IBPs without IBPs - Intersection theory and the vector space of Feynman integrals

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Introduction

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Decomposition of Feynman integrals on the maximal cut by intersection numbers

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Abstract: We elaborate on the recent idea of a direct decomposition of Feynman integrals onto a basis of master integrals on maximal cuts using intersection numbers. We begin by showing an application of the method to the derivation of contiguity relations for special functions, such as the Euler beta function, the Gauss ${}_2F_1$ hypergeometric function, and the Appell F_1 function. Then, we apply the new method to decompose Feynman integrals whose maximal cuts admit 1-form integral representations, including examples that have from two to an arbitrary number of loops, and/or from zero to an arbitrary number of legs. Direct constructions of differential equations and dimensional recurrence relations for Feynman integrals are also discussed. We present two novel approaches to decomposition-by-intersections in cases where the maximal cuts admit a 2-form integral representation, with a view towards the extension of the formalism to n-form representations. The decomposition formulae computed through the use of intersection numbers are directly verified to agree with the ones obtained using integration-by-parts identities.

Keywords: Scattering Amplitudes, Differential and Algebraic Geometry

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PHYSICAL REVIEW LETTERS **123**, 201602 (2019)

Vector Space of Feynman Integrals and Multivariate Intersection Numbers

Hjalte Frellesvig,[1,2,*] Federico Gasparotto,[1,2,†] Manoj K. Mandal,[1,2,‡] Pierpaolo Mastrolia,[1,2,§] Luca Mattiazzi,[2,1,¶] and Sebastian Mizera[3,4,**]

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Feynman integrals obey linear relations governed by intersection numbers, which act as scalar products between vector spaces. We present a general algorithm for the construction of multivariate intersection numbers relevant to Feynman integrals, and show for the first time how they can be used to solve the problem of integral reduction to a basis of master integrals by projections, and to directly derive functional equations fulfilled by the latter. We apply it to the decomposition of a few Feynman integrals at one and two loops, as first steps toward potential applications to generic multiloop integrals. The proposed method can be more generally employed for the derivation of contiguity relations for special functions admitting multifold integral representations.

DOI: 10.1103/PhysRevLett.123.201602

Introduction.—Scattering amplitudes encode crucial information about collision phenomena in our Universe, from the smallest to the largest scales. Within the perturbative field-theoretical approach, the evaluation of multiloop Feynman integrals is mandatory for the determination of scattering amplitudes and related quantities. An exception is made for those cases where a limited number of kinematic invariants yields the use of direct integration techniques; the evaluation of multiloop Feynman integrals requires the exploitation of linear relations among integrals, in order to simplify the otherwise impossible calculations. Those relations can be used both for decomposing scattering amplitudes in terms of a basis of functions, referred to as master integrals (MIs), and for the evaluation of the latter. The standard procedure used to derive relations among Feynman integrals in dimensional regularization makes use of integration-by-parts identities (IBPs) [1], which are found by solving linear systems of equations [2] (see [3,4] and references therein for reviews). Algebraic manipulations in these cases are very demanding, and efficient algorithms for solving large-size systems of linear equations have been devised recently, by making use of finite field arithmetic and rational functions reconstruction [5–8].

In this Letter, we propose a novel, alternative approach for the direct decomposition of Feynman integrals. Our method is based on the identification of algebraic properties obeyed by Feynman integrals, not known until recently, and which we access by means of intersection theory.

In [9], it was shown that intersection numbers [10] of differential forms can be employed to define (what amounts to) a scalar product on a vector space of Feynman integrals in a given family. Using this approach, projecting any multiloop integral onto a basis of MIs is conceptually no different from decomposing a generic vector into a basis of a vector space. Within this new approach, relations among Feynman integrals can be derived avoiding the generation of intermediate, auxiliary expressions which are needed when applying Gauss' elimination, as in the standard IBP-based approaches. In the initial studies, [9,11], this novel decomposition method was applied to the realm of special mathematical functions falling in the class of Lauricella F_D functions, as well as to Feynman integrals on maximal cuts, i.e., with on-shell internal lines, mostly admitting a onefold integral representation. Those results concerned a partial construction of Feynman integral relations, mainly limited to the determination of the coefficients of the MIs with the same number of denominators as the decomposed integral, which was achieved by means of intersection numbers for univariate forms.

