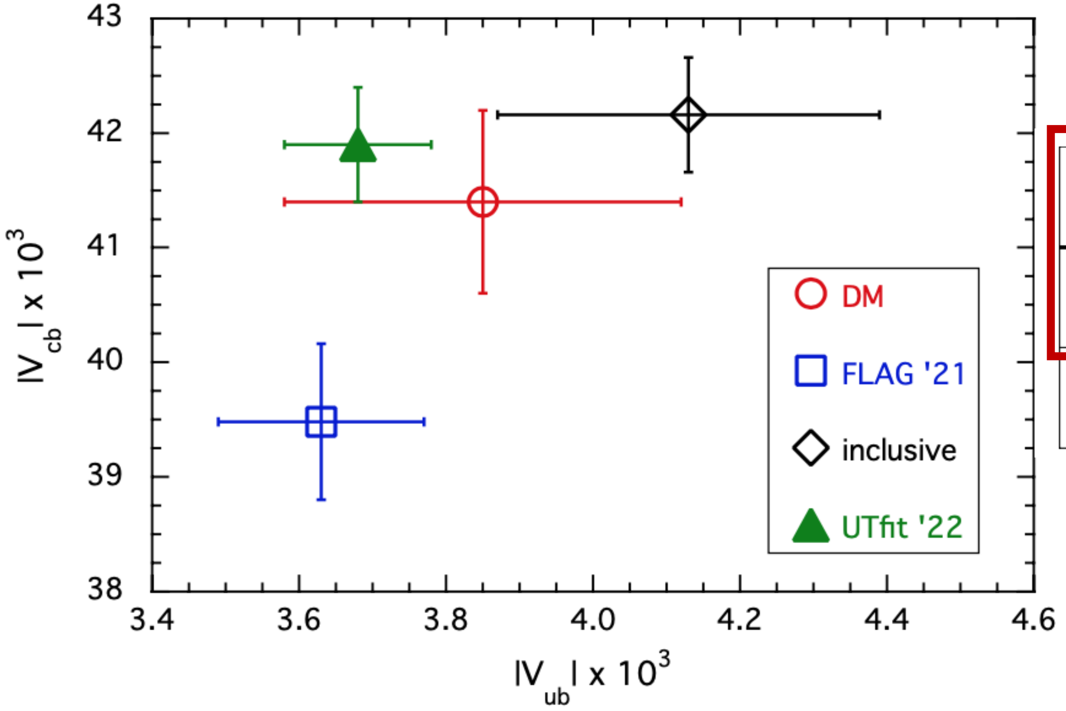
Important: *we still* ***keep separate the theoretical calculations and the experimental data*** *for describing the shape of the FFs!*

Summary plots/tables



	decays	DM	FLAG '21	inclusive
$\|V_{cb}\| \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.4 (8)	39.48 (68)	42.16 (50)
$\|V_{ub}\| \cdot 10^3$	$B_{(s)} \to \pi, K$	3.85 (27)	3.63 (14)	4.13 (26)

Summary plots/tables



See Silvano's talk for the details about the DM value of Vcb!

