Perturbative Methods in Holography vs. Supersymmetric Localization

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Introduction

- This talk will be about a recent progress on supergravity computation, to be compared with large-N result of supersymmetric localization for gauge theory, related via AdS/CFT.
- We will address examples in various dimensions: from $D = 3$ to $D = 5$ mass-deformed CFTs.
- It is about relations between multi-variable integrals (CFT) and solutions of non-linear ODEs (AdS) through holography.

Plan

1. AdS4 : Mass deformation of ABJM to $\mathcal{N} = 2$
2. AdS5 : Mass deformation of $\mathcal{N} = 4$ SYM to $\mathcal{N} = 2^*$ and $\mathcal{N} = 1^*$
3. AdS6 : Mass deformation of Brandhuber-Oz theory

Holography of ABJM with mass deformation

Localization: the case of S^3

- For concreteness and simplicity let us first deal with the path integral for ABJM model.
- To recall, it is the theory on M2-branes and dual to M-theory in $AdS_4 \times S^7$ background.
- Chern-Simons-matter theory with gauge group $U(N) \times U(N)$, CS levels $(k, -k)$ and quartic superpotential for 4 bi-fundamental chiral multiplets.
- $k \neq 1$ leads to orbifolding: $S^7/\mathbb{Z}_k$ and susy from $\mathcal{N} = 8$ to $\mathcal{N} = 6$.
- S^3 localization for partition function and Wilson loops developed by Kapustin, Willett, Yaakov (2009) and later generalized to less susy or different backgrounds such as squashed sphere Jafferis (2010) Hama, Hosomichi, Lee (2011), general $U(1)$ fibration over Riemann surface etc Closset, Kim, Willett (2017).

ABJM partition function

- Z as a function of N_1, N_2, k and an ordinary integral over eigenvalues.

$$Z_{ABJM} = \frac{1}{N_1!N_2!}\int \prod_i^{N_1} \frac{d\mu_i}{2\pi} \prod_j^{N_2} \frac{d\nu_j}{2\pi} e^{\frac{ki}{4\pi}(\sum \mu_i^2 - \sum \nu_j^2)} \frac{\prod_{i<j}(2\sinh(\mu_i-\mu_j))^2 \prod_{i<j}(2\sinh(\nu_i-\nu_j))^2}{\prod_{i,j}(2\cosh(\mu_i-\nu_j)/2)^2}$$

- Drukker, Marino, Putrov (2010,2011): The integral at hand is related to Lens space matrix model whose exact solution is already known: Shown that $F \equiv -\log|Z| \sim k^{1/2}N^{3/2}$. (free energy)
- For more general cases (e.g. with less susy), one can employ the matrix model technique developed by Herzog, Klebanov, Pufu, Tesileanu (2010) and others including Martelli, Sparks, Cheon, Kim, Kim, Jafferis, Klebanov, Pufu, Safdi (2011)

F-maximization

- More precisely, the action depends on the R-charge assignments of chiral multiplets which then affects the free energy.
- The correct free energy is obtained via F-maximization, as a function of R-charge Δ.
- For ABJM, $F = \frac{4\sqrt{2}\pi N^{3/2}}{3}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$ with constraint $\sum \Delta_i = 2$.
- Question: Can we extend the correspondence for general Δ?
- This involves non-conformal holography, since there are Δ-dependent mass terms in field theory action.

Sugra dual of ABJM on S^3

- Euclidean action with 3 complex scalars and BPS equations obtained from $D = 4, \mathcal{N} = 8$ sugra. Freedman, Pufu (2013)

$$S = \frac{1}{8\pi G_4} \int d^4x \sqrt{g} \left[-\frac{1}{2} R + \sum_{\alpha=1}^{3} \frac{\partial_\mu z^\alpha \partial^\mu \tilde{z}^\alpha}{(1 - z^\alpha \tilde{z}^\alpha)^2} + \frac{1}{L^2} \left(3 - \sum_{\alpha=1}^{3} \frac{2}{1 - z^\alpha \tilde{z}^\alpha} \right) \right]$$