In this Letter, we make an important step further, and address the complete integral reduction, by determining all coefficients, including those associated to MIs corresponding to subgraphs. In the current Letter, we show its application to a few paradigmatic cases at one and two loops. Generic Feynman integrals admit multifold integral representations. Their complete decomposition requires the evaluation of intersection numbers for multivariate rational

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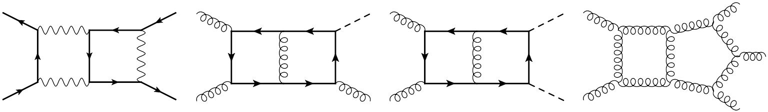
Introduction



1) Write down all Feynman diagrams
2) Perform Dirac (gamma-matrix) algebra and Lorentz algebra
3) Express in terms of scalar Feynman integrals

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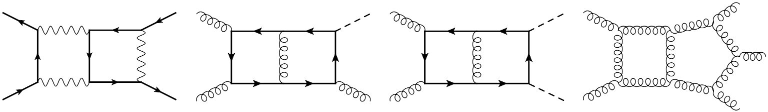

1) Write down all Feynman diagrams
2) Perform Dirac (gamma-matrix) algebra and Lorentz algebra
3) Express in terms of scalar Feynman integrals

$$I_{a_1 \cdots a_P; \cdots a_n} = \int \frac{\mathrm{d}^d k_i}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) D_2^{a_2}(k) \cdots D_P^{a_P}(k)}$$

The Ds are propagators of the form $D_i = (k+p)^2 - m^2$,
k and p are d-dimensional momenta (internal and external),
$N(k) = \prod_{i=P+1}^{n} D_i^{a_i}(k)$ is a numerator function,
P is the number of propagators,
L and E are the numbers of loops and (independent) legs,
$n = L(L+1)/2 + EL$ is the number of independent scalar products,
a_i are integer powers.

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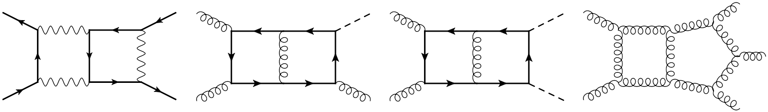
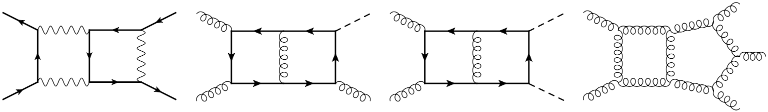


For state-of-the art two-loop scattering amplitude calculations
Feynman diagrams $\rightarrow \mathcal{O}(10000)$ Feynman integrals

Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*



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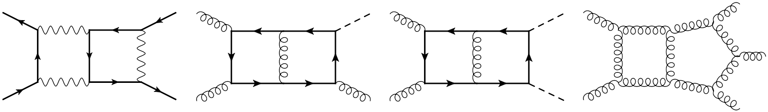
Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = 0$$

Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.



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Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.

The linear relations are often informally referred to as IBPs as well.

The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Subsectors are sub-spaces

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Not all vector spaces are *inner product spaces*

$$\begin{aligned} \langle v| &= \sum_i \langle v v_j^* \rangle \left(C^{-1}\right)_{ji} \langle v_i| \qquad \text{with} \qquad C_{ij} = \langle v_i v_j^* \rangle \\ &= \sum_i c_i \, \langle v_i| \qquad\qquad \left(c_i = \langle v v_i^* \rangle \;\; \text{if} \;\; C_{ij} = \delta_{ij}\right) \end{aligned}$$

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If only there were a way to define an inner product for Feynman integrals...

