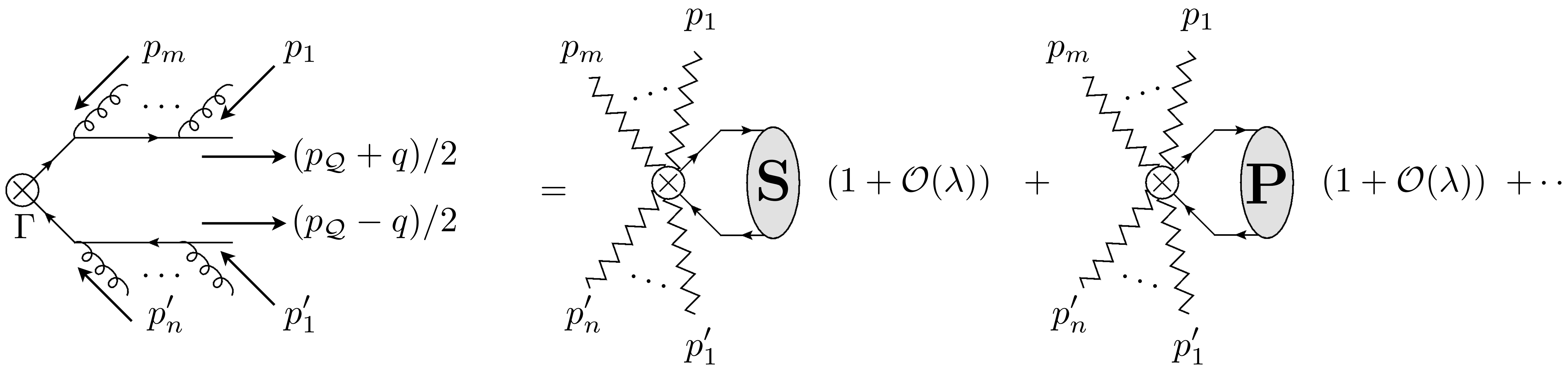
Quarkonium **TMD** factorization with **EFTs**

Yiannis Makris

Diagrammatical analysis at tree level



$$d_\Gamma(m,n) \quad = \quad d_\Gamma^{(0)}(m,n)\ (1+\mathcal{O}(\lambda)) \quad + \ d_\Gamma^{(1)}(m,n)\ (1+\mathcal{O}(\lambda)) \quad + \cdots$$

S-wave:
simple result

$$d_\Gamma^{(0)} = \left(u^{(0)}\right)^\dagger S_v^\dagger\, \Gamma^{(0)}\, S_v v^{(0)}$$

For S-wave, **color-singlet** heavy quark pair the **soft** gluon contribution, at leading order, **cancels.**

For color-octet contributions new soft wilson-lines enter the soft matrix element.

Echevarria vs FMM

TMD Shape Functions for Quarkonia

From Tom's talk

$pp \to \eta_c + X$ small p M. G. Echevarria, JHEP 1910 (2019) 144

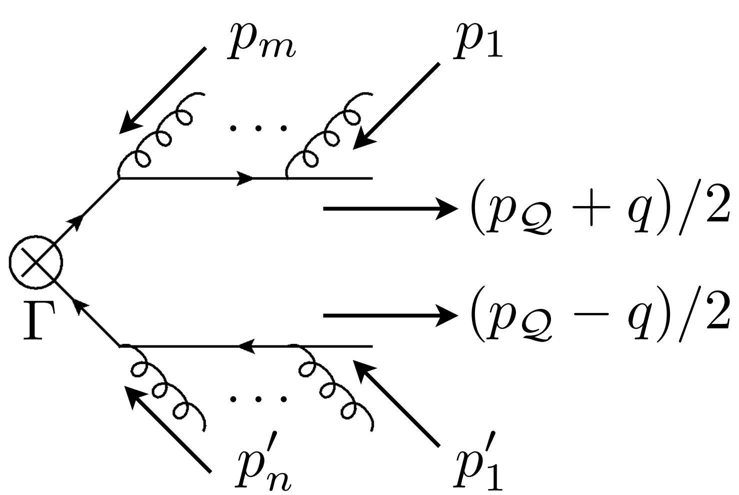
$$\frac{d\sigma}{dy\,d^2q_\perp} = \frac{4M^4\,H(M^2,\mu^2)}{2sM^2(N_c^2-1)}\,\Gamma^*_{\rho\sigma}\Gamma_{\mu\nu}(2\pi)\int d^2\boldsymbol{k}_{n\perp}d^2\boldsymbol{k}_{\bar{n}\perp}d^2\boldsymbol{k}_{s\perp}\,\delta^{(2)}\left(\boldsymbol{q}_\perp - \boldsymbol{k}_{n\perp} - \boldsymbol{k}_{\bar{n}\perp} - \boldsymbol{k}_{s\perp}\right)$$
$$\times\, G^{\sigma\nu}_{g/A}(x_A,\boldsymbol{k}_{n\perp},S_A;\zeta_A,\mu)\,G^{\rho\mu}_{g/B}(x_B,\boldsymbol{k}_{\bar{n}\perp},S_B;\zeta_B,\mu)\,S_{\eta_Q}\left[{}^1S_0^{[1]}\right](\boldsymbol{k}_{s\perp};\mu)\,, \qquad (16)$$

TMD Quarkonium Shape Function

$$S^{(0)}_{\eta_Q}\left[{}^1S_0^{[1]}\right] = \frac{1}{N_c^2-1}\int\frac{d^2\boldsymbol{\xi}_\perp}{(2\pi)^2}\,e^{i\boldsymbol{\xi}_\perp\boldsymbol{k}_{s\perp}}\,\langle 0|\left[\mathcal{Y}_n^{\dagger ab}\mathcal{Y}_{\bar{n}}^{bc}\chi^\dagger\psi\right](\boldsymbol{\xi}_\perp)\,a^\dagger_{\eta_Q}a_{\eta_Q}\left[\mathcal{Y}_{\bar{n}}^{\dagger cd}\mathcal{Y}_n^{da}\psi^\dagger\chi\right](0)\,|0\rangle\;.$$

- No disagreement between the two papers.

Diagrammatical analysis at tree level



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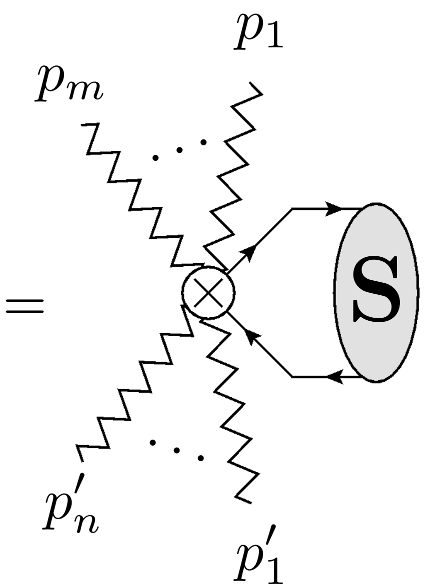
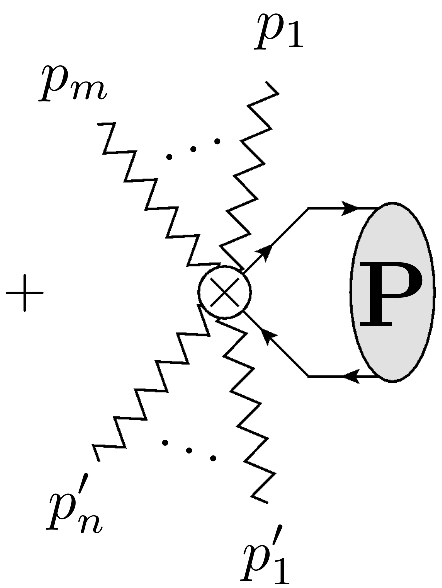


$(1+\mathcal{O}(\lambda)) \; +$



$(1+\mathcal{O}(\lambda)) \; + \cdots$

$$d_\Gamma(m,n) \quad = \quad d_\Gamma^{(0)}(m,n)\;(1+\mathcal{O}(\lambda)) \quad + \; d_\Gamma^{(1)}(m,n)\;(1+\mathcal{O}(\lambda)) \quad + \cdots$$

P-wave: Not so simple result

$$B_s^\mu = -\frac{1}{g} S_v^\dagger \left[(\mathcal{P}^\mu - gA^\mu) S_v \right]$$

$$d_\Gamma^{(1)} = \frac{g}{2m} \left(u^{(0)}\right)^\dagger \left\{ S_v^\dagger \Gamma^{(0)} S_v, \left[\frac{1}{v\cdot\mathcal{P}} \mathbf{q}\cdot\mathbf{B}_s \right] \right\} v^{(0)} + \left(u^{(0)}\right)^\dagger S_v^\dagger\, \mathbf{q}\cdot\left(\boldsymbol{\Gamma}^{(1)} - \frac{1}{4m}\left\{\Gamma^{(0)}, \boldsymbol{\gamma}\right\}\right) S_v v^{(0)}$$

Also through RPI transformations

Quarkonium production

Quarkonium recoils against soft and collinear init. state.

