Cosmological Relaxation from Dark Fermion Production

Ui Min (KAIST)

with Kenji Kadota, Fang Ye, Minho Son
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Outline

- Introduction
  1. GKR model
  2. Hook model
- Models
  1. Non-QCD model
  2. Double scanner model
- Prospect for dark matter
- Conclusion

Introduction

- Success of the SM: Discovery of Higgs in 2012
- Unsolved Problems in the SM
    1. Naturalness problem
    2. Dark matter (DM)
- Any Applications of Fermion Production?

Introduction

Naturalness Problem

- Hierarchy between the electroweak scale and Plank scale

$$v_{EW} \ll M_P$$

- Solutions

1. Symmetries: SUSY, Composite Higgs, etc.

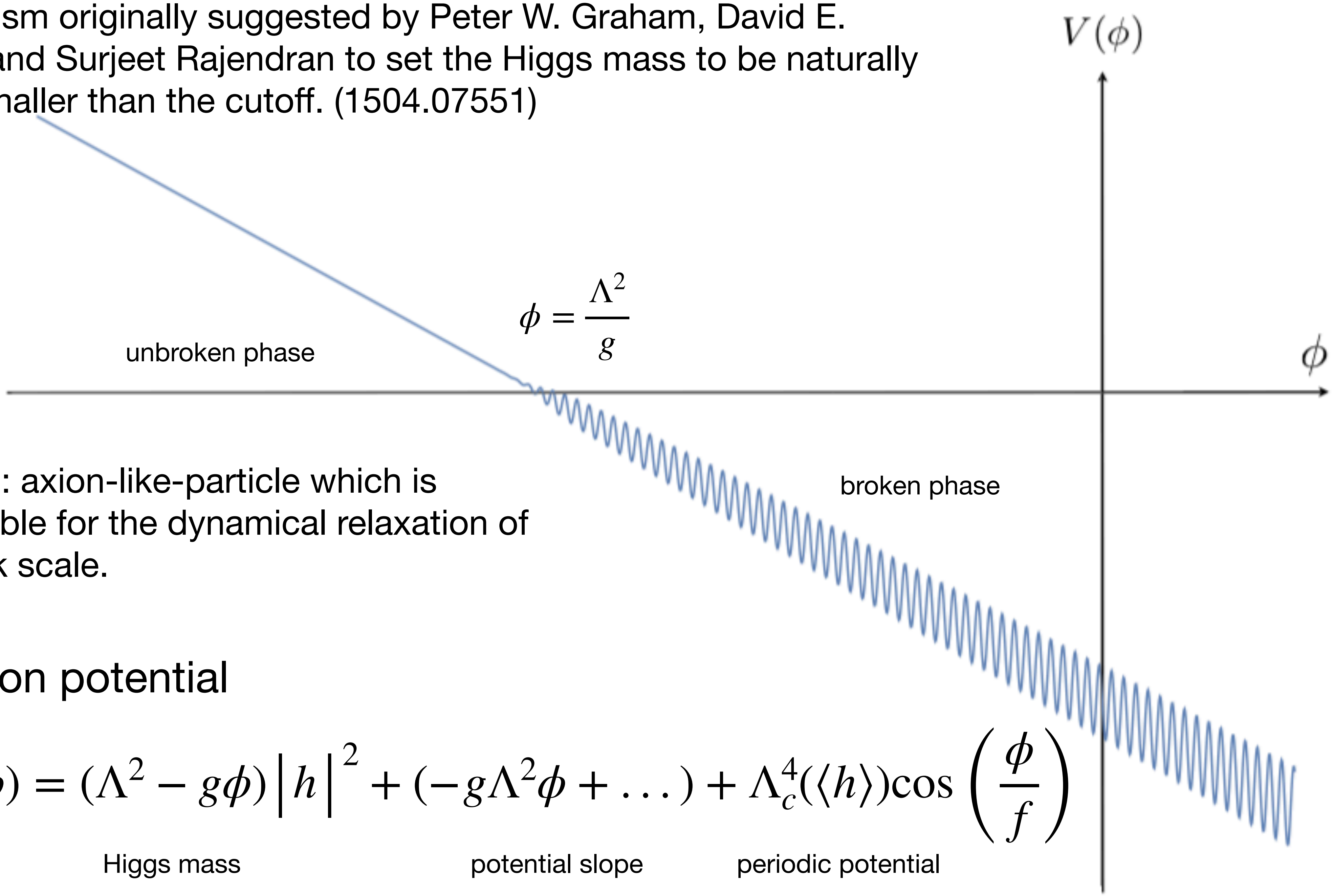
   new particles are expected to be observed at LHC.

2. Dynamics: relaxation

   no signal for new particles at LHC motivates such a scenario

Relaxation

Mechanism originally suggested by Peter W. Graham, David E. Kaplan and Surjeet Rajendran to set the Higgs mass to be naturally much smaller than the cutoff. (1504.07551)



Relaxion: axion-like-particle which is responsible for the dynamical relaxation of the weak scale.

- Relaxion potential

$$V(\phi) = (\Lambda^2 - g\phi)\left|h\right|^2 + (-g\Lambda^2\phi + \dots) + \Lambda_c^4(\langle h \rangle)\cos\left(\frac{\phi}{f}\right)$$

Higgs mass — potential slope — periodic potential

- Its shift symmetry is softly broken by a small dimensionful coupling to the Higgs.

GKR Model

$$V(\phi) = (\Lambda^2 - g\phi)\left|h\right|^2 + (-g\Lambda^2\phi + \ldots) + \Lambda_c^4(\langle h \rangle)\cos\left(\frac{\phi}{f}\right)$$

Conditions

- trapping: $g\Lambda^2 \leq \frac{\Lambda_c^4(v)}{f}$
- slow-roll: $3H\dot{\phi} + \frac{dV}{d\phi} \simeq 0 \quad \rightarrow \quad \dot{\phi} \sim \frac{g\Lambda^2}{H} \sim const$ Hubble expansion as major friction
- field excursion: $\Delta\phi \sim \mathcal{O}\left(\frac{\Lambda^2}{g}\right)$
- e-folding number: $\Delta\phi \sim \dot{\phi}\Delta t \sim \dot{\phi}\frac{N_e}{H} \sim \frac{g\Lambda^2}{H^2}N_e > \frac{\Lambda^2}{g} \quad \rightarrow \quad N_e > \frac{H^2}{g^2}$
- vacuum energy > relaxion energy: $H^2M_P^2 > \Lambda^4$

GKR Model

$$V(\phi) = (\Lambda^2 - g\phi)\left|h\right|^2 + (-g\Lambda^2\phi + \ldots) + \Lambda_c^4(\langle h \rangle)\cos\left(\frac{\phi}{f}\right)$$

Problems

- O(1) theta parameter: O(f) shift of the local minimum of the QCD relaxion

$$\delta\phi_{min} \sim \mathcal{O}(f) \quad \rightarrow \quad \theta_{QCD} \sim \mathcal{O}(1)$$

- After the relaxion is trapped into its local minimum, the reheating era starts. The reheating temperature should be low enough not to melt its potential barriers. Otherwise, that can lead to the second rolling.