Baikov representation

$$I = \int \frac{\mathrm{d}^d k_1}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} \; = \; K \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^{\gamma}(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

The x_i are Baikov variables, $\mathcal{B}$ is the Baikov Polynomial, $\mathcal{C} = \{\mathcal{B} > 0\}$.

$$n = L(L{+}1)/2{+}EL \qquad \gamma = (d{-}E{-}L{-}1)/2$$

P. Baikov: *Nucl. Instrum. Meth.A* **389** (1997) 347–349, [hep-ph/9611449]

Theory

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The loop-by-loop version of Baikov representation can often decrease n

$$I = \tilde{K} \int_{\mathcal{C}} \mathrm{d}^{\tilde{n}} x \, \frac{\left(\prod_{j=1}^{2L-1} \mathcal{B}_j^{\gamma_j}(x) \right) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

HF and C. Papadopoulos, *JHEP* **04** (2017) 083, [arXiv:1701.07356]

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HF and C. Papadopoulos, *JHEP* **04** (2017) 083, [arXiv:1701.07356]

Baikov representation is suitable for *generalized unitarity cuts*

$\int \mathrm{d}x \to \oint \mathrm{d}x$. Preserve linear relations.

J. Bosma, M. Søgaard, Y. Zhang, *JHEP* **08** (2017) 051, [arXiv:1704.04255]

$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^{\gamma}(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u \phi$$

$u = \mathcal{B}^{\gamma}$ is a multivalued function in $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

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$\omega = \mathrm{d}\log(u)$ is *the twist*

$\langle \phi | \mathcal{C}]_\omega$ is a pairing of a *twisted cycle* ($\mathcal{C}$) and a *twisted cocycle* (ϕ) (equivalence classes of contours and integrands respectively)

P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, JHEP **1902** (2019) 139

dim of the set of ϕs, is the number of master integrals.

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$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u\phi = \langle \phi | \mathcal{C}]_\omega$$

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Lee Pomeransky criterion:

nr. of master integrals $=$ nr. of solutions to $\omega = 0$

R. Lee and A. Pomeransky, *JHEP* **11** (2013) 165, [arXiv:1308.6676].

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Lives up to all criteria for being a scalar product.

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When there is one integration variable z (ϕ and ξ are one-forms)

$$\langle\phi|\xi\rangle_\omega = \sum_{p\in\mathcal{P}} \text{Res}_{z=p}(\psi_p\xi) \qquad (\mathrm{d}+\omega)\psi_p = \phi$$

$\mathcal{P}$ is the set of poles of ω.

$(\mathrm{d}+\omega)\psi_p = \phi$ can be solved with a series ansatz around $z=p$
$\psi_p = \sum_i \psi_p^{(i)}(z-p)^i$. The exact expressions are not needed for the Res.

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K. Cho and K. Matsumoto, *Intersection theory for twisted cohomologies and twisted Riemann's period relations*, Nagoya Math. J. **139** (1995) 67-86

K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Osaka J. Math. **35** (1998) no. 4 873-893

S. Mizera, *Scattering Amplitudes from Intersection Theory*, Phys. Rev. Lett. **120** (2018) no. 14 141602

Summary:

$$I = \sum_{i \in \text{masters}} c_i I_i \quad \Leftrightarrow \quad \langle \phi | \mathcal{C}] = \sum_i c_i \langle \phi_i | \mathcal{C}]$$

with $I = \int_{\mathcal{C}} u\phi$. u is multivalued function,
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solve with series ansatz

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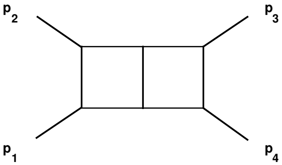
Massless double box:



$$D_1 = k_1^2,\;\; D_2 = (k_1 - p_1)^2,\;\; D_3 = (k_1 - p_1 - p_2)^2,\;\; D_4 = (k_1 - k_2)^2,$$
$$D_5 = (k_2 - p_1 - p_2)^2,\;\; D_6 = (k_1 - p_1 - p_2 - p_3)^2,\;\; D_7 = k_2^2$$

$$n_{\mathsf{std}} = L(L+1)/2 + LE = 9$$

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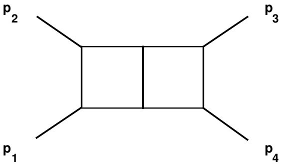