$\mathbf{p}_\psi \ll M_\psi$

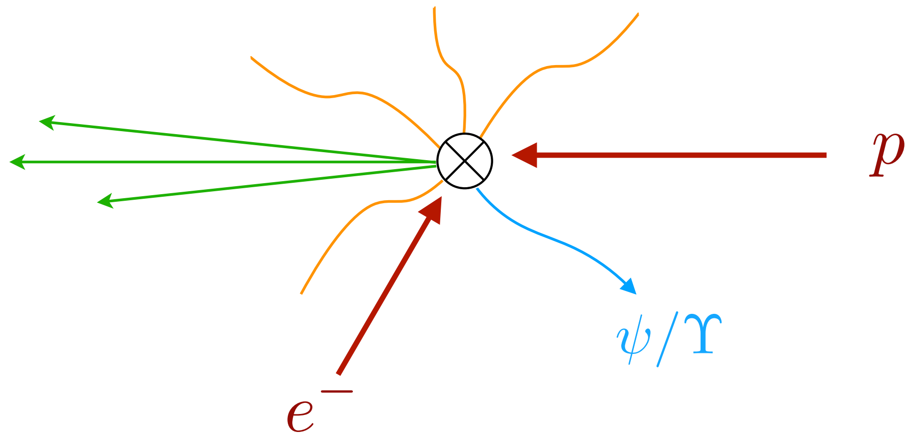
Away from the kinematic endpoint and no additional measurement

Fixed order (traditional) NRQCD approach

Soft-collinear sensitive measurement

New SCET-like approach is needed similar to in-jet factorization

Example: Semi-Inclusive DIS (photo/lepto-production)



Particularly interesting processes for accessing the momentum of gluons in the IS.

The "extra" logs

From Pieter's talk

$\ell + p \to \ell + J/\psi + X$:

matching of collinear and TMD calculation

Collinear calculation at small q_T:

$$\frac{d\sigma}{dy\, dx_B\, dz\, dq_T^2\, d\phi_\psi} = \frac{\alpha^2}{y}\left[\frac{1+(1-y)^2}{Q^2} F_{UU,T} + 4(1-y)\, F_{UU,L} + (1-y)\cos 2\phi_\psi\, F_{UU}^{\cos 2\phi_\psi}\right]$$

transverse γ^* longitudinal γ^* lin. pol. γ^*

TMD calculation at high q_T:

$$\frac{d\sigma}{dy\, dx_B\, dz\, d\boldsymbol{q}_T^2\, d\phi_\psi} = \frac{\alpha^2}{y}\left[\frac{1+(1-y)^2}{Q^2} \mathcal{F}_{UU,T} + 4(1-y)\, \mathcal{F}_{UU,L} + (1-y)\cos 2\phi_\psi\, \mathcal{F}_{UU}^{\cos 2\phi_\psi}\right]\delta(1-z)$$

In kinematic region of overlap:

$$\mathcal{F}_{UU,T} = F_{UU,T} - \sigma_{UU,T}\, C_A \ln\left(\frac{Q^2+M_\psi^2}{q_T^2}\right)$$

$$\mathcal{F}_{UU,L} = F_{UU,L} - \sigma_{UU,L}\, C_A \ln\left(\frac{Q^2+M_\psi^2}{q_T^2}\right)$$

$$\mathcal{F}_{UU}^{\cos 2\phi_\psi} = F_{UU}^{\cos 2\phi_\psi}$$

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From Tom's talk

Some Important Points

IR Safety - checked to NLO

${}^3S_1^{(8)}, {}^3P_J^{(1)}$ both required for IR safety (old NRQCD story)

p_T shape functions linked by RPI (new story)

octet mechanisms - final state radiation from soft Wilson line introduce additional logs

Evolution is slightly different than TMD resummation, than e.g. Drell-Yan

Referring to the same thing

The “extra” logs

From Pieter’s talk

$\ell + p \to \ell + J/\psi + X$:

matching of collinear and TMD calculation

Therefore, assuming both TMD factorization and enforcing the matching:

$$\mathcal{F}_{UU,T} = \sum_n \mathcal{H}^{[n]}_{UU,T} \mathcal{C}\left[f_1^g \Delta^{[n]}\right](x, \mathbf{q}_T^2)$$

$$\mathcal{F}_{UU,L} = \sum_n \mathcal{H}^{[n]}_{UU,L} \mathcal{C}\left[f_1^g \Delta^{[n]}\right](x, \mathbf{q}_T^2)$$

$$\mathcal{F}_{UU}^{\cos 2\phi_\psi} = \sum_n \mathcal{H}^{[n],\cos 2\phi_\psi}_{UU,} \mathcal{C}\left[w h_1^{\perp g} \Delta^{[n]}\right](x, \mathbf{q}_T^2)$$

we have to introduce the following ‘smearing’ function:

$$\Delta^{[n]}(\boldsymbol{k}_T^2, \mu^2) = \frac{\alpha_s}{2\pi^2 \boldsymbol{k}_T^2}\, C_A \,\langle 0|\mathcal{O}_c(n)|0\rangle \,\ln \frac{\mu^2}{\boldsymbol{k}_T^2}$$

Are these the perturbative tails of the ‘shape’ functions introduced by Miguel, Tom & Yannis?

Echevarria (2019); Fleming, Makris, Mehen (2019)

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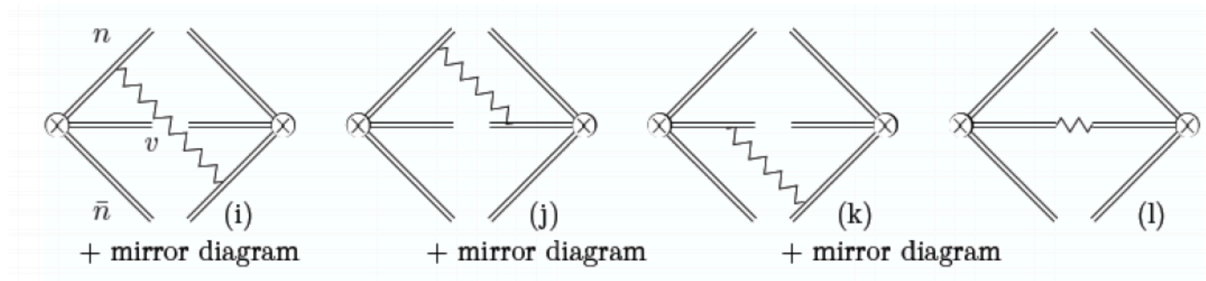
From Tom’s talk



$$d_{(\mathrm{i}+\bar{\mathrm{i}})+(\mathrm{j}+\bar{\mathrm{j}})+(\mathrm{k}+\bar{\mathrm{k}})+\mathrm{l}} = \frac{2}{3}\langle {}^3S_1^{[8]}\rangle_{\mathrm{LO}}\left(S_{\mathrm{DY}}^{\perp} + \frac{\alpha_s C_A}{2\pi}\left\{\frac{1}{\epsilon}\delta^{(2)}(\mathbf{p}_T) - 2\mathcal{L}_0(\mathbf{p}_T^2, \mu^2)\right\}\right)$$

$$S_{\mathrm{DY}}^{\perp} = \frac{\alpha_s C_F}{2\pi}\left\{\frac{4}{\eta}\left[2\mathcal{L}_0(\mathbf{p}_T^2,\mu^2) - \frac{1}{\epsilon}\delta^{(2)}(\mathbf{p}_T)\right] + \frac{2}{\epsilon}\left[\frac{1}{\epsilon} - \ln\left(\frac{\nu^2}{\mu^2}\right)\right]\delta^{(2)}(\mathbf{p}_T) - \frac{\pi^2}{6}\delta(\mathbf{p}_T) - 4\mathcal{L}_1(\mathbf{p}_T^2,\mu^2) + 4\mathcal{L}_0(\mathbf{p}_T^2,\mu^2)\ln\left(\frac{\nu^2}{\mu^2}\right)\right\}$$