GKR Model

Problems

- small soft-breaking coupling for the QDC relaxion: $g \sim 10^{-31} GeV$

  1. large e-folding number: $N_e \sim \frac{H^2}{g^2} \gg 100$
  2. super-Plankian field excursion: $\Delta\phi \sim \frac{\Lambda^2}{g} \gg M_P$

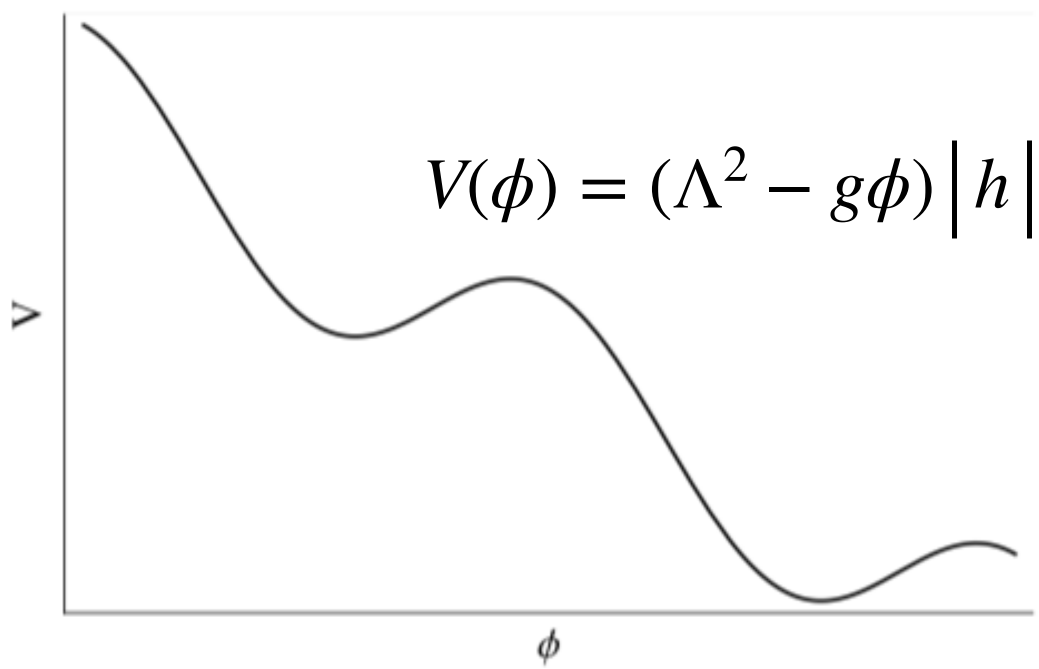
Solution

- Hubble friction is inefficient to dissipate the relaxion energy.
- we need a way for more efficient dissipation: particle production.

Hook Model

Differences from GKR model

- start from the broken phase, Higgs-independent potential barrier



$$V(\phi) = (\Lambda^2 - g\phi)\left|h\right|^2 + (g\Lambda^2\phi + \ldots) + \Lambda_c^4 \cos\left(\frac{\phi}{f}\right) + \frac{\phi}{f'}F\tilde{F}$$

suggested by Anson Hook and
Gustavo Marques-Tavares (1607.01786)

- exponential particle production: tachyonic mode of gauge bosons

$$\ddot{A}_\pm + \left(k^2 + m_A^2 \pm \frac{k\dot{\phi}}{f'}\right)A_\pm = 0 \quad \rightarrow \quad A_\pm \sim exp[i\omega_\pm t],\ \omega_\pm^2 = k^2 + m_A^2 \pm \frac{k\dot{\phi}}{f'}$$

Hook Model

$$V(\phi) = (\Lambda^2 - g\phi)\left|h\right|^2 + (g\Lambda^2\phi + \ldots) + \Lambda_c^4 \cos\left(\frac{\phi}{f}\right) + \frac{\phi}{f'}F\tilde{F}$$

- Exponential particle production is so strong that Hook model does not need slow-roll.

Suggestion!

- Is there any way to achieve slow-roll with particle production?
- Fermions cannot be produced exponentially due to Pauli-blocking.

  Peter Adshead,a Lauren Pearce,a Marco Peloso,b Michael A. Roberts,c Lorenzo Sorboc (1803.04501)

- Friction from fermion production may support slow-rolling of relaxion.

Goals

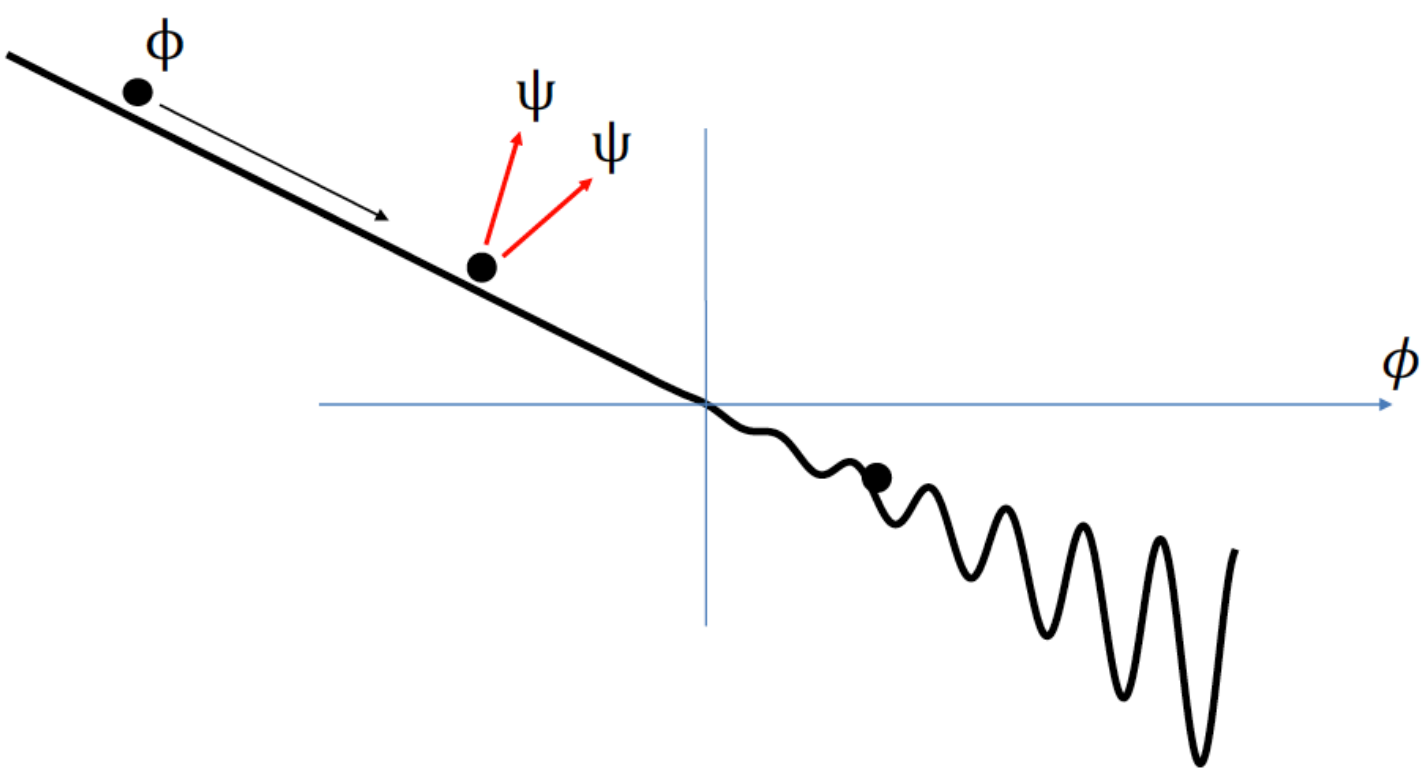
- Fermion production is major friction for slow-roll.
- To solve naturalness problem
- Not too small coupling
- To avoid super-Plankian excursion and large e-folding
- Reheating temperature not to melt the periodic potential

Model

- Fermion action with derivative coupling to axial current

$$\Delta S = \int d^4x\sqrt{-g}\left[\bar{\psi}\left(ie_a^{\mu}\gamma^a D_{\mu} - m_{\psi} - \frac{\partial_{\mu}\phi}{f_{\psi}}e_a^{\mu}\gamma^a\gamma^5\right)\psi\right]$$

with FRW metric for simplicity: $ds^2 = dt^2 - a^2\, dx^2 = a^2\left(d\tau^2 - dx^2\right)$



We expect the fermion production through the derivative coupling.