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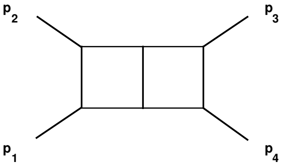
$$n_{\mathsf{std}} = L(L+1)/2 + LE = 9 \quad \text{but} \quad n_{\mathsf{LBL}} = 8$$

$$I = \int \mathrm{d}^8 x \, \frac{u N(x)}{x_1^{a_1} \cdots x_7^{a_1}} \quad \to \quad I_{7\times\mathsf{cut}} = \int u_{7\times\mathsf{cut}} \, \phi \qquad u_{7\times\mathsf{cut}} = z^{d/2-3} (z+s)^{2-d/2} (z-t)^{d-5}$$

Example (double box)

Massless double box:



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$$\omega = d\log(u) = \left(\frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t} \right) \mathrm{d}z \qquad \Rightarrow \qquad \nu = 2$$

Example (double box)

We want to reduce

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower}$$

$$c_i = \langle\phi|\xi_j\rangle(C^{-1})_{ji} \qquad \text{with} \qquad C_{ij} = \langle\phi_i|\xi_j\rangle$$

$\phi = z^2\,\mathrm{d}z\,, \quad \phi_1 = 1\,\mathrm{d}z\,, \quad \phi_2 = z\,\mathrm{d}z$

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The ξ basis is (almost) arbitrary. A dlog form is convenient

Example (double box)

We want to reduce

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Example (double box)

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Let us calculate

$$\langle \phi_1 | \xi_1 \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} (\psi \xi_1) \qquad\qquad (\mathrm{d} + \omega) \psi = \phi_1$$

$$\omega = \left(\frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t} \right) \mathrm{d}z \quad \to \quad \text{Poles of } \omega\text{: } \mathcal{P} = \{0, -s, t, \infty\}$$

Let us start with $z = 0$

Example (double box)

$$\phi_1 = 1\,\mathrm{d}z\,,\;\; \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,,\;\; \mathcal{P} = \{0, -s, t, \infty\}\,,\;\; \langle \phi_1 | \xi_1 \rangle = \sum_{p\in\mathcal{P}} \mathop{\mathsf{Res}}_{z=p}(\psi \xi_1)\,,\;\; (\mathrm{d}+\omega)\psi = \phi_1$$

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$$(\mathrm{d} + \omega)\psi_0 = \phi \qquad \Rightarrow \qquad \left(\partial_z + \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\psi_0 = 1$$

$$\psi_0 = \sum_i \kappa_i z^i$$

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$$\psi_0 = \sum_i \kappa_i z^i \qquad \kappa_{\leq 0} = 0\,, \quad \kappa_1 = \frac{2}{d-4}\,, \quad \kappa_2 = \frac{4(d-5)s + 2(d-4)t}{(d-4)(d-2)st}\,, \ldots$$

$$\mathsf{Res}_{z=0}\left(\left(\tfrac{2}{d-4} z + \mathcal{O}(z^2)\right)\left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\right) = 0$$

Example (double box)

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Likewise in $-s$ and t: $\mathsf{Res}_{z\in\{-s,t\}}(\psi\xi_1) = 0$

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For $z = \infty$ we use $y = z^{-1}$ giving $\xi_1 = \frac{-s}{1+sy} dy$

$$(\mathrm{d} + \omega)\psi_\infty = \phi \qquad \Rightarrow \qquad \left(\partial_y + \frac{-1}{2y}\left((d-6) - \frac{d-4}{1+sy} + \frac{2(d-5)}{1-ty}\right)\right)\psi_\infty = \frac{-1}{y^2}$$

$$\psi_\infty = \sum_i \kappa_i y^i \qquad \kappa_{\leq -2} = 0\,, \quad \kappa_{-1} = \frac{1}{d-5}\,, \quad \kappa_0 = \frac{-((d-4)s + 2(d-5)t)}{2(d-6)(d-5)}\,, \ldots$$

$$\operatorname{Res}_{y=0}\left(\left(\tfrac{1}{d-5}\tfrac{1}{y} + \mathcal{O}(y^0)\right)\left(\tfrac{-s}{1+sy}\right)\right) = \frac{-s}{d-5}$$