Yes! well… almost

The “extra” logs

From Pieter’s talk

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we have to introduce the following ‘smearing’ function:

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From my talk@REF-2019

Color octet contributions

$$d\sigma \sim f_{g/H}^{\perp}(x, \boldsymbol{b}) \sum_n H_n \times S_n^{\perp}(\boldsymbol{b})$$

- Half the rapidity divergences
- CO-logarithm same as in case-1
- No operator mixing at NLL

The shape functions

$$S_{Q\bar{Q}[n]\to\mathcal{Q}} \sim \sum_X \left\langle O_2^{[n]} \mathcal{S}_v^{ba} \mathcal{S}_n^{bc} \middle| \mathcal{Q} + X \right\rangle \left\langle \mathcal{Q} + X \middle| \mathcal{S}_n^{cd} \mathcal{S}_v^{ed} O_2^{[n]\dagger} \right\rangle$$

$$= \langle O_n^{\mathcal{Q}} \rangle_{\rm LO} \left(\delta^{(2)}(\boldsymbol{q}_\perp) + \frac{\alpha_s^2 C_A}{2\pi} \left\{ 4 \ln\left(\frac{\nu}{\mu}\right) \mathcal{L}_0(q_\perp^2, \mu^2) - 2\mathcal{L}_1(q_\perp^2, \mu^2) - 2\mathcal{L}_0(q_\perp^2, \mu^2) - \frac{\pi}{12} \delta^{(2)}(\boldsymbol{q}_\perp) \right\} \right) + \mathcal{O}(\alpha_s^2)$$

The hard functions

$$H_n = 1 + \frac{\alpha_s C_A}{2\pi} \left\{ 2D(n) - \frac{\pi^2}{12} - \ln\left(\frac{\mu^2}{s}\right) - \frac{\beta_0}{2C_A} \ln\left(\frac{\mu^2}{s}\right) - \frac{1}{2} \ln^2\left(\frac{\mu^2}{s}\right) \right\} + \mathcal{O}(\alpha_s^2)$$

arXiv:hep-ph/9708349 (F. Maltoni, M. L. Mangano, and A. Petrelli)

U. of Pavia: 26/11/2019 24

Yes! well… almost

The “extra” logs

Advantages of shape function formalism:

- These logs are the same form and accuracy as the ones TMD factorization resums.

  Shape function formulation resums those logarithms.

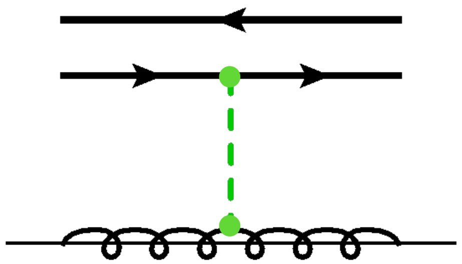
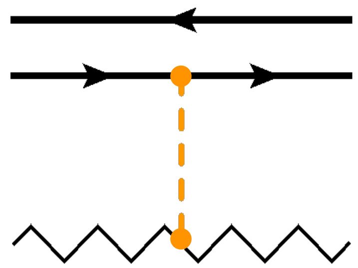
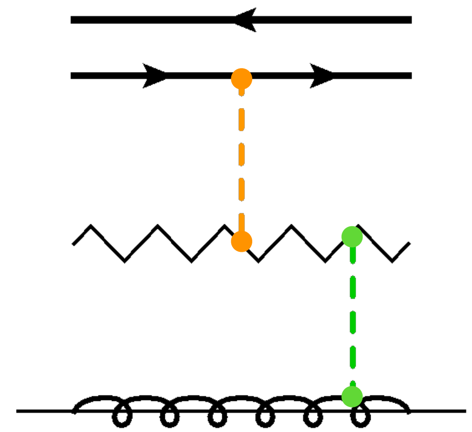
- Makes it much cleaner to classify non-perturbative contributions.

  Shape functions in non-perturbative regime are the “smearing” function.

- Perturbative calculations are far more easier.

- Otherwise “hard to catch” operators become explicit.

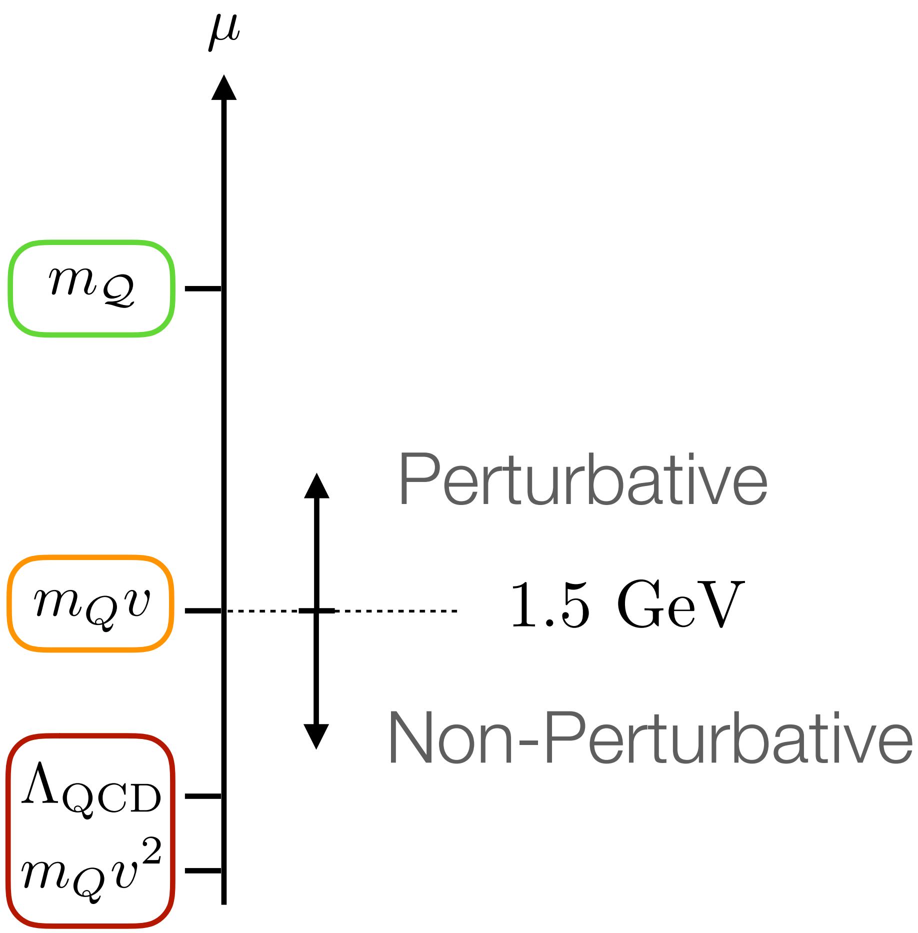
The hadroproduction of η

 Glauber	 Coulomb		What does it mean in terms of factorization?
✗			No factorization
		✗	
✓	✗	✓	Factorization with shape fun.
✓	✓	✓	Factorization with soft fun.