Fermion production as Drag force

Model

$$\Delta S = \int d^4x\sqrt{-g}\left[\bar{\psi}\left(ie_a^\mu\gamma^a D_\mu - m_\psi - \frac{\partial_\mu\phi}{f_\psi}e_a^\mu\gamma^a\gamma^5\right)\psi\right]$$

- with proper scaling and rotation for calculations,

$$\psi \to a^{-3/2}\psi, \quad \psi \to e^{-i\gamma^5\phi/f_\psi}\psi$$

- Fermionic occupation number and energy density are well-defined in this new basis without derivative coupling.

$$\mathscr{H} = \bar{\psi}\left(-i\gamma^i\nabla_i + m_R - im_I\gamma^5\right)\psi \qquad \Delta\mathscr{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_r + im_I\gamma^5\right)\psi$$

$$m_R = m_\psi a\cos\left(\frac{2\phi}{f_\psi}\right), \quad m_I = m_\psi a\sin\left(\frac{2\phi}{f_\psi}\right)$$

massless fermion cannot be produced from the relaxion rolling.

Model

$$V(\phi) = (-\Lambda^2 + g\phi)\left|h\right|^2 + (g\Lambda^2\phi + \dots) + \Lambda_c^4 \cos\left(\frac{\phi}{f}\right)$$

- Equation of motion and fermionic back-reaction

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi}(\phi) = \mathscr{B}, \quad \dot{\phi} \equiv \frac{\partial \phi}{\partial t}$$

$$\mathscr{B} = \frac{2m_\psi}{a^3 f}\langle\psi\left[\sin\left(\frac{2\phi}{f_\psi}\right) + i\gamma^5 \cos\left(\frac{2\phi_\psi}{f}\right)\right]\psi\rangle$$

- In the certain limit $\mu^2 \equiv \frac{m_\psi^2}{H^2} \ll \xi, \quad \xi \equiv \frac{\dot{\phi}}{2Hf_\psi} \gg 1,$

$$\mathscr{B} \sim -\frac{H^4\mu^2\xi^2}{f_\psi}$$

heavier fermions are more efficiently produced in this limit.
(this statement is no longer valid for too heavy fermion.)

Model

$$V(\phi) = (-\Lambda^2 + g\phi)\left|h\right|^2 + (g\Lambda^2\phi + \ldots) + \Lambda_c^4 \cos\left(\frac{\phi}{f}\right)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi}(\phi) = \mathscr{B}$$

- Strong back-reaction: fermion production is major friction.

$$\mathscr{B} > 3H\dot{\phi} \quad \rightarrow \quad m_\psi > 6\frac{Hf_\psi^{3/2}}{\Lambda\, g^{1/2}}$$

- Slow-roll condition: $\dfrac{\partial V}{\partial \phi} \simeq \mathscr{B} \quad \rightarrow \quad \dot{\phi} \sim \dfrac{g^{1/2}\Lambda f_\psi^{3/2}}{m_\psi} \sim const$

the size of fermionic back-reaction is determined by potential slope g.
(significant fermion production is related to not-too-small g.)

Relaxtion trapped at $g\Lambda^2 = \dfrac{\Lambda_c^4}{f}$

Model

- Constraints

1. strong back-reaction: $\mathscr{B} > 3H\dot{\phi} \quad \rightarrow \quad m_\psi > 6\frac{H f_\psi^{3/2}}{\Lambda\, g^{1/2}}$

2. EFT validity: $\dot{\phi} < \Lambda^2 \quad \rightarrow \quad m_\psi > 2\frac{g^{1/2} f_\psi^{3/2}}{\Lambda}$

3. fermion energy density < total energy density:

$$\rho_\psi \sim 16\pi^2 H^4 \mu^2 \xi^3 < H^2 M_P^2 \quad \rightarrow \quad m_\psi > \frac{\Lambda^3\, g^{3/2} f_\psi^{3/2}}{H^3 M_P^2}$$

4. relaxion kinetic energy density < total energy density:

$$\dot{\phi}^2 < H^2 M_P^2 \quad \rightarrow \quad m_\psi > 2\frac{\Lambda\, g^{1/2} f_\psi^{3/2}}{H M_P}$$

automatically satisfied by $\Lambda^4 < H^2 M_P^2$

Model

- Constraints

1. e-folding number: $\Delta\phi \sim \dot{\phi}\Delta t \sim \dot{\phi}\frac{N_e}{H} > \frac{\Lambda^2}{g} \quad \rightarrow \quad m_\psi < 2N_e\frac{g^{3/2}f_\psi^{3/2}}{H\Lambda}$

2. sub-Plankian field excursion: $M_P > \Delta\phi \quad \rightarrow \quad m_\psi > 2N_e\frac{\Lambda g^{1/2}f_\psi^{3/2}}{HM_P}$

3. classical rolling > quantum spreading:

$$\dot{\phi}\Delta t > H \quad \rightarrow \quad m_\psi < \frac{\Lambda\, g^{1/2}f_\psi^{3/2}}{H^2}$$

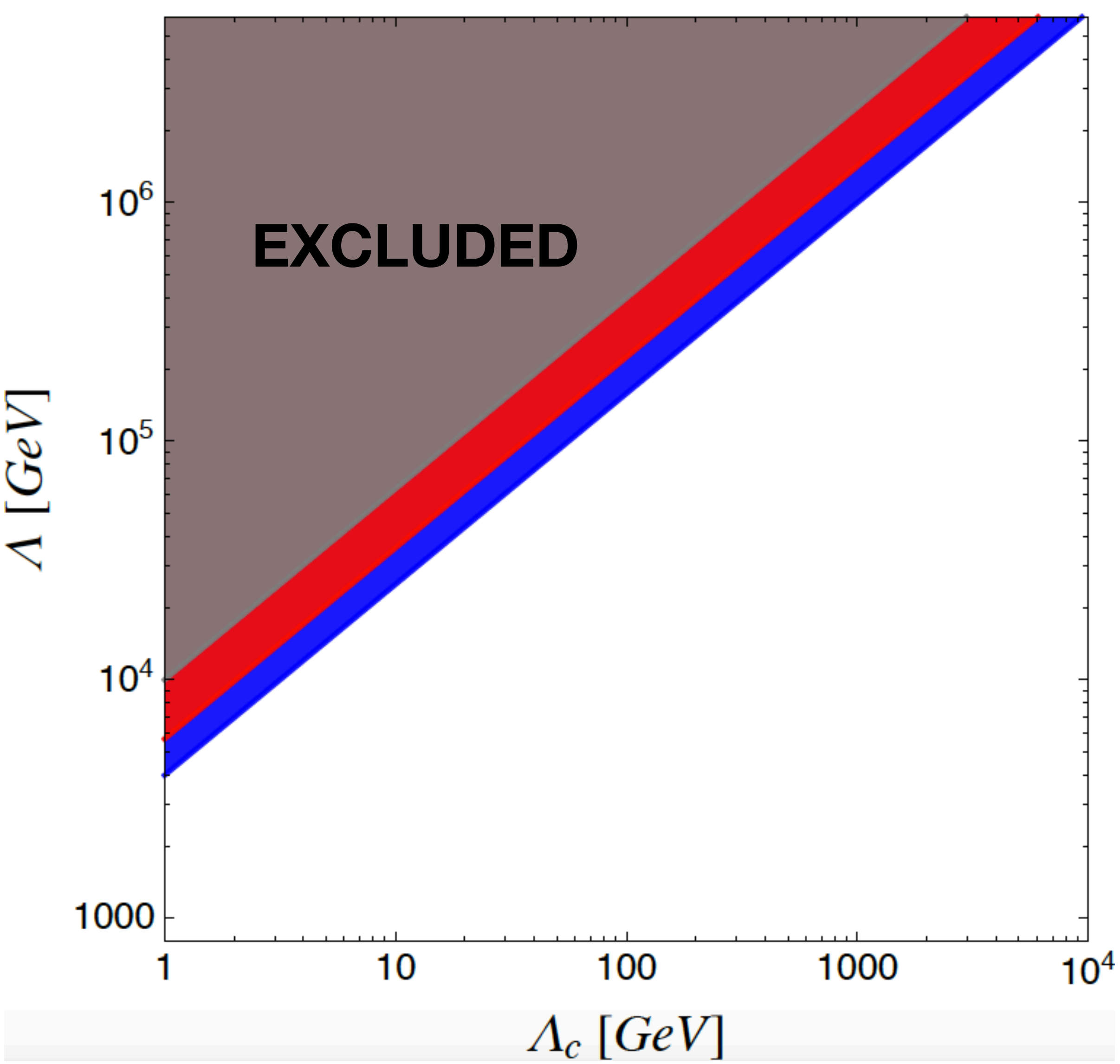
4. precision of Higgs mass: $\Delta m_h^2 \sim g\Delta\phi \sim 2\pi g f < m_h^2 \sim (100GeV)^2$