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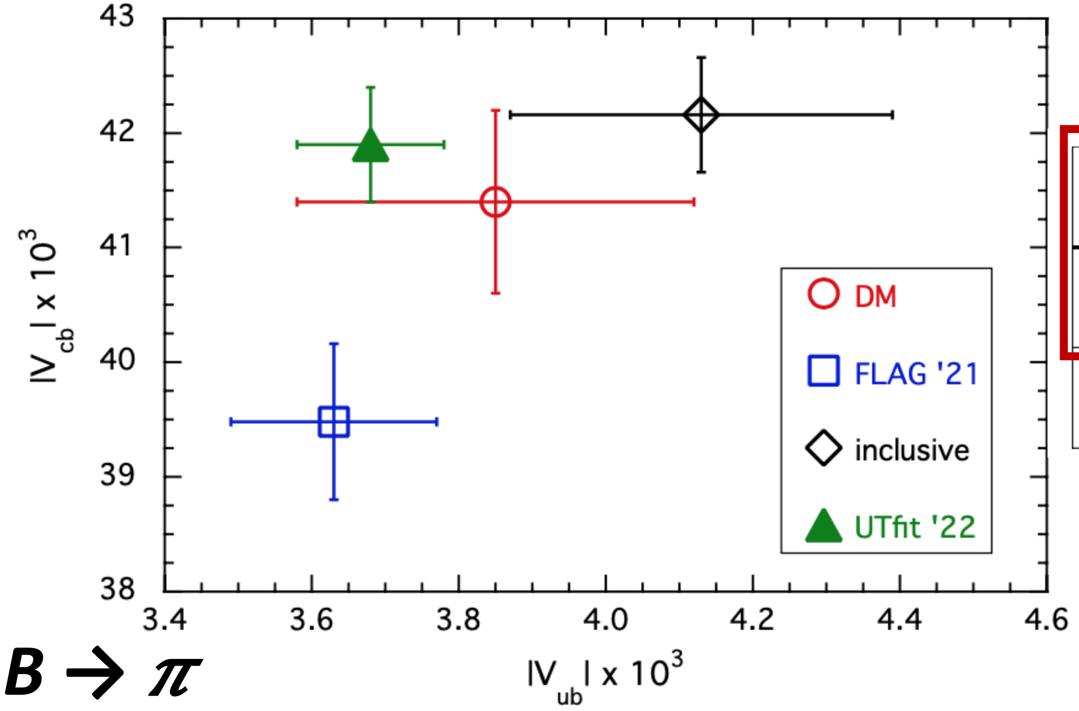

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$B \to \pi$

	RBC/UKQCD	FNAL/MILC	combined
$R_\pi^{\tau/\mu}$	0.767(145)	0.838(75)	0.793(118)
$\bar{A}_{FB}^{\mu,\pi}$	0.0043(39)	0.0018(14)	0.0034(31)
$\bar{A}_{FB}^{\tau,\pi}$	0.219(25)	0.221(19)	0.220(24)
$\bar{A}_{polar}^{\mu,\pi}$	0.985(11)	0.991(4)	0.988(9)
$\bar{A}_{polar}^{\tau,\pi}$	0.294(87)	0.309(82)	0.301(86)

$B_s \to K$

	RBC/UKQCD	FNAL/MILC	HPQCD	combined
$R_K^{\tau/\mu}$	0.845(122)	0.816(64)	0.680(134)	0.755(138)
$\bar{A}_{FB}^{\mu,K}$	0.0032(18)	0.0024(12)	0.0059(29)	0.0046(28)
$\bar{A}_{FB}^{\tau,K}$	0.257(14)	0.246(14)	0.278(19)	0.262(23)
$\bar{A}_{polar}^{\mu,K}$	0.990(5)	0.992(4)	0.982(8)	0.986(7)
$\bar{A}_{polar}^{\tau,K}$	0.172(54)	0.254(64)	0.112(79)	0.172(91)

THANKS FOR YOUR ATTENTION!

BACK-UP SLIDES

A methodological break: comparison with BGL/BCL

What is the **main improvement** with respect to BGL/BCL parametrization?

Boyd, Grinstein and Lebed, Phys. Lett. B353, 306 (19
Boyd, Grinstein and Lebed, Nucl. Phys. B461, 493 (19
Boyd, Grinstein and Lebed, Phys. Rev. D 56, 6895 (19

Basics of BGL: the hadronic FFs corresponding to definite spin-parity can be represented as an expansion, originating from unitarity, analyticity and crossing symmetry, in terms of the conformal variable *z*, for instance

$$g(z) = \frac{1}{\sqrt{\chi_{1^-}(q_0^2)}} \frac{1}{\phi_g(z, q_0^2)\, P_{1^-}(z)} \sum_{n=0}^{\infty} a_n\, z^n$$

Unitarity:

$$\sum_{n=0}^{\infty} a_n^2 \leq 1$$

Basics of BCL: similar to BGL, the expansion series has a simpler form, for instance

$$f_+(z) = \frac{1}{1 - q^2/m_{B^*}^2} \sum_{n=0}^{N_z - 1} a_k \left[z^n - (-1)^{n-N_z} \frac{n}{N_z} z^{N_z} \right],$$

$$f_0(z) = \sum_{n=0}^{N_z-1} b_k z^k.$$

Bourrely, Caprini and Lellouch, Phys. Rev. D 79, 013008 (2009

Unitarity:

$$\sum_{i,j=0}^{N_z} B_{mn}^{+} a_m a_n \leq 1, \qquad \sum_{i,j=0}^{N_z} B_{mn}^{0} b_m b_n \leq 1$$