Additional slides

NRQCD in brief (scales)

NRQCD = Non-Relativistic QCD



$b\bar{b}: \quad v^2 \sim 0.1$ bottomonium

$c\bar{c}: \quad v^2 \sim 0.3$ charmonium

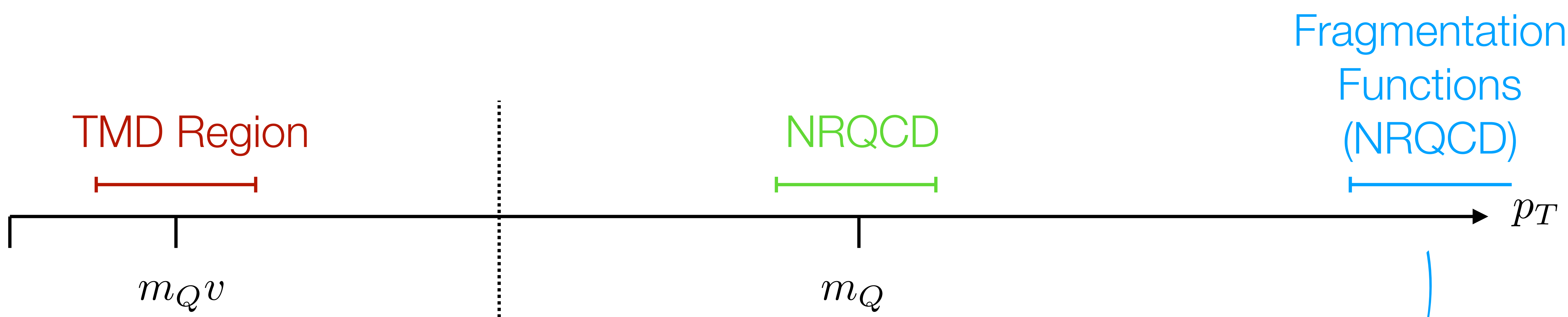
Relative velocity of the heavy quark and antiquark in the quarkonium

typical momentum of heavy quark: $|\mathbf{p}_Q| \sim m_Q v$ (soft)

typical kinetic energy of heavy quark: $K_Q \sim m_Q v^2$ (ultra-soft)

NRQCD in brief (regimes)

Quarkonium spectrum vs EFT regions



Expansion in v and in λ

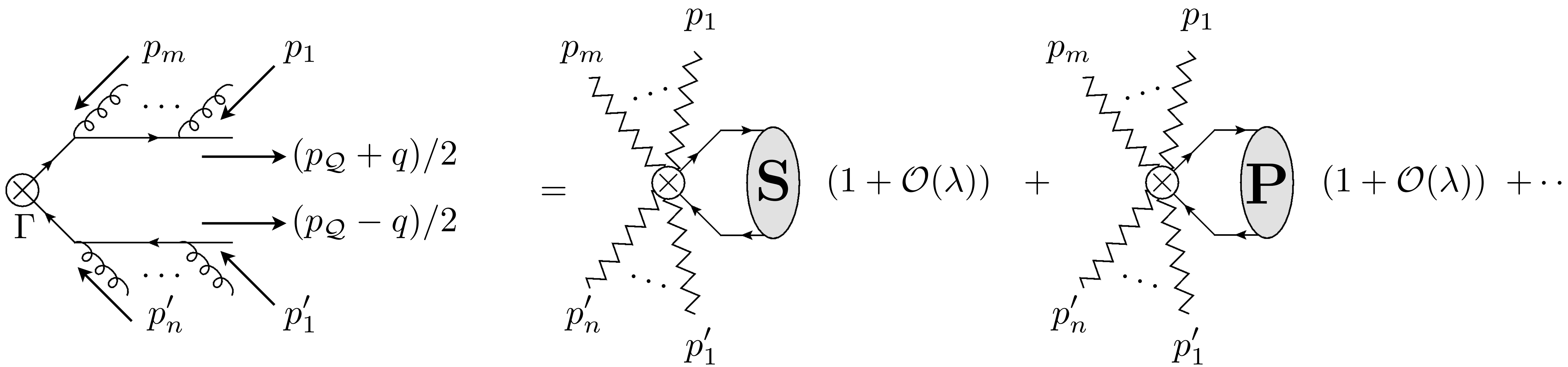
$$\lambda = q_\perp / M$$

will properly incorporate quarkonium recoiling against soft radiation

Expansion in the relative velocity

Note: it is possible to probe TMDs in this region as well but the additional measurement introduces a new soft scale which alters the factorization: e.g., TMD fragmentation functions

Diagrammatical analysis at tree level



$$d_\Gamma(m,n) = d_\Gamma^{(0)}(m,n)\ (1 + \mathcal{O}(\lambda)) + d_\Gamma^{(1)}(m,n)\ (1 + \mathcal{O}(\lambda)) + \cdots$$

S-wave:
simple result

$$S_v = \sum_n \sum_{\text{perms}} \frac{g^n}{n!} \prod_{s=1}^{n} \left[\frac{A^0_{n+1-s}}{p^0_t(s)} \right]$$

$$d_\Gamma^{(0)} = \left(u^{(0)}\right)^\dagger S_v^\dagger\, \Gamma^{(0)}\ S_v v^{(0)}$$

$$S_v(x, -\infty) = \mathrm{P}\Big[\exp\Big(-ig \int_{-\infty}^{0} d\tau\ v \cdot A_{soft}(x^\mu + v^\mu \tau) \Big) \Big]$$

Re-parameterization transformations

$$\psi^\dagger_{\mathbf{p}} S^\dagger_v \, \Gamma^{(0)} \, S_v \chi_{\mathbf{p}}$$

$$v^\mu_\pm = \left(\sqrt{1+\frac{\mathbf{q}^2}{4m^2}}, \pm\frac{\mathbf{q}}{2m} \right) = v^\mu + \frac{q^\mu}{2m} + O\left(\frac{q^2}{m^2}\right)$$

$$\slashed{v}_\pm \psi_{\mathbf{p},\pm} = \psi_{\mathbf{p},\pm}$$
$$\slashed{v}_\pm \chi_{\mathbf{p},\pm} = -\chi_{\mathbf{p},\pm}$$

$$\psi^\dagger_{\mathbf{p},+} S^\dagger_{v_+} \, \Gamma^{(0)} \, S_{v_-} \chi_{\mathbf{p},-}$$

$$\psi_{\mathbf{p},\pm} = \psi_{\mathbf{p}} \pm \frac{\slashed{q}}{4m}\psi_{\mathbf{p}} \qquad \chi_{\mathbf{p},\pm} = \chi_{\mathbf{p}} \mp \frac{\slashed{q}}{4m}\chi_{\mathbf{p}}$$

$$\left(v + \frac{q}{2m}\right) \cdot (\mathcal{P} - gA)(S_v + \delta S_v) = 0$$

$$\begin{aligned} S_{v,+} &= S_v + \frac{1}{2m}\frac{1}{v\cdot(\mathcal{P}-gA)} q\cdot(\mathcal{P}-gA)S_v \\ &= S_v + \frac{1}{2m} S_v \frac{1}{v\cdot\mathcal{P}} S^\dagger_v q\cdot(\mathcal{P}-gA)S_v \\ &= S_v - \frac{g}{2m} S_v \frac{1}{v\cdot\mathcal{P}} q\cdot B\,. \end{aligned}$$

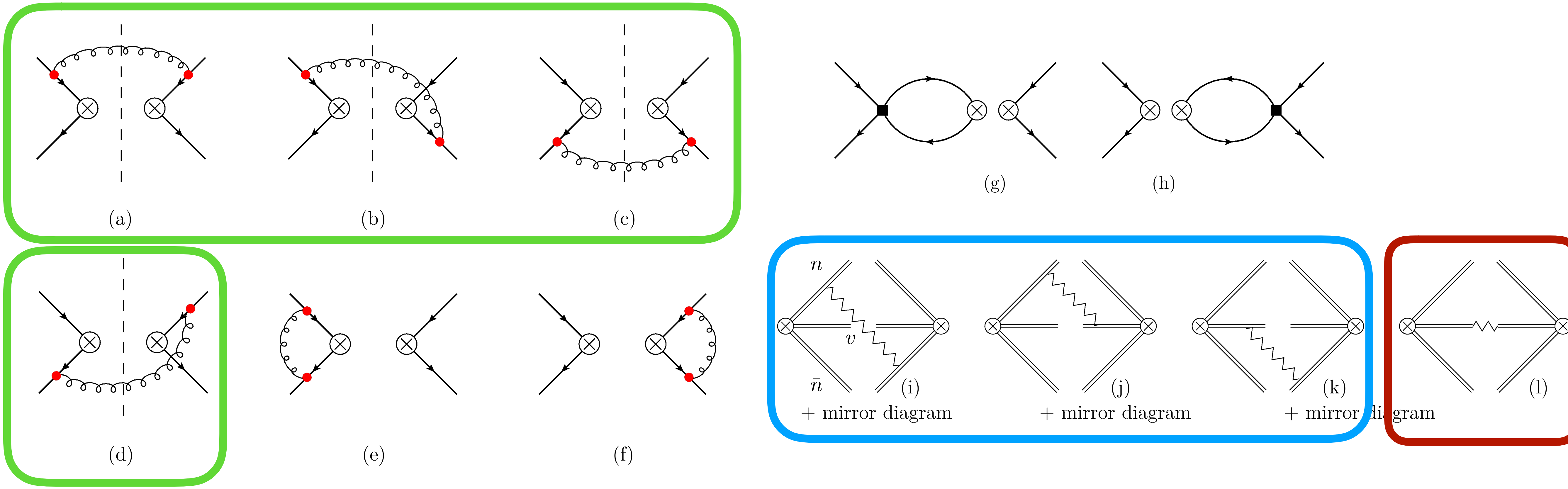
HQET:

arXiv: hep-ph/9205228 (M. E. Luke and A. V. Manohar)