   to select the effective Higgs mass with the enough precision not to overshoot the electroweak scale.

5. T(SM) < Higgs VEV not to scan termal Higgs mass

Model

- By combining upper and lower bounds on fermion mass, we can obtain inequalities of cut-off scale.



$$\Lambda < \min\left[(N_e/3)^{1/10} M_p^{1/5} \Lambda_c^{4/5},\ (1/6)^{1/7} M_P^{3/7} \Lambda_c^{4/7},\ N_e^{1/5} M_P^{1/5} \Lambda_c^{4/5}\right]$$

with O(100)-size of e-folding number $N_e \sim 100$

for little hierarchy, the cut-off scale fixed to

$$\Lambda \sim 10^{4-5} GeV$$

Model

- Different new physics models predict different forms of the periodic potentials.

    model-dependent Λ_c

- with single scalar

    1. QCD relaxion model: strong CP problem, fixed barrier height

    2. non-QCD relaxion model: model-dependent barrier height

- with multiple scalars: double scanner model

    introducing extra scalar in addition to relaxion

Non-QCD Model (1504.07551)

- new massive fermions charged under new gauge group which gets strongly coupled in the low energy scale

$$\Delta\mathscr{L}_{non-QCD} = m_L L L^c + m_N N N^c + y h L N^c + \tilde{y} h^\dagger L^c N + (\phi/f) G' \tilde{G}'$$

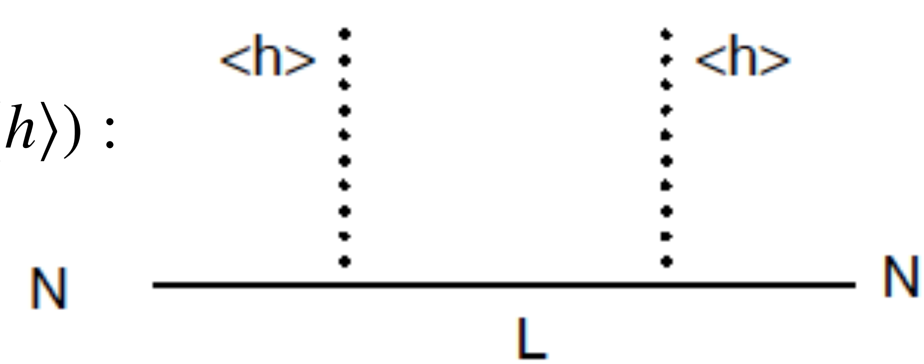
L: lepton doublet, N: right-handed neutrino with $m_L > v_{EW}, \quad m_L \gg f_{\pi'} \gg m_N$

$\Lambda \gg 4\pi f_{\pi'}$ to suppress Higgs-independent contribution to cosine potential

- Very light right-handed neutrino forms a condensate below the confinement scale.

$$m_N^{eff} e^{i\phi/f} N N^c + h.c. \quad \rightarrow \quad m_N^{eff} \langle N N^c \rangle \cos(\phi/f) \sim m_N^{eff} (4\pi f_{\pi'}^3) \cos(\phi/f)$$

$m_N(\langle h \rangle):$

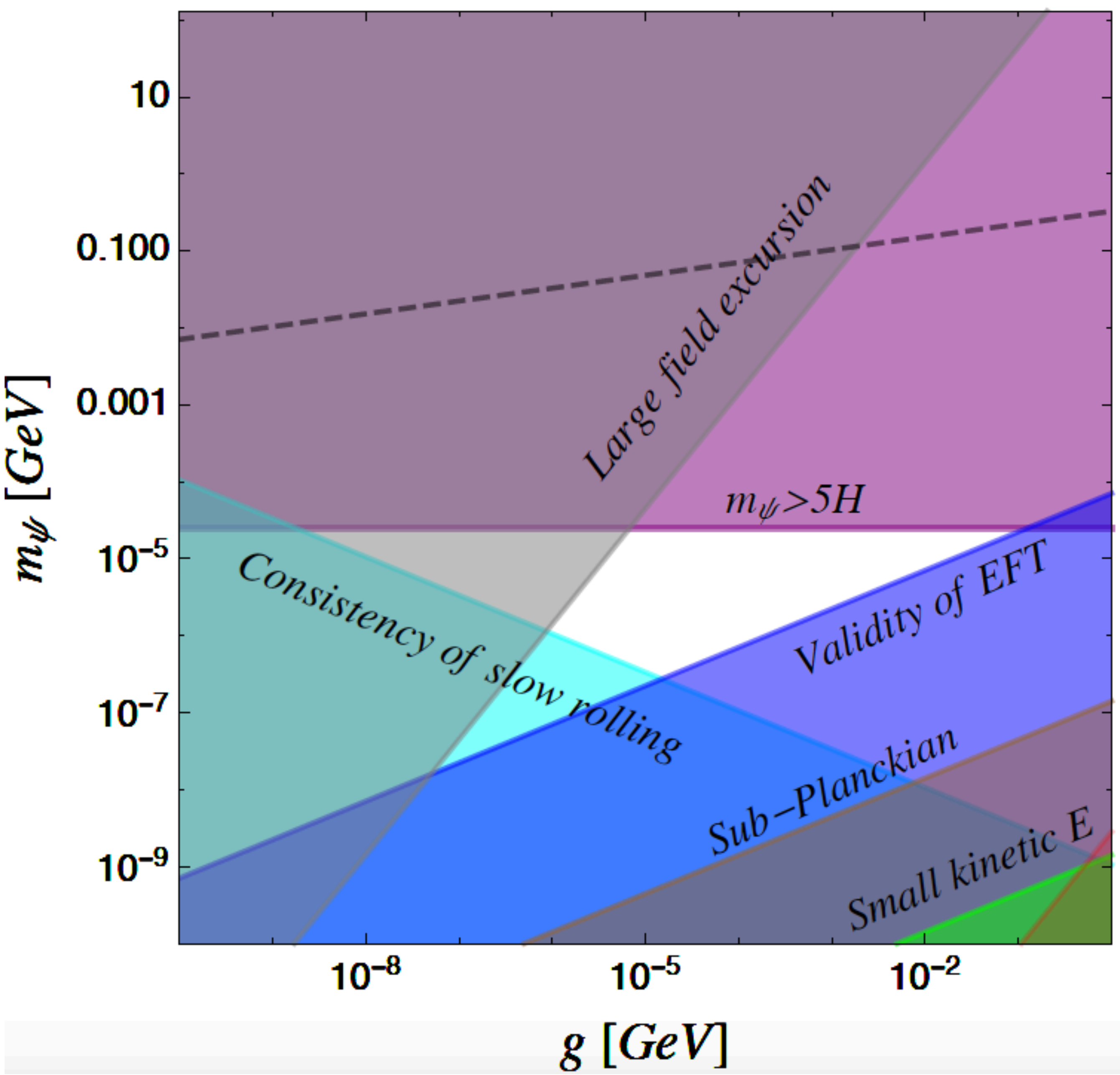


$$\Lambda_c^4 = 4\pi f_{\pi'}^3 m_N^{eff} \sim 4\pi f_{\pi'}^3 \frac{y\tilde{y}\langle h \rangle^2}{m_L}$$