- Metric in conformal gauge $ds^2 = e^{2A}(dr^2/r^2 + ds^2(S^3))$

$$\begin{aligned} r(1 + \tilde{z}^1 \tilde{z}^2 \tilde{z}^3) z^{\alpha\prime} &= (\pm 1 - rA')(1 - z^\alpha \tilde{z}^\alpha) \left(z^\alpha + \tfrac{\tilde{z}^1 \tilde{z}^2 \tilde{z}^3}{\tilde{z}^\alpha} \right), \\ r(1 + z^1 z^2 z^3) \tilde{z}^{\alpha\prime} &= (\mp 1 - rA')(1 - z^\alpha \tilde{z}^\alpha) \left(\tilde{z}^\alpha + \tfrac{z^1 z^2 z^3}{z^\alpha} \right), \\ -1 &= -r^2 (A')^2 + e^{2A} \tfrac{(1 + z^1 z^2 z^3)(1 + \tilde{z}^1 \tilde{z}^2 \tilde{z}^3)}{\prod_{\beta=1}^{3} (1 - z^\beta \tilde{z}^\beta)}. \end{aligned}$$

Exact solutions and holographic free energy

- Scalars $z^\alpha(r) = c_\alpha f(r), \quad \tilde{z}^\alpha(r) = -\frac{c_1 c_2 c_3}{c_\alpha} f(r)$, with $f(r) = \frac{1-r^2}{1+c_1 c_2 c_3 r^2}$
- Metric $e^{2A} = \frac{4r^2(1+c_1 c_2 c_3)(1+c_1 c_2 c_3 r^4)}{(1-r^2)^2(1+c_1 c_2 c_3 r^2)^2}$
- 3 integration constants, eventually related to Δ_i.
- To evaluate holographic free energy, one evaluates on-shell action, add Gibbons-Hawking and counterterms, and through Legendre transformation w.r.t. UV asymptotics coefficients: the result matches with the field theory.
- Freedman and Pufu just presented the solution. Is there a general method to tackle similar problems?
- Idea: Treat c_α as small parameters and solve perturbatively!

Solving Perturbatively

- Introduce an expansion parameter ϵ and write

$$z^\alpha(r) = \sum_{k=1}^{\infty} \epsilon^k z_k^\alpha(r)\,, \quad \tilde{z}^\alpha(r) = \sum_{k=1}^{\infty} \epsilon^k \tilde{z}_k^\alpha(r)\,,$$
$$e^{2A(r)} = \frac{4r^2}{(1-r^2)^2}\left(1 + \sum_{k=1}^{\infty} \epsilon^k a_k(r)\right)\,.$$

- Then at leading order, a_1 should be zero (due to regularity) and

$$z_1^{\alpha\prime} + \frac{2r}{1-r^2} z_1^\alpha = 0, \qquad \tilde{z}_1^{\alpha\prime} + \frac{2}{r(1-r^2)}\tilde{z}_1^\alpha = 0.$$

- $z_1^\alpha = c(1-r^2), \tilde{z}_1^\alpha = \tilde{c}(1-r^{-2})$. We want regular solutions, so should set $\tilde{c} = 0$.
- Continuing this, one can construct the solutions found by Freedman and Pufu.

Holography of Mass-deformed $\mathcal{N} = 4$ SYM

Mass-deformed $\mathcal{N} = 4$ super Yang-Mills

- $\mathcal{N} = 4$ SYM with (susy-compatible) mass terms for adjoint chiral multiplets are called $\mathcal{N} = 2^*$ or $\mathcal{N} = 1^*$ theories.
- For $\mathcal{N} = 2^*$ localization formula is available and the large-N limit gives Pestun (2007), Buchel, Russo, Zarembo (2013)

$$\frac{d^3 F_{S^4}}{d(ma)^3} = -2N^2 \frac{ma(m^2 a^2 + 3)}{(m^2 a^2 + 1)^2}$$

- The supergravity dual of $\mathcal{N} = 2^*$ on S^4 was constructed by Bobev, Elvang, Freedman, Pufu (2013) considering a subsector of $\mathcal{N} = 8, D = 5$ gauged supergravity.

BPS system for sugra dual of $\mathcal{N} = 2^*$ SYM

- Action

$$L = \frac{1}{16\pi G_5}\left[-R + \frac{\partial_\mu \eta \partial^\mu \eta}{\eta^2} + \frac{4\partial_\mu z \partial^\mu \tilde{z}}{(1 - z\tilde{z})^2} + V\right]$$

$$V = -\frac{4}{L^2}\left(\frac{1}{\eta^4} + 2\eta^2 \frac{1 + z\tilde{z}}{1 - z\tilde{z}} + \frac{\eta^8}{4}\frac{(z - \tilde{z})^2}{(1 - z\tilde{z})^2}\right)$$