SCET:

arXiv: hep-ph/0204229 (A. V. Manohar, T. Mehen, D. Pirjol and I. W. Stewart)

Shape functions at NLO



$$S^{\perp,\mathrm{NLO}}_{\chi\to{}^3S_1^{[8]}}(\mathbf{k}_\perp;\mu,\nu) = \frac{d-2}{d-1}\Bigg\{\Bigg[S^\perp_{\mathrm{DY}}(\mathbf{k}_\perp) + \frac{\alpha_s C_A}{2\pi}\left(\frac{1}{\epsilon}\delta^{(2)}(\mathbf{k}_\perp) - 2\mathcal{L}_0(\mathbf{k}_\perp^2,\mu^2)\right)\Bigg]\langle {}^3S_1^{[8]}\rangle_{\mathrm{LO}}$$

$$+\delta^{(2)}(\mathbf{k}_\perp)\left[\frac{4\alpha_s}{3\pi m^2}\left(C_F\sum_J\langle {}^3P_J^{[1]}\rangle_{\mathrm{LO}} + B_F\sum_J \langle{}^3P_J^{[8]}\rangle_{\mathrm{LO}}\right)\left(\frac{1}{\epsilon_{\mathrm{UV}}}-\frac{1}{\epsilon_{\mathrm{IR}}}\right)\right]\Bigg\}$$

RGEs and their solution

$$\frac{d}{d\ln\mu}\tilde{C}_n(b;\mu,\nu) = \sum_m \left(\tilde{\gamma}_\mu^S \delta^{nm} + \gamma_C^{nm}\right)\tilde{C}_m(b;\mu,\nu)\,, \qquad \frac{d}{d\ln\mu}\langle\mathcal{O}_{\chi_J}^{[n]}\rangle^{(\mu)} = \sum_m \gamma_{\mathcal{O}}^{nm}\langle\mathcal{O}^{[m]}\rangle_{\chi_J}^{(\mu)}$$

Consistency relations confirmed at NLO

$$\gamma_C^T = -\gamma_{\mathcal{O}}$$

$$\gamma_\mu^S = -(\gamma_\mu^H + 2\gamma_{\mu,q}^D) = -2\frac{\alpha_s C_F}{\pi}\ln\left(\frac{\nu^2}{\mu^2}\right) + \frac{\alpha_s C_A}{\pi}, \qquad \gamma_C = -\frac{8\alpha_s(\mu)C_F}{3\pi m^2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

Yields CSS kernel   Additional term (octets only)

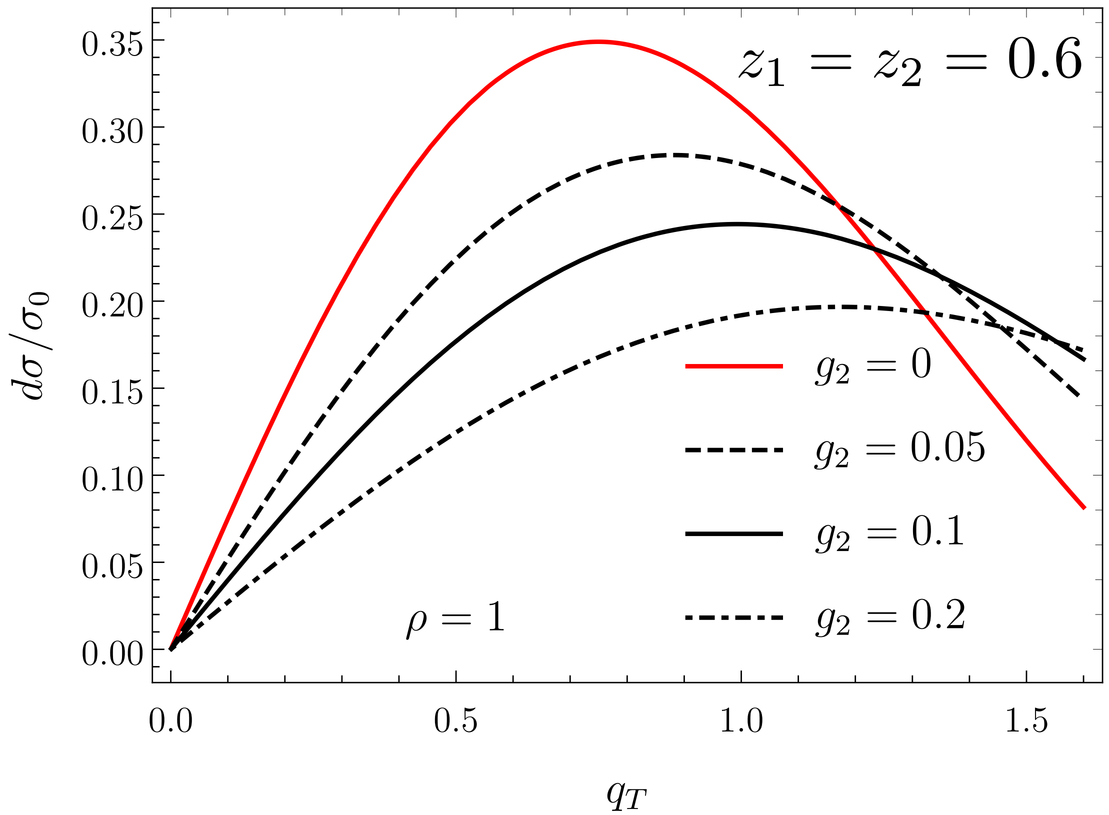
$$\mathcal{U}_{^3S_1^{[8]}}(\mu,\mu_f) = \begin{pmatrix}1 & \omega_{^3S_1^{[8]}}(\mu,\mu_f)\\ 0 & 1\end{pmatrix}\,, \qquad \omega_{^3S_1^{[8]}}(\mu,\mu_f) = -\frac{8C_F}{3m^2\beta_0}\ln\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_f)}\right)$$

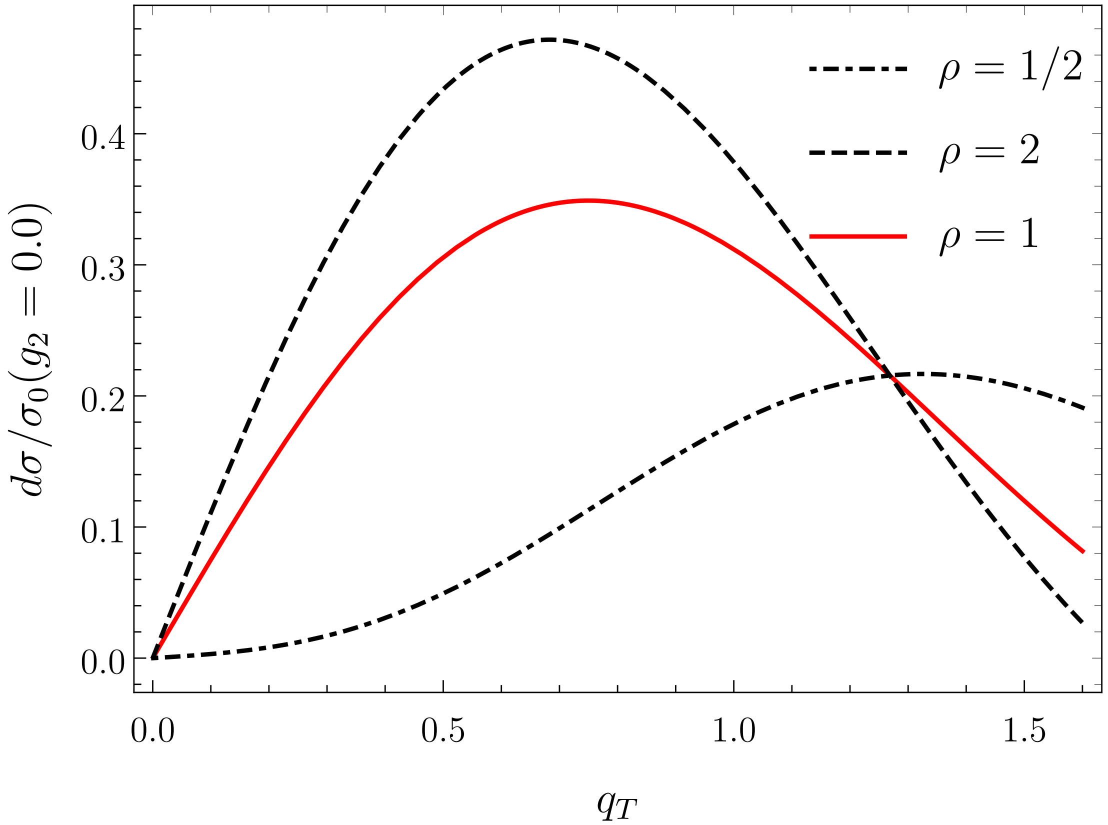
$$\langle{}^3S_1^{[8]}\rangle_{\chi_J}^{(\mu)} = \mathcal{U}_{^3S_1^{[8]}}^{nm}(\mu,\mu_f)\langle\mathcal{O}^{[m]}\rangle_{\chi_J}^{(\mu_f)}$$