LFU in semileptonic B → π decays

Fit	$N_z = 3$	$N_z = 4$	$N_z = 5$
χ^2/dof	2.5	0.64	0.73
dof	6	4	2
p	0.02	0.63	0.48
$\sum B^+_{mn} b^+_m b^+_n$	0.11(2)	0.016(5)	1.0(2.3)
$\sum B^0_{mn} b^0_m b^0_n$	0.33(8)	2.8(1.7)	8(19)
$f(0)$	0.00(4)	0.20(14)	0.36(27)
b^+_0	0.395(15)	0.407(15)	0.408(15)
b^+_1	−0.93(11)	−0.65(16)	−0.60(21)
b^+_2	−1.6(1)	−0.5(9)	−0.2(1.4)
b^+_3		0.4(1.3)	3(4)
b^+_4			5(5)
b^0_0	0.515(19)	0.507(22)	0.511(24)
b^0_1	−1.84(10)	−1.77(18)	−1.69(22)
b^0_2	−0.14(25)	1.3(8)	2(1)
b^0_3		4(1)	7(5)
b^0_4			3(9)

Table XIII
of ***arXiv:1503.07839***
(FNAL/MILC Coll.)

$$f^\pi(q^2 = 0)|_{\mathrm{RBC/UKQCD}} = -0.06 \pm 0.25$$

$$f^\pi(q^2 = 0)|_{\mathrm{FNAL/MILC}} = -0.01 \pm 0.16$$

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The DM approach is independent of this issue!!!

LFU in semileptonic $B \to \pi$ decays

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Table XIX
*of **arXiv:1501.05363***
(RBC/UKQCD Coll.)

		$f_+^{B\pi}$						$f_0^{B\pi}$						
K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	K	$b^{(0)}$	$b^{(1)}/b^{(0)}$	$b^{(2)}/b^{(0)}$	$b^{(3)}/b^{(0)}$	$\sum B_{mn}b_m b_n$	$f(q^2=0)$	χ^2/dof	p
1	0.447(36)				0.00394(63)							0.447(36)	4.02	2%
2	0.410(39)	-1.30(52)			0.0120(59)							0.241(83)	0.30	58%
3	0.420(43)	-1.46(59)	-4.7(7.2)		0.15(42)							0.07(32)		
						1	0.460(61)				0.0225(60)	0.460(61)	90.1	0%
						2	0.516(61)	-4.09(55)			0.408(63)	-0.074(73)	0.03	87%
						3	0.516(61)	-3.94(97)	0.7(3.8)		0.32(41)	-0.02(28)		
2	0.366(37)	-2.79(54)			0.0337(85)	2	0.587(58)	-3.33(38)			0.346(55)	0.040(65)	6.18	0%
3	0.427(40)	-1.62(46)	-7.7(1.5)		0.38(15)	2	0.521(60)	-4.03(52)			0.404(62)	-0.066(70)	0.10	91%
2	0.410(39)	-1.24(51)			0.0113(56)	3	0.520(60)	-3.12(42)	4.5(1.3)		0.41(17)	0.248(82)	0.58	56%
3	0.424(41)	-1.50(57)	-6.0(5.0)		0.24(38)	3	0.519(60)	-3.81(81)	1.2(3.4)		0.27(25)	0.01(24)	0.07	79%

LFU in semileptonic $B \to \pi$ decays

Same considerations developed for the FNAL/MILC case…

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Important issue: *the DM method equivalent to the results of* ***all*** *possible fits which satisfy unitarity and at the same time reproduce exactly the input data*

How to build up the ***combined*** case

FFs with mean values $x_i^{(k)}$ and uncertainties $\sigma_i^{(k)}$ $(k = 1, \cdots, N)$