- BPS equations

$$z' = \frac{3\eta'(z\tilde{z} - 1)\left[2(z + \tilde{z}) + \eta^6(z - \tilde{z})\right]}{2\eta\left[\eta^6(\tilde{z}^2 - 1) + \tilde{z}^2 + 1\right]}$$

$$\tilde{z}' = \frac{3\eta'(z\tilde{z} - 1)\left[2(z + \tilde{z}) - \eta^6(z - \tilde{z})\right]}{2\eta\left[\eta^6(z^2 - 1) + z^2 + 1\right]}$$

$$(\eta')^2 = \frac{\left[\eta^6(z^2 - 1) + z^2 + 1\right]\left[\eta^6(\tilde{z}^2 - 1) + \tilde{z}^2 + 1\right]}{9\eta^2(z\tilde{z} - 1)^2}$$

UV asymptotics

- Unlike Mass-deformed ABJM, exact solutions are not available.
- One may resort to numerical solutions.
- Holographic renormalization requires UV expansion. Metric in Fefferman-Graham coordinates $ds^2 = d\rho^2/\rho^2 + e^{2f(\rho)} ds^2_{S^4}$ (BPS solutions, but not regular for general μ, v)

$$
\begin{aligned}
e^{2f} &= \frac{1}{4\rho^2} + \frac{1}{6}(\mu^2 - 3) + \mathcal{O}(\rho^2 \log^2 \rho) \\
\eta &= 1 + \rho^2 \left[-\frac{2\mu^2}{3} \log \rho + \frac{\mu(\mu + v)}{3} \right] + \mathcal{O}(\rho^4 \log^2 \rho) \\
(z + \tilde{z})/2 &= \rho^2(-2\mu \log \rho + v) + \mathcal{O}(\rho^4 \log^2 \rho) \\
(z - \tilde{z})/2 &= \mp \mu\rho \mp \rho^3 \left[-\frac{4}{3}\mu(\mu^2 - 3) \log z + \frac{1}{3}\left(2v(\mu^2 - 3) + \mu(4\mu^2 - 3) \right) \right] \\
&\quad + \mathcal{O}(\rho^5 \log^2 \rho)
\end{aligned}
$$

Holographic renormalization

- Following usual procedure of considering bulk action, Gibbons-Hawking term, and adding counterterms, one obtains that the scheme-independent part is

$$\frac{d^3 F}{d\mu^3} = -N^2 v''(\mu)$$

- Here the UV expansion of BPS equations do NOT fix a relation between μ, v. It is determined as one imposes regularity at IR ($r = 0$).
- The numerical results suggest $v(\mu) = -2\mu - \mu \log(1 - \mu^2)$, which is consistent with localization formula. Bobev et al. (2013)

Applying perturbative approach

- We expand scalar/warp factor functions

$$z(r) = \sum_{k=1}^{\infty} \epsilon^k z_k(r), \quad \tilde{z}(r) = \sum_{k=1}^{\infty} \epsilon^k \tilde{z}_k(r),$$
$$\eta(r) = \sum_{k=2}^{\infty} \epsilon^k \eta_k(r),$$
$$e^{A(r)} = \frac{2r}{1-r^2}\left(1 + \sum_{k=2}^{\infty} \epsilon^k a_k(r)\right).$$

- At leading order we have a coupled 1st order ODE of $z_1, \tilde{z}_1$ which can be easily solved to give

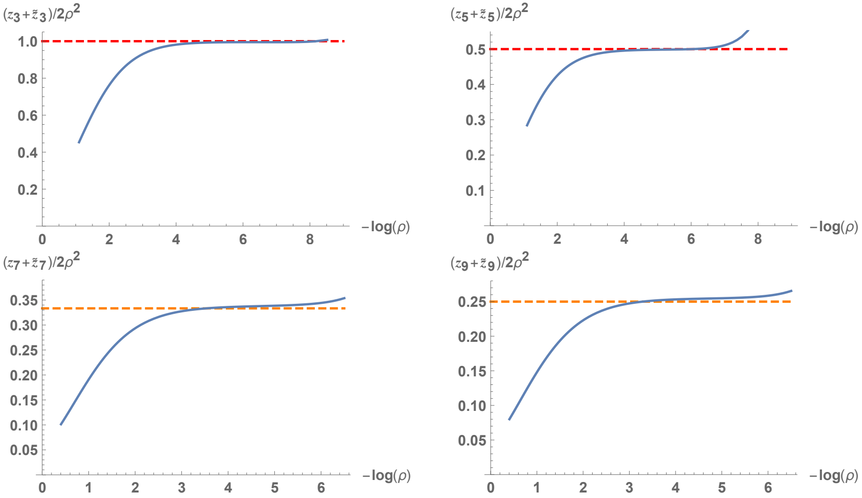
$$z_1 = \frac{c_1(1-r^2)^2}{r^3} + \frac{c_2(1-r^2)}{r^3}\left[2r - (1-r^2)\log\left(\frac{1+r}{1-r}\right)\right],$$
$$\tilde{z}_1 = \frac{c_1(1-r^2)^2}{r} - \frac{c_2(1-r^2)}{r}\left[2r + (1-r^2)\log\left(\frac{1+r}{1-r}\right)\right].$$