The mixing effect in the evolution

$$\frac{1}{\Gamma_0}\frac{d\Gamma^{\chi_J}}{d^2q_\perp dz_1 dz_2} = \int_0^\infty bdb\; J_0(bq_\perp)\,\mathcal{U}_H(\mu_b, M_\chi)\mathcal{V}_S(M_\chi,\mu_b,\mu_b)D_{q/H_1}(z_1,\mu_b)D_{\bar{q}/H_2}(z_2,\mu_b)$$

$$\times\left[1+\frac{\langle^3P_J^{[1]}\rangle}{\langle^3S_1^{[8]}\rangle}\omega_{\mathcal{O}}(\mu_b, M_\chi)\right]$$

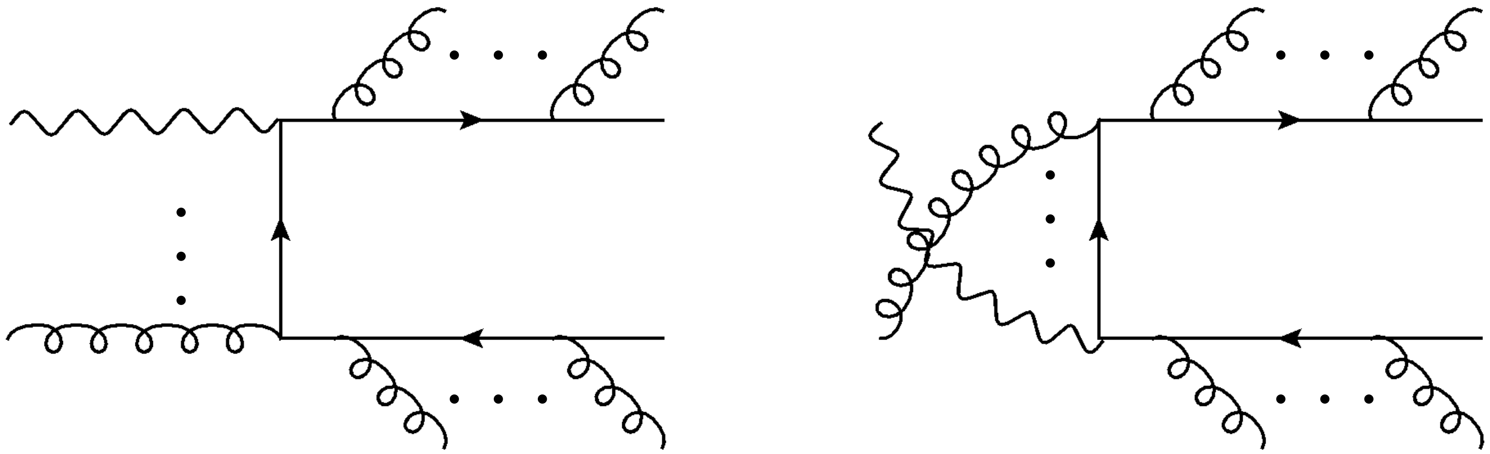




$$\rho = \langle^3S_1^{[8]}\rangle m_b^2/\langle^3P_J^{[1]}\rangle$$

Case-2: Photoproduction

Photoproduction processes relevant for accessing gluon TMDs in DIS:



S-wave octet: ${}^1S_0^{(8)}$

$$(\psi^\dagger T^a \chi)\mathcal{S}_v^{ba} B_{n\perp}^{c,i} S_n^{cb} \epsilon_\perp^k$$

P-wave octet: ${}^3P_{0/2}^{(8)}$

$$(\psi^\dagger \frac{\sigma^j \overleftrightarrow{\mathcal{P}}^m}{2} T^a \chi)\mathcal{S}_v^{ba} B_{n\perp}^{c,i} S_n^{cb} \epsilon_\perp^k$$

S-wave singlet: ${}^3S_1^{(1)}$

$$(\psi^\dagger \sigma^m \chi)\left[\frac{B_{n\perp}^{a,i} B_{n\perp}^{a,j}}{\bar{n}\cdot\mathcal{P}}\right]\epsilon_\perp^k$$

The color singlet operator is suppressed in the λ power counting but enhanced in the relative velocity, v.

See also: arXiv:hep-ph/0211303 (S. Fleming and A. K. Leibovich)

Case-2: Photoproduction

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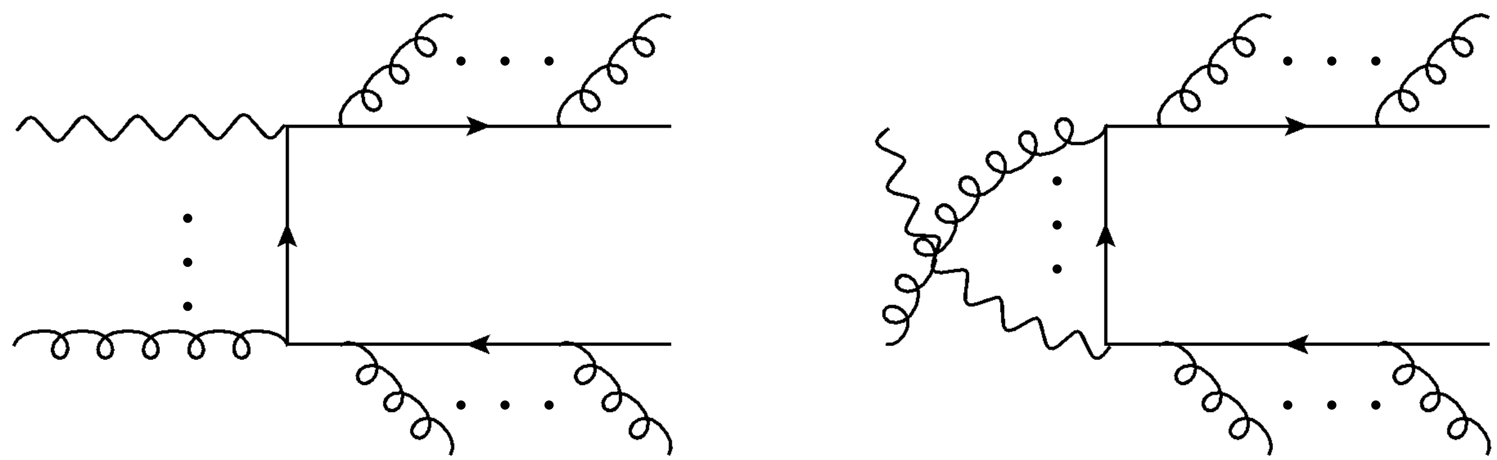
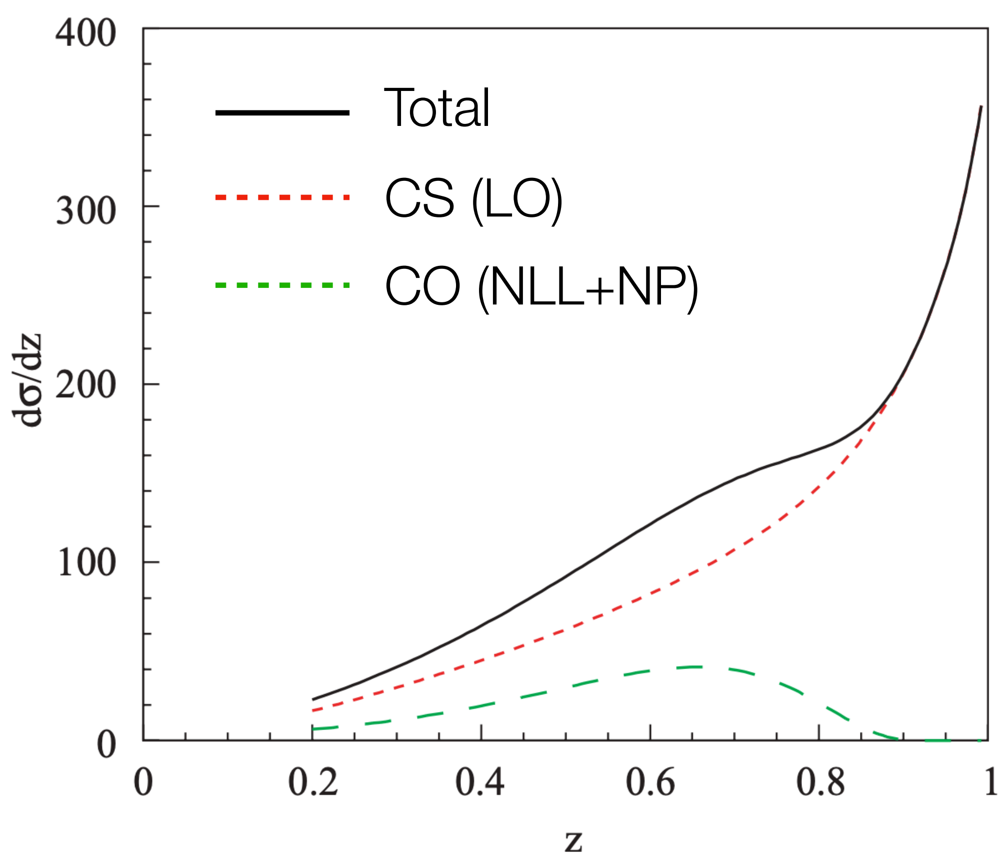