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$$\phi_1 = 1\,\mathrm{d}z\,,\;\; \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,,\;\; \mathcal{P} = \{0, -s, t, \infty\}\,,\;\; \langle \phi_1 | \xi_1 \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p}(\psi \xi_1)\,,\;\; (\mathrm{d}+\omega)\psi = \phi_1$$

$$(\mathrm{d}+\omega)\psi_0 = \phi \qquad \Rightarrow \qquad \left(\partial_z + \frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t}\right)\psi_0 = 1$$

$$\psi_0 = \sum_i \kappa_i z^i \qquad \kappa_{\leq 0} = 0\,, \quad \kappa_1 = \frac{2}{d-4}\,, \quad \kappa_2 = \frac{4(d-5)s + 2(d-4)t}{(d-4)(d-2)st}\,, \ldots$$

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Example (double box)

$$\phi = z^2\,\mathrm{d}z\,, \quad \phi_1 = 1\,\mathrm{d}z\,, \quad \phi_2 = z\,\mathrm{d}z\,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s}\right)\mathrm{d}z\,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t}\right)\mathrm{d}z\,,$$

We needed the intersection numbers: $\Big\{\langle\phi|\xi_1\rangle,\ \langle\phi|\xi_2\rangle,\ \langle\phi_1|\xi_1\rangle,\ \langle\phi_1|\xi_2\rangle,\ \langle\phi_2|\xi_1\rangle,\ \langle\phi_2|\xi_2\rangle\Big\}$

Using $\langle\phi|\xi\rangle = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}(\psi_p \xi)$ with $(\mathrm{d}+\omega)\psi_p = \phi$, we get

$$\langle\phi|\xi_1\rangle = \frac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)}\,,$$

$$\langle\phi|\xi_2\rangle = \frac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)}\,,$$

$$\langle\phi_1|\xi_1\rangle = \frac{-s}{d-5}\,, \qquad \langle\phi_1|\xi_2\rangle = \frac{s+t}{d-5}\,,$$

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$$c_i = \langle\phi|\xi_j\rangle(\mathbf{C}^{-1})_{ji} \qquad \text{with} \qquad \mathbf{C}_{ij} = \langle\phi_i|\xi_j\rangle$$

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$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower} \qquad c_0 = \frac{(d-4)st}{2(d-3)}\,,\quad c_1 = \frac{2t - 3(d-4)s}{2(d-3)}\,,$$

in agreement with FIRE

Further examples

We did $\mathcal{O}(30)$ examples in arXiv:1901.11510

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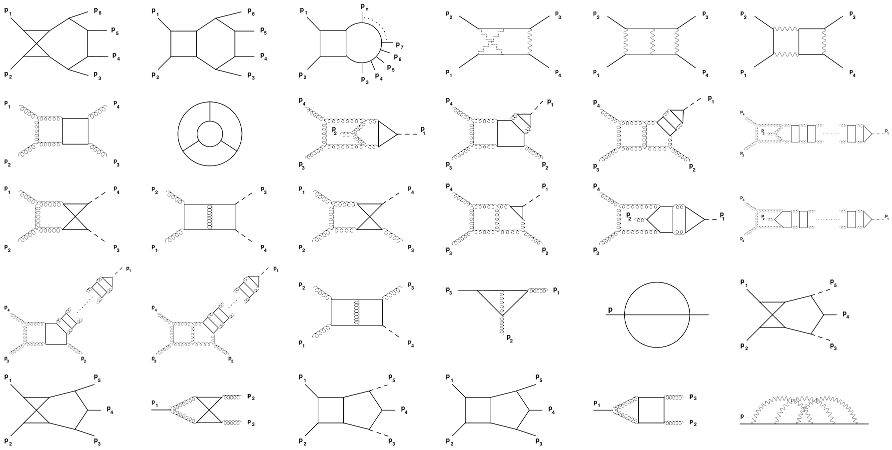
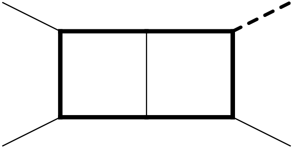