- Imposing regularity at IR ($r = 0$), we set $c_1 = 0$. $\epsilon = \mu$ if $c_2 = -1/8$.

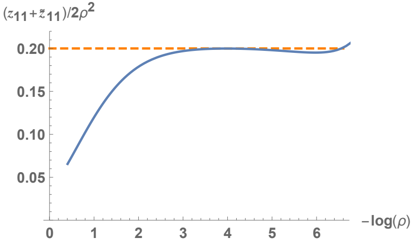
Higher orders

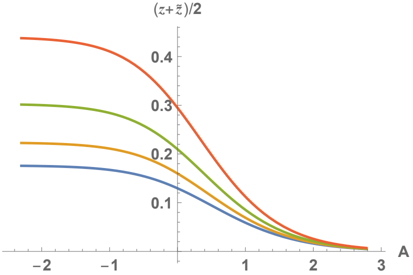
- Going to higher orders is in principle straightforward, but it involves integration of complicated functions involving log, polylog etc.
- At 3rd order we checked indeed the coefficient of μ^3 is 1, but couldn't do the integration explicitly.
- We instead solved the ODEs at each order by series expansion, at $r = 0$. (IR regular)

Plots from series expansion solutions



Plots continued





$\mathcal{N} = 1^*$ results

- The supergravity side BPS equations are constructed in Bobev, Elvang, Kol, Olson, Pufu (2016) and they calculated a few coefficients in the series expansion of sphere free energy using numerical solutions.
- We applied our perturbative prescription and fixed the leading nontrivial order coefficients exactly. NK, Se-Jin Kim (2019)

$$
\begin{aligned}
F_{S^4}/N^2 &= A_1(\mu_1^4 + \mu_2^4 + \mu_3^4) + A_2(\mu_1^2 + \mu_2^2 + \mu_3^2)^2 \\
&\quad + B_1(\mu_1^6 + \mu_2^6 + \mu_3^6) + B_2(\mu_1^2 + \mu_2^2 + \mu_3^2)^3 + B_3\mu_1^2\mu_2^2\mu_3^2 + \mathcal{O}(\mu^8). \\
A_1 &= (105 - 16\pi^4)/4200 \approx -0.346082 \\
A_2 &= (8\pi^4 - 315)/4200 \approx 0.110541
\end{aligned}
$$

Holography of mass deformed Brandhuber-Oz theory

Mass deformation of an AdS_6 example

- Although YM theory is not renormalizable in $D = 5$, string theory implies there do exist superconformal field theories.
- massive IIA theory allows AdS_6 solution as D4-D8-O8 system. Can be described using $D = 6, F(4)$ gauged supergravity.
- Dual theory has $USp(2N)$ gauged group with N_f matter hypermultiplets in fundamental rep, and one in antisymmetric tensor rep. $N^{5/2}$ dof scaling matched using localization formula Brandhuber, Oz (1999) Jafferis, Pufu (2012).

$$F = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8 - N_f}}$$

- Can be uplifted to IIB solutions as well. Hong, Liu, Mayerson (2018) etc.

Sugra action for mass-deformed AdS_6/CFT_5

- One can consider adding mass to matter in fundamental rep. Action and BPS equations found by Gutperle, Kaidi, Raj (2018)

$$S = \frac{1}{4\pi G_6} \int d^6x \sqrt{g} \left(-\frac{1}{4} R + \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{4} G_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + V(\sigma, \phi^i) \right)$$

$$G_{ij} = \text{diag}(\cosh^2 \phi^1 \cosh^2 \phi^2 \cosh^2 \phi^3, \cosh^2 \phi^2 \cosh^2 \phi^3, \cosh^2 \phi^3, 1).$$

$$\begin{aligned} V(\sigma, \phi^i) = &- g^2 e^{2\sigma} + \frac{1}{8} m e^{-6\sigma} \Big[- 32 g e^{4\sigma} \cosh \phi^0 \cosh \phi^1 \cosh \phi^2 \cosh \phi^3 + 8 m \cosh^2 \phi^0 \\ &+ m \sinh^2 \phi^0 \Big(- 6 + 8 \cosh^2 \phi^1 \cosh^2 \phi^2 \cosh(2\phi^3) + \cosh(2(\phi^1 - \phi^2)) \\ &+ \cosh(2(\phi^1 + \phi^2)) + 2 \cosh(2\phi^1) + 2 \cosh(2\phi^2) \Big) \Big] \qquad (1) \end{aligned}$$