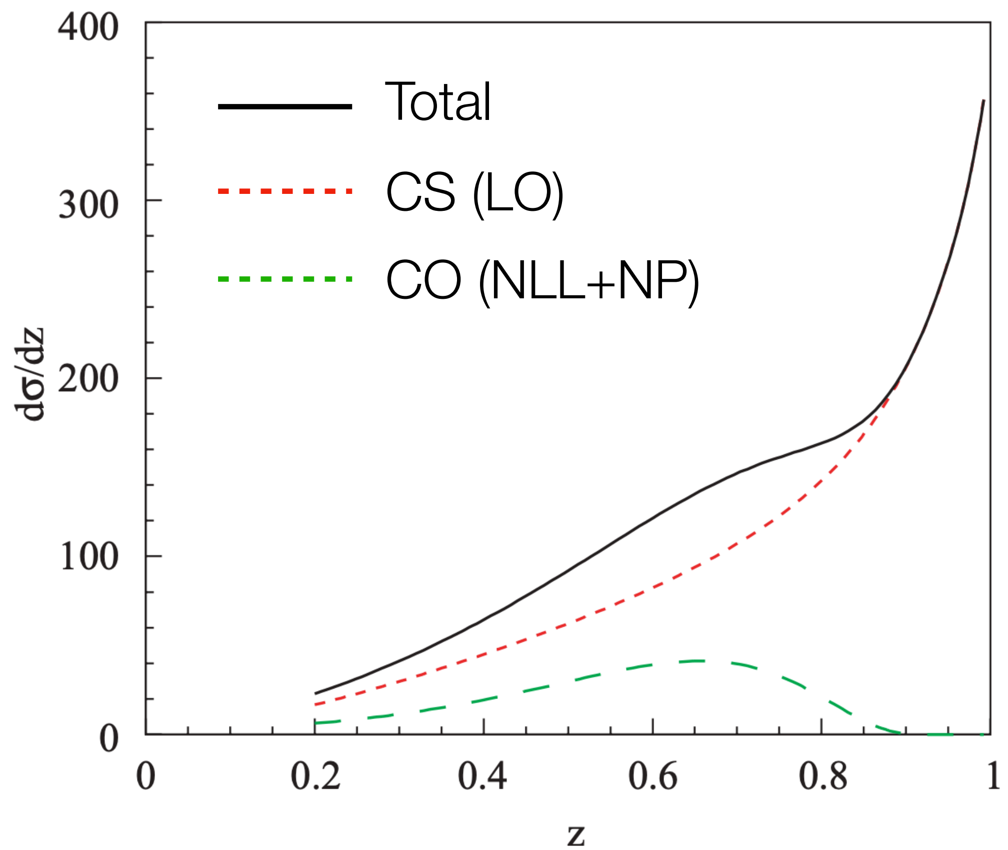
S-wave singlet: $^3S_1^{(1)}$

$$(\psi^\dagger \sigma^m \chi)\left[\frac{B_{n\perp}^{a,i} B_{n\perp}^{a,j}}{\bar{n}\cdot\mathcal{P}}\right]\epsilon_\perp^k$$

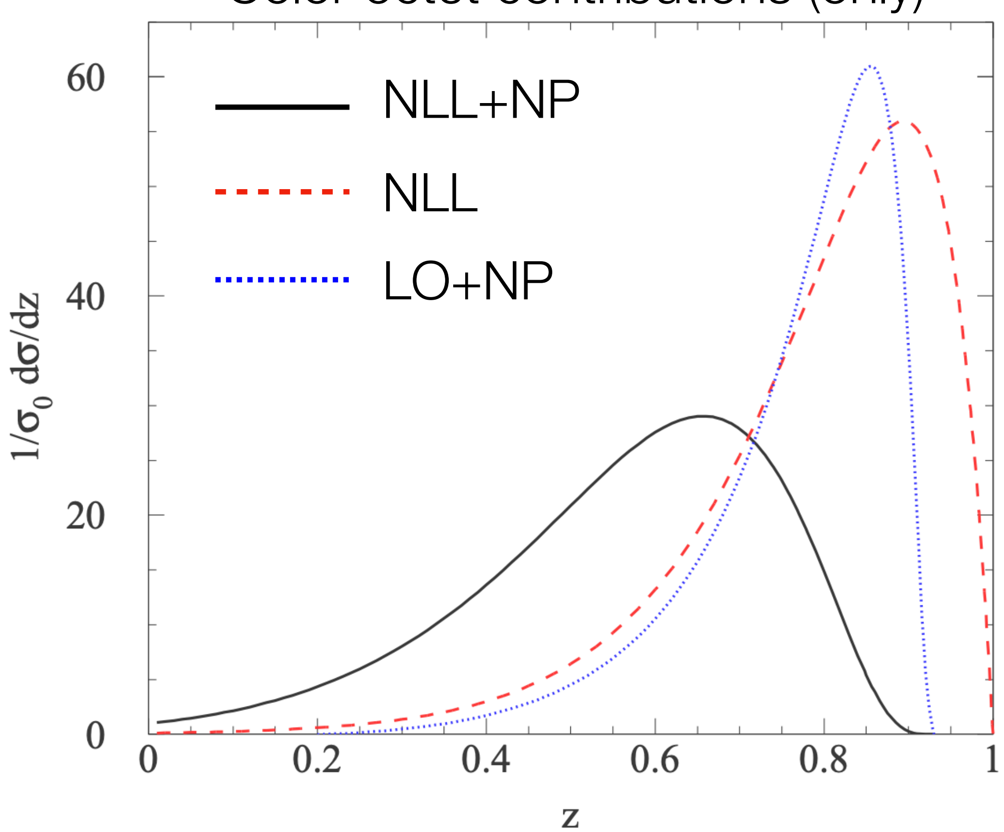
The color singlet operator is suppressed in the λ power counting but enhanced in the relative velocity, v.

arXiv:hep-ph/0607121 (S. Fleming, A. K. Leibovich, and T. Mehen)

Color singlet resumed vs fixed order



Color octet contributions (only)



arXiv:hep-ph/0607121 (S. Fleming, A. K. Leibovich, and T. Mehen)

Color octet contributions

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- Half the rapidity divergences
- CO-logarithm same as in case-1
- No operator mixing at NLL