A planar integral contributing to NLO Higgs+jet production

$H + j$ "Family A":



$$D_1 = k_2^2 - m_t^2\,, \quad D_2 = (k_2 + p_1)^2 - m_t^2\,, \quad D_3 = (k_2 + p_1 + p_2)^2 - m_t^2\,, \quad D_4 = (k_1 + p_1 + p_2)^2 - m_t^2\,,$$
$$D_5 = (k_1 + p_1 + p_2 + p_3)^2 - m_t^2\,, \quad D_6 = k_1^2 - m_t^2\,, \quad D_7 = (k_1 - k_2)^2$$

A planar integral contributing to NLO Higgs+jet production

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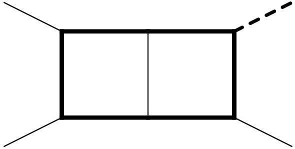

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$$u = z^{d-5}(z^2+sz+m_t^2 s)^{\frac{4-d}{2}} \Big((m_H^2-s)^2 z^2 + 2(m_H^2-s)stz + st\big(4m_t^2(m_H^2-s-t)+st\big)\Big)^{\frac{d-5}{2}}$$

There are four master integrals.

A planar integral contributing to NLO Higgs+jet production

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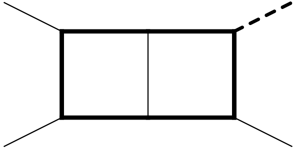

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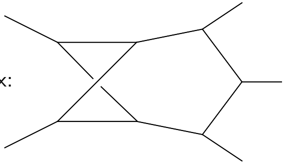
There are four master integrals.

$$I_{1111111;-1} = c_1 I_{1111111;0} + c_2 I_{1211111;0} + c_3 I_{1111211;0} + c_4 I_{1111112;0} + \text{lower}$$

The intersection procedure gives cs in agreement with Kira.

A non-planar integral contributing to NNLO 3-jet production

First non-planar pentabox: 

$$D_1 = k_1^2\,, \quad D_2 = (k_1+p_1)^2\,, \quad D_3 = (k_1-k_2-p_2)^2\,, \quad D_4 = (k_1-k_2)^2\,, \quad D_5 = (k_2+p_1+p_2)^2\,,$$
$$D_6 = (k_2+p_1+p_2+p_3)^2\,, \quad D_7 = (k_2+p_1+p_2+p_3+p_4)^2\,, \quad D_8 = (k_2)^2\,; \quad D_9 = (k_2+p_1)^2 = z\,.$$

$$u = \Big(z\,(z+s_{12})\,\big(s_{35}z^2 + (s_{51}s_{12}+s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51})z - s_{51}s_{12}s_{23}\big)\Big)^{\frac{d-6}{2}}$$

The Lee-Pomeransky criterion gives three master integrals in agreement with the literature.

$$I_{11111111;-3} = c_0 I_{11111111;0} + c_1 I_{11111111;-1} + c_2 I_{11111111;-2} + \text{lower}$$

Again the intersection procedure gives cs in agreement with the codes.

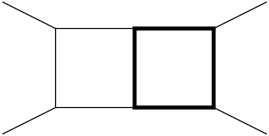
Further examples

An example of apparent discrepancy:

Further examples

An example of apparent discrepancy:

Internally massive double-box:



$$D_1 = k_1^2\,, \quad D_2 = (k_1+p_1)^2\,, \quad D_3 = (k_1+p_1+p_2)^2\,, \quad D_4 = (k_2+p_1+p_2)^2 - m^2\,,$$
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Further examples

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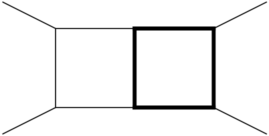

$$D_1 = k_1^2\,, \quad D_2 = (k_1{+}p_1)^2\,, \quad D_3 = (k_1{+}p_1{+}p_2)^2\,, \quad D_4 = (k_2{+}p_1{+}p_2)^2 - m^2\,,$$
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The Lee-Pomeransky criterion gives three master integrals, but the literature mentions four!

$$I_{1111111;0}\,, \quad I_{1211111;0}\,, \quad I_{1111211;0}\,, \quad I_{1111112;0}\,.$$

Further examples

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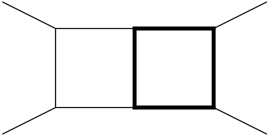