UV expansion

- UV expansion of BPS equations for σ, ϕ^0, ϕ^3 subsystem contain three constants: α, β, f_k.
- IR regularity restricts to a one-parameter family of solutions.
- Metric in Fefferman-Graham coordinates: $ds^2 = d\rho^2/\rho^2 + e^{2f(\rho)} ds^2_{S^5}$

$$
\begin{aligned}
f &= -\log\rho + f_k - \left(\frac{1}{4}e^{-2f_k} + \frac{1}{16}\alpha^2\right)\rho^2 + O(\rho^4)\,,\\
\sigma &= \frac{3}{8}\alpha^2\rho^2 + \frac{1}{4}e^{f_k}\alpha\beta\rho^3 + O(\rho^4)\,,\\
\phi^0 &= \alpha\rho - \left(\frac{5}{4}\alpha e^{-2f_k} + \frac{23}{48}\alpha^3\right)\rho^3 + O(\rho^4)\,,\\
\phi^3 &= e^{-f_k}\alpha\rho^2 + \beta\rho^3 + O(\rho^4)\,.
\end{aligned}
$$

- Holographic renormalization gives Gutperle, Kaidi, Raj (2018)

$$
\frac{dF}{d\alpha} = \frac{\pi^2}{8G_6}\beta e^{4f_k}\left(4 - \alpha\frac{df_k}{d\alpha}\right).
$$

Using perturbative approach

- In terms of conformal metric, $e^{2A}(dr^2/r^2 + ds^2(S^5))$ we find that at all orders the solutions are given as polynomials of r $(0 \le r \le 1)$
- Although we don't see a simple pattern from the solutions and sum the series, it turns out $e^{f_k(\alpha)} = 1/2$ and

$$\beta(\alpha) = -4\alpha - \frac{\alpha^3}{2} + \frac{\alpha^5}{32} - \frac{\alpha^7}{256} + \frac{5\alpha^9}{8192} - \frac{7\alpha^{11}}{65536} + \frac{21\alpha^{13}}{1048576} - \frac{33\alpha^{15}}{8388608} + \frac{429\alpha^{17}}{536870912} - \frac{715\alpha^{19}}{4294967296} + \frac{2431\alpha^{21}}{68719476736} - \frac{4199\alpha^{23}}{549755813888} + \frac{29393\alpha^{25}}{17592186044416} + \cdots,$$

- It is the same as $\beta(\alpha) = -4\alpha\sqrt{1 + \alpha^2/4}$!
- So in this example, although we could not find exact solutions of BPS equations, we can exactly compute holographic free energy.

Comparison with field theory

- Free energy: $F(\alpha) - F(0) = \frac{\pi^2}{3G_6}\left[1 - \left(1 + \frac{\alpha^2}{4}\right)^{3/2}\right]$
- Field theory gives:
  $F(\mu) = \frac{\pi}{135}\left((N_f - 1)|\mu|^5 - \sqrt{\frac{2}{8-N_f}}(9 + 2\mu^2)^{5/2}\right) N^{5/2}$
- Since the mass term is subleading in $1/N$, we expect they should match only at leading order in $\mu \sim \alpha$.
- We find $\mu/\alpha = \frac{3\sqrt{30}}{20} \approx 0.821584$ and agrees reasonably well with the numerical analysis of Gutperle et al.

Discussions

- Perturbative prescription of AdS/CFT on supergravity side works well, when the worldvolume is curved (sphere or hyperbolic space)
- Can apply to a number of other examples of AdS/CFT
    - mABJM (dual of $SU(3) \times U(1)$ symmetric point in $D = 4$ supergravity): obtained when one integrates out a chiral multiplet in ABJM. $F = \frac{4\sqrt{2}}{3} N^{3/2} \sqrt{\Delta_1 \Delta_2 \Delta_3}$, proven holographically when there is a specific relation between UV parameters. Bobev, Min, Pilch, Rosso (2018) NK, S-J Kim (2019)
    - Janus, Black Holes etc. (future work)