The shape functions

$$S_{Q\bar{Q}[n]\to\mathcal{Q}} \sim \sum_X \left\langle O_2^{[n]} \mathcal{S}_v^{ba} \mathcal{S}_n^{bc} \middle| \mathcal{Q}+X \right\rangle \left\langle \mathcal{Q}+X \middle| \mathcal{S}_n^{cd} \mathcal{S}_v^{ed} O_2^{[n]\,\dagger} \right\rangle$$

$$= \langle O_n^{\mathcal{Q}} \rangle_{\mathrm{LO}} \left(\delta^{(2)}(\boldsymbol{q}_\perp) + \frac{\alpha_s^2 C_A}{2\pi} \left\{ 4 \ln\left(\frac{\nu}{\mu}\right) \mathcal{L}_0(q_\perp^2, \mu^2) - 2\mathcal{L}_1(q_\perp^2, \mu^2) - 2\mathcal{L}_0(q_\perp^2, \mu^2) - \frac{\pi}{12} \delta^{(2)}(\boldsymbol{q}_\perp) \right\} \right) + \mathcal{O}(\alpha_s^2)$$

The hard functions

$$H_n = 1 + \frac{\alpha_s C_A}{2\pi} \left\{ 2D(n) - \frac{\pi^2}{12} - \ln\left(\frac{\mu^2}{s}\right) - \frac{\beta_0}{2C_A} \ln\left(\frac{\mu^2}{s}\right) - \frac{1}{2} \ln^2\left(\frac{\mu^2}{s}\right) \right\} + \mathcal{O}(\alpha_s^2)$$

arXiv:hep-ph/9708349 (F. Maltoni, M. L. Mangano, and A. Petrelli)

Color singlet contribution (collinear)

$$d\sigma(^3S_1^{(1)}) \sim H_{^3S_1^{(1)}}(M,\mu)\mathcal{B}_\perp(z,M,\mu) \otimes S^\perp_{^3S_1^{(1)}}(\mu)$$

$$\mathcal{B}_\perp^{\mu\nu\rho\sigma} \sim \mathrm{Im}\Big[\langle P|T[B^{\mu a}_{n\perp}B^{\nu a}_{n\perp}(x^+,0^-,x^\perp)B^{\rho a}_{n\perp}B^{\sigma a}_{n\perp}(0)]|P\rangle\Big]$$

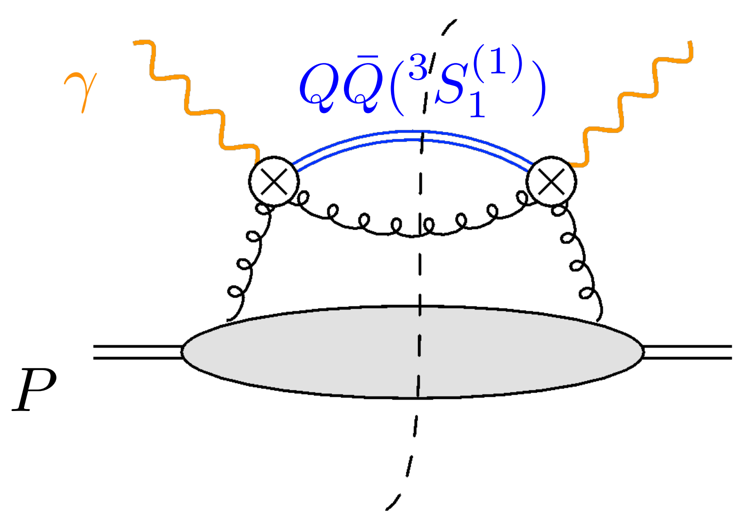
For perturbative values of transverse momentum we can match onto the collinear PDF.

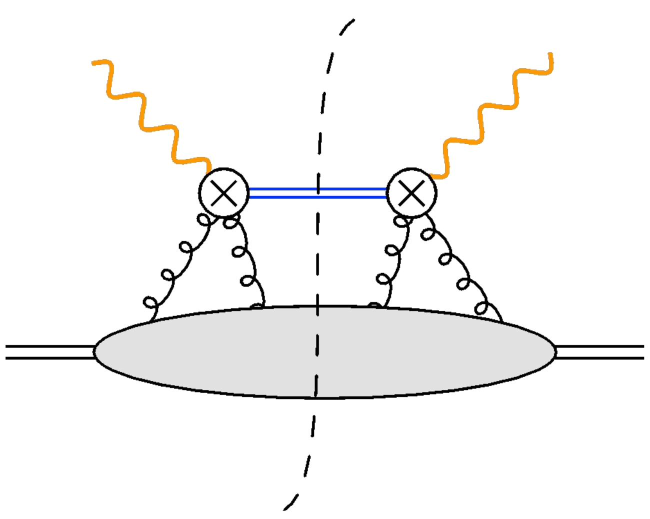
$$\mathcal{B}_\perp(z,M,\mu) = \int_0^{1-z} dy \int_{z+y}^1 \frac{dx}{x} C_\perp^{(1)}(\frac{z+y}{x},y,\mu) \otimes f_{g/P}(x,\mu)$$

Note: For non-perturbative values of the transverse momentum then one has to match onto a higher twist collinear function(s).

$$\mathcal{B}_\perp(z,M,\mu) = C_\perp^{(0)}(x_1,x_2) \otimes G(x_1,x_2)$$

Leading order contributions:





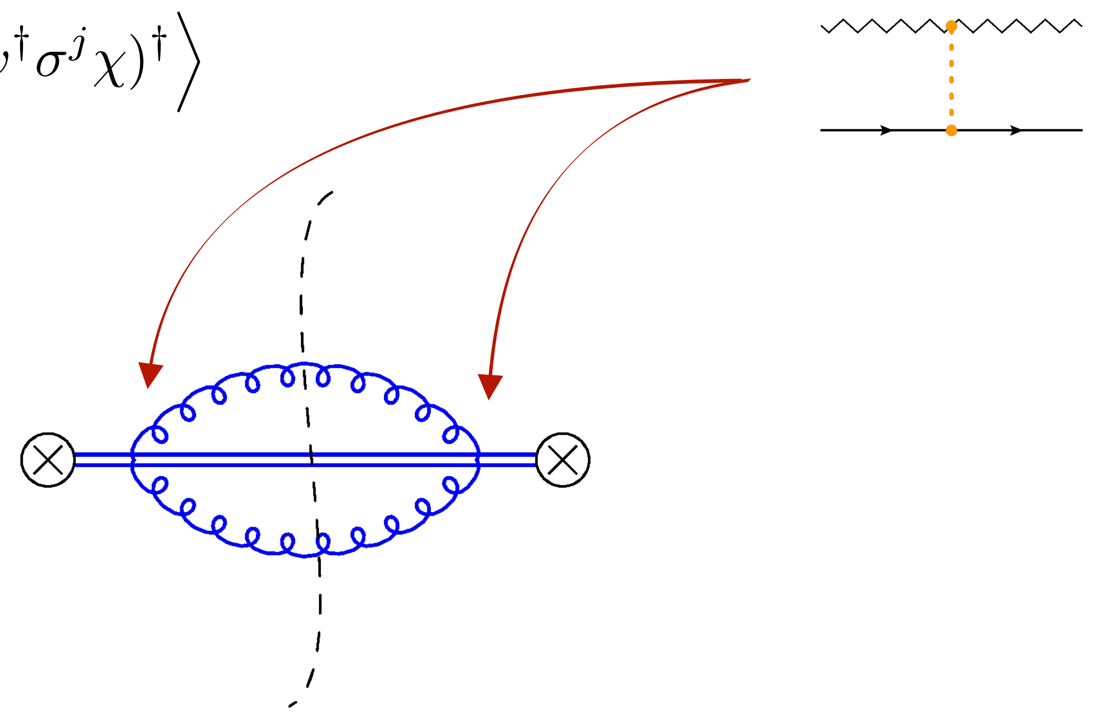
Color singlet contribution (soft)

$$d\sigma({}^3S_1^{(1)}) \sim H_{{}^3S_1^{(1)}}(M,\mu)\mathcal{B}_\perp(z,M,\mu)\otimes S^\perp_{{}^3S_1^{(1)}}(\mu)$$

$$S^\perp_{{}^3S_1^{(1)}} \sim \sum_X \left\langle (\psi^\dagger\sigma^i\chi)\middle|\mathcal{Q}+X\right\rangle\left\langle\mathcal{Q}+X\middle|(\psi^\dagger\sigma^j\chi)^\dagger\right\rangle$$

EFT Lagrangian insertions:

First beyond leading order contributions:



- Peculiar form of the first beyond leading order contribution to the shape function: Is “CSS-like” factorization still possible?