Mean values and uncertainties of the ***new combined*** values

$$x_i = \sum_{k=1}^{N} \omega^{(k)} x_i^{(k)},$$

Weights

$$\sigma_i^2 = \sum_{k=1}^{N} \omega^{(k)} (\sigma_i^{(k)})^2 + \sum_{k=1}^{N} \omega^{(k)} (x_i^{(k)} - x_i)^2$$

$$\left[\sum_{k=1}^{N} \omega^{(k)} = 1 \right]$$

$$\omega^{(k)} = 1/N$$

Conservative choice in arXiv:2202.10285

Cov. Matrices of the k-th LQCD computation

Covariance matrix of the ***new combined*** values

$$C_{ij} \equiv \frac{1}{N} \sum_{k=1}^{N} C_{ij}^{(k)} + \frac{1}{N} \sum_{k=1}^{N} (x_i^{(k)} - x_i)(x_j^{(k)} - x_j)$$

The Dispersive Matrix (DM) method

Let us examine the case of the production of a pseudoscalar meson (as for the $B \to D$ case). Supposing to have *n* LQCD data for the FFs at the quadratic momenta $\{t_1, \cdots, t_n\}$ (hereafter $t \equiv q^2$), we define

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

$$\langle h_1 | h_2 \rangle = \int_{|z|=1} \frac{dz}{2\pi i z} \bar{h}_1(z) h_2(z)$$

$$g_t(z) \equiv \frac{1}{1 - \bar{z}(t) z}$$

CENTRAL REQUIREMENT:

$$\det \mathbf{M} \geq 0$$

The conformal variable z is related to the momentum transfer as:

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

Two advantages:

1. *z is real*
2. *1-to-1 correspondence:*

$[0, t_{max} = t_-] \Rightarrow [z_{max}, 0]$

A lot of work in the past:

L. Lellouch, NPB, 479 (1996), p. 353-391
C. Bourrely, B. Machet, and E. de Rafael, NPB, 189 (1981), pp.
E. de Rafael and J. Taron, PRD, 50 (1994), p. 373-380

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle \phi f|\phi f\rangle & \langle \phi f|g_t\rangle & \langle \phi f|g_{t_1}\rangle & \cdots & \langle \phi f|g_{t_n}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle \end{pmatrix}$$

We also have to define the kinematical functions

$$\phi_0(z,Q^2) = \sqrt{\frac{2n_I}{3}}\sqrt{\frac{3t_+t_-}{4\pi}\frac{1}{t_+-t_-}\frac{1+z}{(1-z)^{5/2}}}\left(\beta(0)+\frac{1+z}{1-z}\right)^{-2}\left(\beta(-Q^2)+\frac{1+z}{1-z}\right)^{-2},$$

$$\phi_+(z,Q^2) = \sqrt{\frac{2n_I}{3}}\sqrt{\frac{1}{\pi(t_+-t_-)}\frac{(1+z)^2}{(1-z)^{9/2}}}\left(\beta(0)+\frac{1+z}{1-z}\right)^{-2}\left(\beta(-Q^2)+\frac{1+z}{1-z}\right)^{-3}, \quad \beta(t)\equiv\sqrt{\frac{t_+-t}{t_+-t_-}}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, *@ $\{t_1,\dots,t_n\}$*: from Cauchy's theorem (for generic *m*)

$$\langle g_{t_m}|\phi f\rangle = \phi(t_m,Q^2)f(t_m) \qquad \langle g_{t_m}|g_{t_l}\rangle = \frac{1}{1-\bar{z}(t_l)z(t_m)}$$

LQCD data!

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \langle \phi f|\phi f\rangle$$

Since the susceptibilities are computed on the lattice, we can in principle use whatever value of Q^2 !

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle\phi f|\phi f\rangle & \langle\phi f|g_t\rangle & \langle\phi f|g_{t_1}\rangle & \cdots & \langle\phi f|g_{t_n}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle \end{pmatrix}$$

In the presence of **poles @** $t_{P1}, t_{P2}, \cdots ..., t_{PN}$:

$$\phi(z,q^2) \to \phi_P(z,q^2) \equiv \phi(z,q^2) \times \frac{z - z(t_{P1})}{1 - \bar{z}(t_{P1})z} \times \cdots \times \frac{z - z(t_{PN})}{1 - \bar{z}(t_{PN})z}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ $\{t_1,...,t_n\}$: from Cauchy's theorem (for generic *m*)

$$\langle g_{t_m}|\phi f\rangle = \phi(t_m, Q^2) f(t_m)$$

LQCD data!