Higgs-dependent potential barrier

Non-QCD Model

- Excluded region by constraints



Benchmark point

$$\Lambda = 10^4 GeV, \quad H = 5 \times 10^{-6} GeV$$

$$m_\phi = 5 \times 10^{-2} GeV, \quad f = 3.4 \times 10^4 GeV$$

$$m_\psi = 10^{-6} GeV, \quad f_\psi = 0.5 GeV$$

$$g = 10^{-6} GeV, \quad y\tilde{y} = 1.5 \times 10^{-2}$$

$$m_L = 300 GeV, \quad f_{\pi'} = 45 GeV$$

Non-QCD Model

- Problems

1. EFT inconsistency between two scales: $f_\psi \ll \Lambda$

2. reheating: the energy released from fermion production can be transferred to the visible sector, and then the barrier may disappear.

Fermion is quickly thermalized to achieve its thermal bath

Fermion thermalizes relaxion sector, then relaxion thermalizes SM sector.

The second rolling ruins relaxation mechanism.

solution: double scanner model.

Double Scanner Model

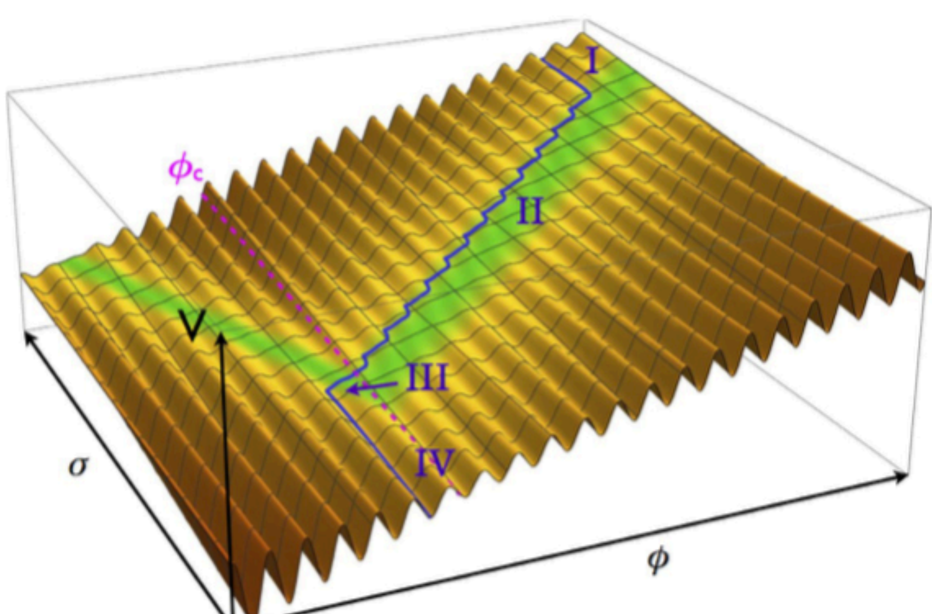
- Suggested by J.R. Espinosa, C. Grojean, G. Panico, A. Pomarol, O. Pujolas, G. Servant to solve the thermalization problem during reheating.

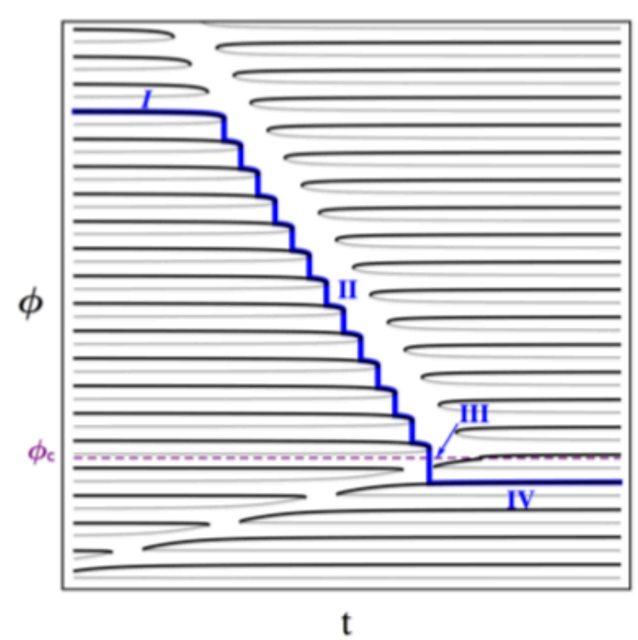
$$V(\phi, \sigma) = (-\Lambda^2 + g\phi)\left|h\right|^2 + (g\Lambda^2\phi + g_\sigma\Lambda^2\sigma) + A(\phi, \sigma, h)\ \cos(\phi/f)$$

effective Higgs mass   linear potential slope   periodic potential

confinement scale ~ cut-off scale: barrier height: $A(\phi, \sigma, h) \equiv \epsilon\Lambda^4\left(\beta + c_\phi\frac{g\phi}{\Lambda^2} - c_\sigma\frac{g_\sigma\sigma}{\Lambda} + \frac{|h|^2}{\Lambda^2}\right)$

- We add fermion sector: $\frac{\partial_\mu\phi}{f_\psi}\bar{\psi}\gamma^\mu\gamma^5\psi + \frac{\partial_\mu\sigma}{f_\sigma}\bar{\psi}\gamma^\mu\gamma^5\psi + m_\psi\bar{\psi}\psi$