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The Lee-Pomeransky criterion gives three master integrals, but the literature mentions four!

$$I_{1111111;0}\,, \quad I_{1211111;0}\,, \quad I_{1111211;0}\,, \quad I_{1111112;0}\,.$$

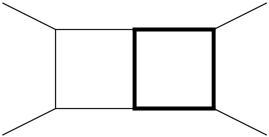
There is an extra relation relating 7-propagator sectors:

$$I_{1111211;0} = I_{1111112;0} - 2I_{01111111;1} - \tfrac{d-4}{2m^2} I_{1111011;1} + \tfrac{d-4}{m^2} I_{1111110;1} + \text{lower}$$

The intersection theory knows this relation!
On the $7\times$cut there are three (checked numerically)

An example of apparent discrepancy:

Internally massive double-box:



$$D_1 = k_1^2\,,\quad D_2 = (k_1+p_1)^2\,,\quad D_3 = (k_1+p_1+p_2)^2\,,\quad D_4 = (k_2+p_1+p_2)^2 - m^2\,,$$
$$D_5 = (k_2-p_4)^2 - m^2\,,\quad D_6 = k_2^2 - m^2\,,\quad D_7 = (k_1-k_2)^2 - m^2\,;\quad D_8 = (k_1-p_4)^2 = z\,.$$

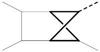
The Lee-Pomeransky criterion gives three master integrals, but the literature mentions four!

$$I_{1111111;0}\,,\quad I_{1211111;0}\,,\quad I_{1111211;0}\,,\quad I_{1111112;0}\,.$$

There is an extra relation relating 7-propagator sectors:

$$I_{1111211;0} = I_{1111112;0} - 2I_{01111111;1} - \frac{d-4}{2m^2} I_{1111011;1} + \frac{d-4}{m^2} I_{1111110;1} + \text{lower}$$

The intersection theory knows this relation!
On the $7\times$cut there are three (checked numerically)

This also holds for $H+j$ fam. F  $(6 \to 4)$

(see arXiv:1907.13156 for fam. F.)

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but now $\langle \phi | \xi \rangle$ is a *multivariate intersection number*

K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Osaka J. Math. **35** (1998) no. 4 873-893

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$$_{\mathbf{n}}\langle \phi^{(\mathbf{n})} | \xi^{(\mathbf{n})} \rangle = - \sum_{p \in \mathcal{P}_n} \operatorname*{Res}_{z_n = p} \Big({}_{\mathbf{n-1}}\langle \phi^{(\mathbf{n})} | h_i^{(\mathbf{n-1})} \rangle \, \psi_i^{(n)} \Big) \; ,$$

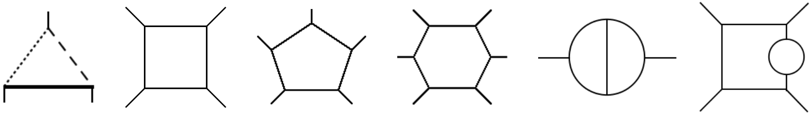
$$\Big(\delta_{ij} \, \partial_{z_n} - \hat{\mathbf{\Omega}}_{ij}^{(n)} \Big) \, \psi_j^{(n)} = \hat{\xi}_i^{(n)} \; ,$$

$$\hat{\mathbf{\Omega}}_{ij}^{(n)} = - \big(\mathbf{C}_{(\mathbf{n-1})}^{-1} \big)_{ik} \; {}_{\mathbf{n-1}}\langle e_k^{(\mathbf{n-1})} | (\partial_{z_n} - \hat{\omega}_n) h_j^{(\mathbf{n-1})} \rangle \; ,$$

$$\xi_i^{(n)} = \big(\mathbf{C}_{(\mathbf{n-1})}^{-1} \big)_{ij} \; {}_{\mathbf{n-1}}\langle e_j^{(\mathbf{n-1})} | \xi^{(\mathbf{n})} \rangle \; ,$$

$$\big(\mathbf{C}_{(\mathbf{n-1})} \big)_{ij} \equiv {}_{\mathbf{n-1}}\langle e_i^{(\mathbf{n-1})} | h_j^{(\mathbf{n-1})} \rangle \; .$$