$$\langle g_{t_m}|g_{t_l}\rangle = \frac{1}{1 - \bar{z}(t_l) z(t_m)}$$

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \langle\phi f|\phi f\rangle$$

The DM method

The positivity of the original inner products guarantee that $\det \mathbf{M} \geq 0$: the solution of this inequality can be computed analitically, bringing to

LOWER bound $$\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

UNITARITY FILTER: unitarity is satisfied if γ is semipositive definite, namely if

$$\chi \geq \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j}$$

This is a **parametrization-independent unitarity test** of the LQCD input data

Kinematical Constraints (KCs)

REMINDER: after the unitarity filter we were left with $N_U < N$ *survived events!!!*

Let us focus on the pseudoscalar case. Since by construction the following *kinematical constraint* holds

$$f_0(0) = f_+(0)$$

we will filter only the $N_{KC} < N_U$ events for which the two bands of the FFs intersect each other @ $t = 0$.
Namely, for each of these events we also define

$$\phi_{lo} = \max[F_{+,lo}(t=0), F_{0,lo}(t=0)]$$

$$\phi_{up} = \min[F_{+,up}(t=0), F_{0,up}(t=0)]$$

From WE theorem

$$\langle D(p_D)|\, V^\mu \,|B(p_B)\rangle \;=\; f_+(p_B+p_D)^\mu + f_-(p_B-p_D)^\mu$$

One then defines

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_B^2 - m_D^2} f_-(q^2)$$

$$\langle D(p_D)\,|\,V^\mu\,|\,B(p_B)\rangle = f^+(q^2)\left(p_B^\mu + p_D^\mu - \frac{m_B^2 - m_D^2}{q^2}q^\mu\right) + f^0(q^2)\frac{m_B^2 - m_D^2}{q^2}q^\mu$$

Kinematical Constraints (KCs)

We then consider a **modified matrix**

$$\mathbf{M_C} = \begin{pmatrix} \phi f|\phi f\rangle & \langle \phi f|g_t\rangle & \langle \phi f|g_{t_1}\rangle & \cdots & \langle \phi f|g_{t_n}\rangle & \langle \phi f|g_{t_{n+1}}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle & \langle g_t|g_{t_{n+1}}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle & \langle g_{t_1}|g_{t_{n+1}}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle & \langle g_{t_n}|g_{t_{n+1}}\rangle \\ \langle g_{t_{n+1}}|\phi f\rangle & \langle g_{t_{n+1}}|g_t\rangle & \langle g_{t_{n+1}}|g_{t_1}\rangle & \cdots & \langle g_{t_{n+1}}|g_{t_n}\rangle & \langle g_{t_{n+1}}|g_{t_{n+1}}\rangle \end{pmatrix}$$

with $t_{n+1} = 0$. Hence, we compute the new lower and upper bounds of the FFs in this way. For each of the N_{KC} events, we extract $N_{KC,2}$ values of $f_0(0) = f_+(0) \equiv f(0)$ with uniform distribution defined in the range $[\phi_{lo}, \phi_{up}]$. Thus, for both the FFs and for each of the N_{KC} events we define

$$\begin{aligned} F_{lo}(t) &= \min[F_{lo}^1(t), F_{lo}^2(t), \cdots, F_{lo}^{N_{KC,2}}(t)], \\ F_{up}(t) &= \max[F_{up}^1(t), F_{up}^2(t), \cdots, F_{up}^{N_{KC,2}}(t)] \end{aligned}$$

Non-perturbative computation of the susceptibilities

In **arXiv:2105.07851**, we have presented the results of the first computation on the lattice of the susceptibilities for the $b \to c$ quark transition, using the N_f=2+1+1 gauge ensembles generated by ETM Collaboration.