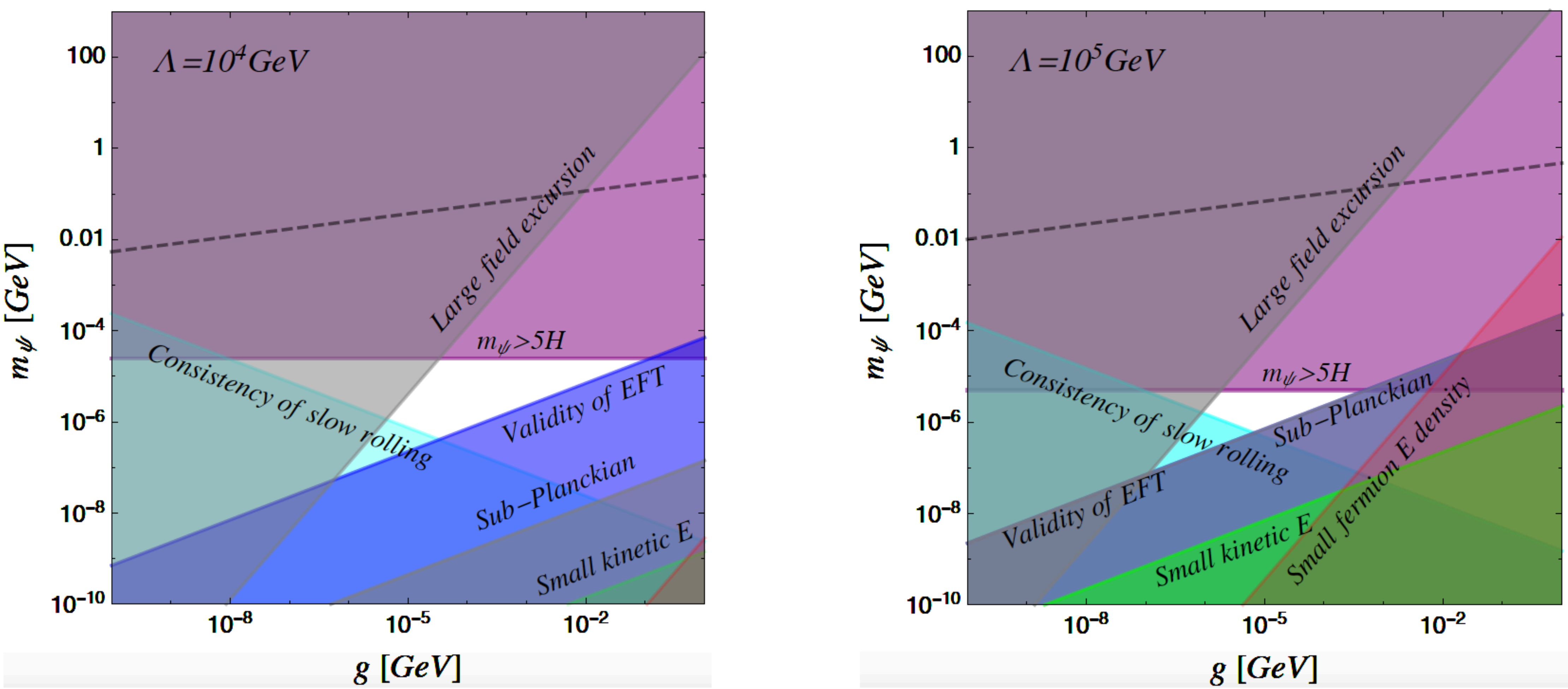


deep potential barriers form after step-4 (sigma stops its rolling)

Potential barriers are not erased during the reheating era due to the existence of extra scalar

Double Scanner Model

- Excluded region by constraints



Λ	H	m_ψ	$f_\psi \sim f_\sigma$	g	$g_\sigma(\sim m_\sigma)$	ϵ	f	m_ϕ
10^4	5×10^{-6}	$1. \times 10^{-6}$	0.5	$1. \times 10^{-5}$	$2. \times 10^{-6}$	$1. \times 10^{-5}$	6.1×10^4	5.2
10^5	1×10^{-6}	$1. \times 10^{-6}$	5	$1. \times 10^{-6}$	$2. \times 10^{-7}$	$2. \times 10^{-6}$	1.2×10^5	1.2×10^2

keV-scale fermion is allowed

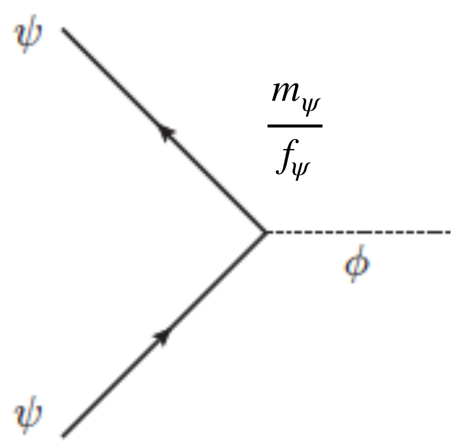
benchmark points in GeV except for epsilon

Prospect for DM

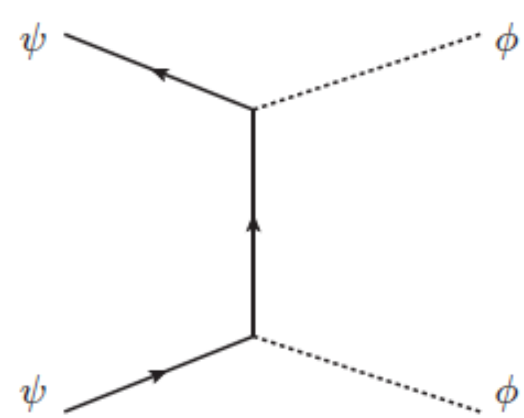
- with the benchmark point in the double scanner model,

$$m_\psi = 1keV, \quad m_\phi = 100GeV \qquad m_\phi^2 = \frac{\epsilon\Lambda^4}{f^2} \sim \frac{g\Lambda^4}{v^2 f}$$

- the relaxion can always decay into the fermion and anti-fermion before the Big Bang Nucleosynthesis.



$$\frac{\partial_\mu \phi}{f_\psi}\bar{\psi}\gamma^\mu\gamma^5\psi \quad \rightarrow \quad \Gamma_{\phi\to\psi\bar{\psi}} = \frac{1}{2\pi}\frac{m_\psi^2}{f_\psi^2}m_\phi\sqrt{1-\frac{4m_\psi^2}{m_\phi^2}}$$



$$\Gamma_{\psi\bar{\psi}\to\phi\phi} \sim \frac{m_\psi^2}{f_\psi^4}T^3, \quad T \gg m_\phi, m_\psi$$

Prospect for DM

- controlling non-universal couplings to scalars

$$f_\sigma = 5GeV \to \Lambda \sim 10^5 GeV, \qquad f_\psi = 5GeV$$

decoupling temperature of fermion: $T_d \sim m_\phi \sim 100GeV$
fermion decouples while it is highly relativistic.

slow-roll of sigma via Hubble friction while that of phi by fermion production

- Lyman-alpha constraint on thermal WDM mass

$$m_{WDM} \geq 5keV \times \left(\frac{g_{*S}(T_d) \sim \mathcal{O}(10^2)}{g_{*S}(T \ll MeV)} \right)^{-1/3} \sim 1keV$$

keV scale WDM is favored

- Fermion relic abundance

$$\Omega_0^\psi h^2 \sim \frac{\rho_\psi}{\rho_c} \sim 0.12 \left(\frac{m_\psi}{3 \times 10^{-3} keV} \right) \times \frac{g_{*S}(T_0)}{g_{*S}(T_d)} \times \frac{1}{S} \sim \mathcal{O}(10^{-1})$$

for keV scale fermion

entropy dilution

- WDM candidate?

Conclusion

- Cosmological relaxation through the fermion production can solve the naturalness problem.

- Fermion production is a more efficient friction source than the Hubble expansion.

    1. no extremely small g

    2. sub-Plankian field excursion, small e-folding number

- Good application of fermion production playing major role

- EFT inconsistency is still generic problem. $f_\psi \ll \Lambda$

Conclusion

- Cosmological relaxation through the fermion production can solve the naturalness problem.
- Fermion production is a more efficient friction source than the Hubble expansion.

    1. no extremely small g
    2. sub-Plankian field excursion, small e-folding number