We have done the full reduction of



multivariate

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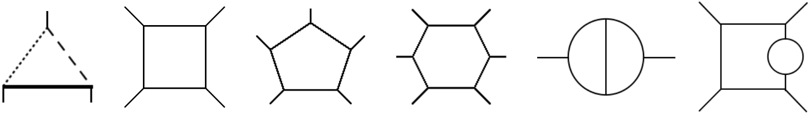
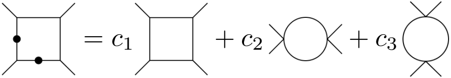


In particular



$$I = \int u(\mathbf{x})\hat{\phi}(\mathbf{x})\,\mathrm{d}^4\mathbf{x} \quad \text{with} \quad u(\mathbf{x}) = \Big((st-sx_4-tx_3)^2 - 2tx_1(s(t+2x_3-x_2-x_4)+tx_3) + s^2x_2^2 + t^2x_1^2 - 2sx_2(t(s-x_3)+x_4(s+2t))\Big)^{\frac{d-5}{2}}$$

We have to introduce *regulators* ρ: $u \to u\prod_i x_i^{\rho_i}$ since all poles of $\hat{\phi}$ must be poles of ω. We now get $\nu = 3$.

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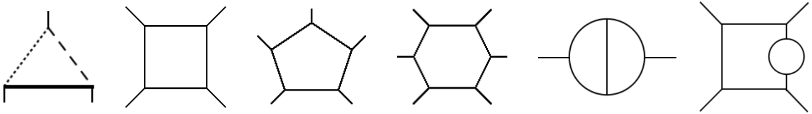

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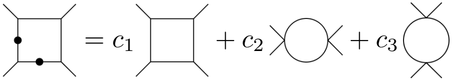

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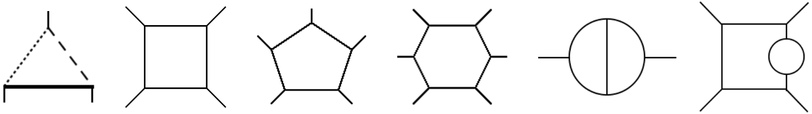

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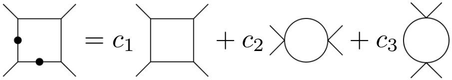

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$$c_1 = \frac{(d-5)(d-6)}{st}, \quad c_2 = \frac{-4(d-5)(d-3)}{s^3 t}, \quad c_3 = \frac{-4(d-5)(d-3)}{st^3},$$

in agreement with FIRE

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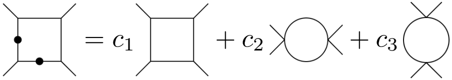
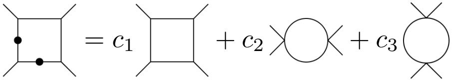
multivariate



The cut of the s-channel bubble: cut $\{x_2, x_4\}$.

$$\int \frac{u\,\mathrm{d}^4 x}{x_1^2 x_2^2 x_3 x_4}\bigg|_{\mathsf{cut}2,4} = \int \frac{\partial_{x_2} u}{x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 = \int \left(u_{\{x_2,x_4\}\to 0}\ \frac{\partial_{x_2} u}{u x_1^2 x_3}\bigg|_{\{x_2,x_4\}\to 0} \mathrm{d}x_1 \mathrm{d}x_3 \right)$$



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$$u_{24\text{cut}} = \left(s^2 t + t(x_1 - x_3)^2 - 2s(2x_1 x_3 + t(x_1 + x_3))\right)^{\frac{d-5}{2}}$$

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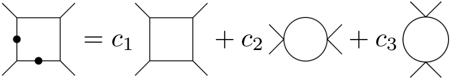

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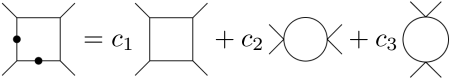

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The $\{x_1, x_3\}$-cut would yield c_3. Combine *spanning cuts*: The bottom-up approch.

Perspectives

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For multivariate forms it is more involved but similar.

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“IBPs without IBPs”

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Thank you for the invitation to speak,
and thank you for listening!

Hjalte Frellesvig

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