How are they defined? The starting point is the HVP tensor:

$$\Pi^V_{\mu\nu}(Q) = \int d^4x \; e^{-iQ\cdot x} \langle 0| T\left[\bar{b}(x)\gamma^E_\mu c(x)\; \bar{c}(0)\gamma^E_\nu b(0)\right]|0\rangle$$
$$= -Q_\mu Q_\nu \Pi_{0+}(Q^2) + (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi_{1-}(Q^2)$$

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0+}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0+}(t) \;, \qquad C_{0+}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0 c(x)\; \bar{c}(0)\gamma_0 b(0)\right]|0\rangle \;,$$

$$\chi_{1-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1-}(t) \qquad C_{1-}(t) = \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{b}(x)\gamma_j c(x)\; \bar{c}(0)\gamma_j b(0)\right]|0\rangle \;,$$

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$$\chi_{1+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1+}(t) \qquad C_{1+}(t) = \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{b}(x)\gamma_j\gamma_5 c(x)\; \bar{c}(0)\gamma_j\gamma_5 b(0)\right]|0\rangle$$

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$$= -Q_\mu Q_\nu \Pi_{0^+}(Q^2) + (\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi_{1^-}(Q^2)$$

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^+}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\; t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0^+}(t')\right]$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1^-}(t)$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^-}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\; t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b + m_c)^2 C_P(t') + Q^2 C_{0^-}(t')\right]$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4 \frac{j_1(Qt)}{Qt}\; C_{1^+}(t)$$

Non-perturbative computation of the susceptibilities

The possibility to compute the χs on the lattice allows us to choose *whatever value of Q^2 !!!!* (i.e. near the region of production of the resonances)

NOT POSSIBLE IN PERTURBATION THEORY!!!

$$(m_b + m_c)\Lambda_{QCD} << (m_b + m_c)^2 - q^2$$

POSSIBLE IMPROVEMENT IN THE STUDY OF THE FFs through our method!

`Work in progress…`

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0^+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^+}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^+}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\; t'^4\,\frac{j_1(Qt')}{Qt'}\left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0^+}(t')\right]$$

$$\chi_{1^-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4\frac{j_1(Qt)}{Qt}\; C_{1^-}(t)$$

$$\chi_{0^-}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0^-}(Q^2)\right] = \int_0^\infty dt\; t^2 j_0(Qt)\; C_{0^-}(t)\;, \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\; t'^4\,\frac{j_1(Qt')}{Qt'}\left[(m_b + m_c)^2 C_P(t') + Q^2 C_{0^-}(t')\right]$$

$$\chi_{1^+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1^+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt\; t^4\frac{j_1(Qt)}{Qt}\; C_{1^+}(t)$$

ETMC ratio method & final results

For the extrapolation to the physical *b*-quark point we have used the ETMC ratio method:

$$R_j(n;a^2,m_{ud}) \equiv \frac{\chi_j[m_h(n);a^2,m_{ud}]}{\chi_j[m_h(n-1);a^2,m_{ud}]} \boxed{\frac{\rho_j[m_h(n)]}{\rho_j[m_h(n-1)]}}$$

→ *to ensure that* $\lim_{n\to\infty} R_j(n) = 1$

$$\rho_{0^+}(m_h) = \rho_{0^-}(m_h) = 1 \ ,$$
$$\rho_{1^-}(m_h) = \rho_{1^+}(m_h) = (m_h^{pole})^2$$

All the details are deeply discussed in *arXiv:2105.07851*. In this way, we have obtained **the first lattice QCD determination of susceptibilities of heavy-to-heavy (and heavy-to-light) transition current densities:**

b → c

	Perturbative	With subtraction	Non-perturbative	With subtraction
$\chi_{V_L}[10^{-3}]$	6.204(81)	—	7.58(59)	—
$\chi_{A_L}[10^{-3}]$	24.1	19.4	25.8(1.7)	21.9(1.9)
$\chi_{V_T}[10^{-4}\ \text{GeV}^{-2}]$	6.486(48)	5.131(48)	6.72(41)	5.88(44)
$\chi_{A_T}[10^{-4}\ \text{GeV}^{-2}]$	3.894	—	4.69(30)	—

Bigi, Gambino PRD '16
Bigi, Gambino, Schacht PLB '17
Bigi, Gambino, Schacht JHEP '17

b → u

Non-perturbative	With subtraction
2.04(20)	—
2.34(13)	—
4.88(1.16)	4.45(1.16)
4.65(1.02)	—