- Good application of fermion production playing major role
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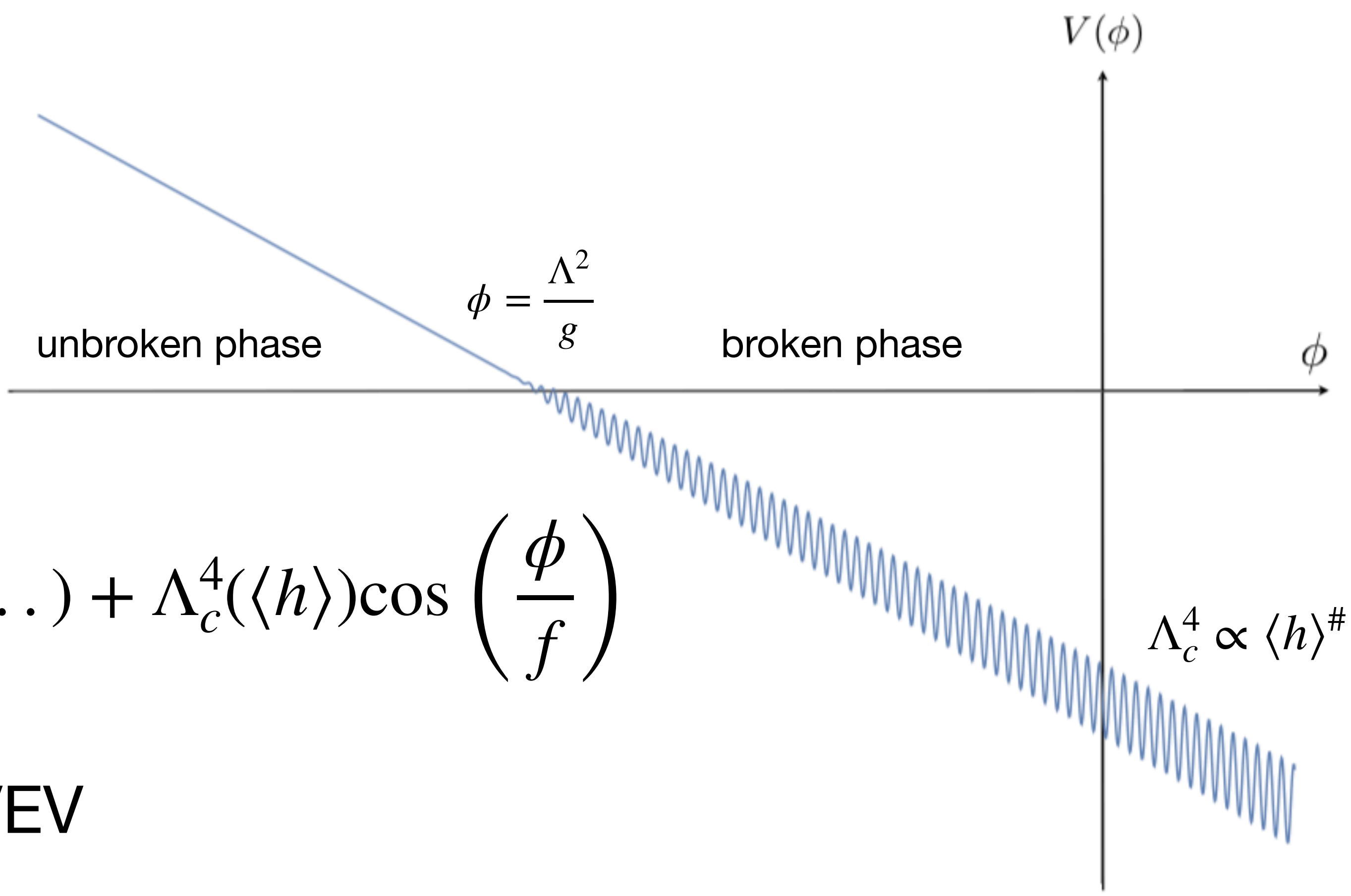
THANK YOU

Back-up slides

Relaxation



$$V(\phi) = (\Lambda^2 - g\phi)\left|h\right|^2 + (-g\Lambda^2\phi + \ldots) + \Lambda_c^4(\langle h \rangle)\cos\left(\frac{\phi}{f}\right)$$

squared Higgs mass and Higgs VEV

unbroken phase

$$\phi < \frac{\Lambda^2}{g} \quad \rightarrow \quad \mu^2 \equiv \Lambda^2 - g\phi > 0,\ \langle h \rangle = 0$$

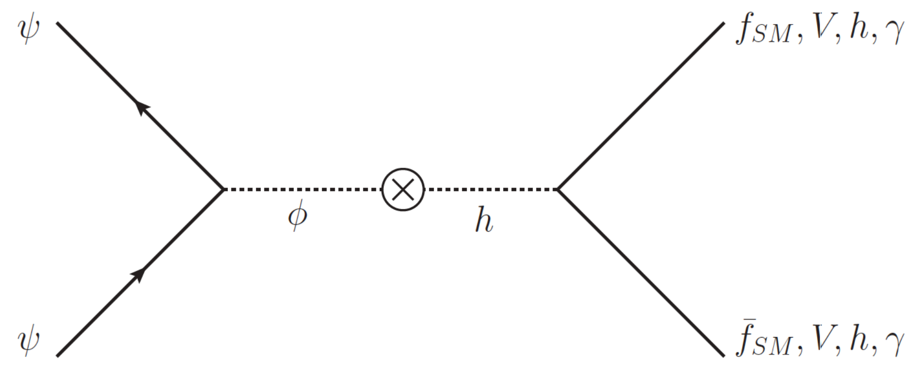
broken phase

$$\phi > \frac{\Lambda^2}{g} \quad \rightarrow \quad \mu^2 \equiv \Lambda^2 - g\phi < 0,\ \langle h \rangle \neq 0,\ \Lambda_c^4 \propto \langle h \rangle^{\#}$$

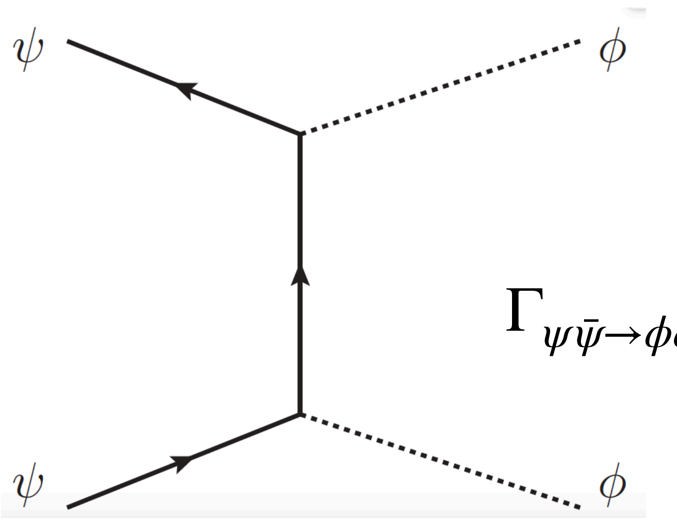
Model

- EFT inconsistency between two scales: $f_\psi \ll \Lambda$

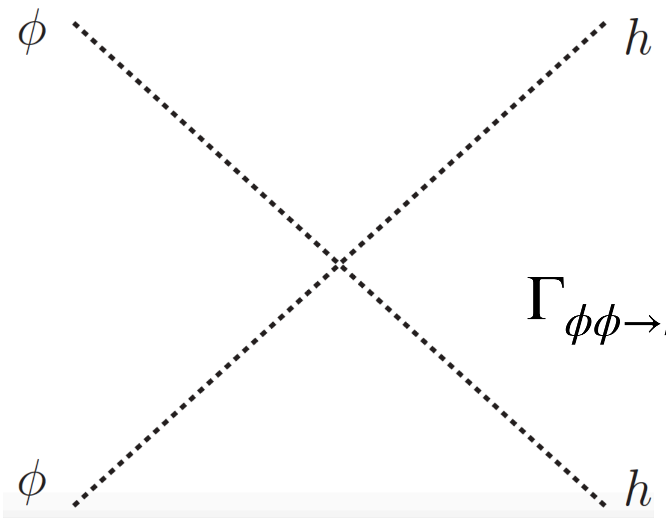
cannot form their own thermal baths respectively



$$\Gamma_{\psi\bar{\psi}\to sm} \sim g^2 v^2 \frac{m_\psi^2}{f_\psi^2} T^{-3} \ll H$$



$$\Gamma_{\psi\bar{\psi}\to\phi\phi} \sim \frac{m_\psi^2}{f_\psi^4} T^3 > H$$



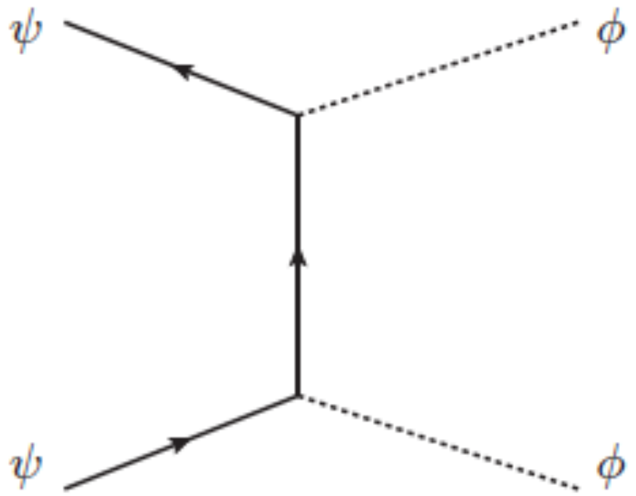
$$\Gamma_{\phi\phi\to hh} \sim \epsilon^2 \frac{\Lambda^4}{f^4} T > H$$

fermions thermalize relaxion, and then relaxtion thermalize Higgs successively.

fermion, relaxion, Higgs are in the same thermal bath.

Prospect for DM

$$m_\psi = 1keV, \quad m_\phi = 100GeV, \quad f_\psi = 5GeV, \quad g = 10^{-6}GeV$$



$$\Gamma_{\psi\bar{\psi}\to\phi\phi} \sim \frac{m_\psi^2}{f_\psi^4} T^3, \quad T \gg m_\phi, m_\psi$$

relaxions can thermally be produced from fermions as,

$$\Gamma_{\psi\bar{\psi}\to\phi\phi} \sim \frac{m_\psi^2}{f_\psi^4} T^3 > H \sim \frac{T^2}{M_P} \quad \to \quad T_d = 10^{-6}GeV \text{ valid only for } T \gg m_\phi, m_\psi$$

$$m_\psi \ll m_\phi \quad \to \quad T_d \sim m_\phi \sim 100GeV \text{ kinematically}$$

Fermion Production

Peter Adshead, Lauren Pearce, Marco Peloso (1803.04501)

Model with derivative coupling to fermion

$$\mathscr{L} = \sqrt{-g}\left[\bar{\psi}\left(ie^{\mu}{}_{a}\gamma^{a}D_{\mu} - m - \frac{1}{f}e^{\mu}{}_{a}\gamma^{a}\gamma^{5}\partial_{\mu}\phi\right)\psi + \frac{1}{2}(\partial_{\mu}\phi)^{2} - V(\phi)\right]$$

for FRW metric with conformal time : $ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2 = a^2\left(d\tau^2 - d\mathbf{x}^2\right)$

After rescaling the field, $\psi \to a^{-3/2}\psi$

$$\mathscr{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - ma - \frac{1}{f}\gamma^{\mu}\gamma^{5}\partial_{\mu}\phi\right)\psi + \frac{1}{2}a^{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - a^{4}V(\phi)$$

Field redefinition

derivative coupling

$$\mathscr{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - ma - \frac{1}{f}\gamma^\mu\gamma^5\partial_\mu\phi\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

Field redefinition or rotation : $\psi \to e^{-i\gamma^5\phi/f}\psi$

mass terms

$$\mathscr{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_R + i\,m_I\,\gamma^5\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

,where $m_R = ma\cos\left(\frac{2\phi}{f}\right)$, $m_I = ma\sin\left(\frac{2\phi}{f}\right)$

Hamiltonian Formalism

Conjugate momenta : $\Pi_\psi = i\psi^\dagger$, $\Pi_\phi = a^2\dot{\phi}$

$$\mathscr{H} = \bar{\psi}\left(-i\gamma^i\partial_i + m_R - i\,m_I\,\gamma^5\right)\psi + \frac{1}{2}a^2\dot{\phi}^2 + a^4V(\phi)$$

$= \mathscr{H}_\psi$: Hamiltonian density only for the fermion

quadratic order of psi

Field quantization

$$\psi = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{r=\pm} \left[U_r(\mathbf{k},\tau)a_r(\mathbf{k}) + V_r(-\mathbf{k},\tau)b_r^\dagger(-\mathbf{k})\right]$$

useful tensor decomposition :

$$U_r(\mathbf{k},\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix} u_r\chi_r \\ r\, v_r\, \chi_r \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} u_r \\ r\, v_r \end{pmatrix} \otimes \chi_r \equiv \underbrace{\xi_r(\mathbf{k},\tau)}_{\text{chiral part}} \otimes \underbrace{\chi_r(\mathbf{k})}_{\text{helicity part}}$$

Matrix diagonalization

$$\mathscr{H}_\psi = \sum_{r=\pm} \int d^3k \left(a_r^\dagger(\mathbf{k}), b_r(-\mathbf{k})\right) \begin{pmatrix} A_r(\tau) & B_r^*(\tau) \\ B_r(\tau) & -A_r(\tau) \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k}) \\ b_r^\dagger(-\mathbf{k}) \end{pmatrix}$$

$$A_r(\tau) = \frac{1}{2}\left[m_r\left(|u_r|^2 - |v_r|^2\right) + 2k\Re\left(u_r^* v_r\right) + 2rm_I\Im\left(u_r^* v_r\right)\right]$$

$$B_r(\tau) = \frac{re^{ir\theta_k}}{2}\left[2m_r u_r v_r - k\left(u_r^2 - v_r^2\right) - irm_I\left(u_r^2 + v_r^2\right)\right]$$

Bogolybov transformation with eigenvalues $\pm\omega, \quad \omega = \sqrt{k^2 + m^2 a^2}$

$$\mathscr{H}_\psi = \sum_{r=\pm} \int d^3k \left(a_r^\dagger(\mathbf{k},\tau), b_r(-\mathbf{k},\tau)\right) \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \begin{pmatrix} a_r(\mathbf{k},\tau) \\ b_r^\dagger(-\mathbf{k},\tau) \end{pmatrix}$$

absorbed time-dependency into the ladder operators

To define a vacuum state

A state which vanishes by annihilation operators at the initial time

$$a_r(k,\tau_0)\,|\,0\rangle = a_r(k)\,|\,0\rangle = 0$$

Fermion number density

defined by ladder operators :

$$n_{r,k}(\tau) = \langle 0\,|\,a_r^\dagger(k,\tau)\,a_r(k,\tau)\,|\,0\rangle$$

$$= \frac{1}{2} - \frac{m_R}{4\omega}\left(|\,u_r|^2 - |\,v_r|^2\right) - \frac{k}{2\omega}\mathfrak{R}\left(u_r^* v_R\right) - \frac{r m_I}{2\omega}\mathfrak{I}\left(u_r^* v_r\right)$$

$$n_{r,k}(\tau_0) = \langle 0\,|\,a_r^\dagger(k,\tau_0)\,a_r(k,\tau_0)\,|\,0\rangle = \langle 0\,|\,a_r^\dagger(k)\,a_r(k)\,|\,0\rangle = 0$$

Without Field Rotation

$$\mathscr{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - ma - \frac{1}{f}\gamma^\mu\gamma^5\partial_\mu\phi\right)\psi + \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - a^4V(\phi)$$

Hamiltonian Formalism

Conjugate momenta : $\Pi_\psi = i\psi^\dagger$, $\Pi_\phi = a^2\dot{\phi} - \frac{1}{f}\bar{\psi}\gamma^0\gamma^5\psi$

$$\mathscr{H} = \bar{\psi}\left(-i\gamma^i\partial_i + ma + \frac{1}{f}\gamma^0\gamma^5\dot{\phi}\right)\psi - \frac{1}{2a^2}\frac{\left(\bar{\psi}\gamma^0\gamma^5\psi\right)^2}{f^2} + \frac{1}{2a^2}\Pi_\phi^2 + a^4V(\phi)$$

$= \mathscr{H}_\psi$: Hamiltonian density only for the fermion

quadratic and quartic orders of psi

we cannot use Bogolybov transformation.

undefined number density, energy density