Computational fluid dynamics for physicists and engineers

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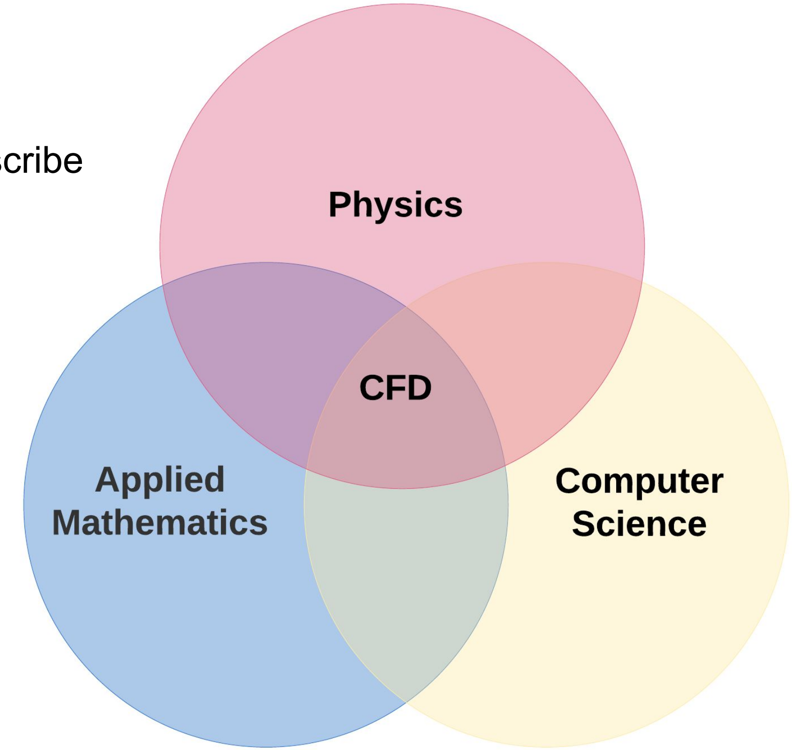
Inverted CERN school of Computing (iCSC), CERN, 28th Sep- 2nd Oct 2020

Content

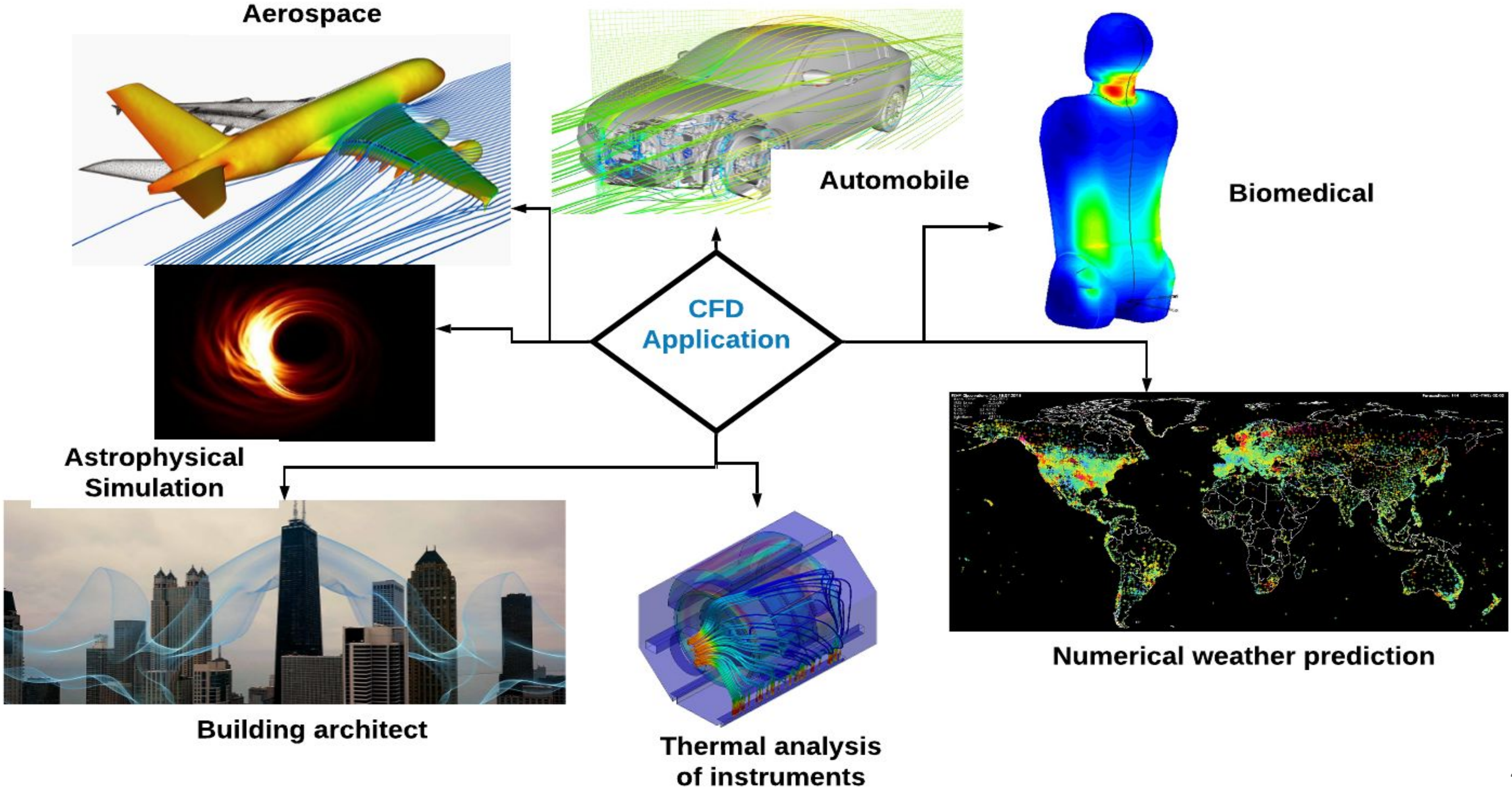
WHAT IS CFD?

Fluid flows are governed by system of **partial differential equations** (PDEs) which describe the conservation of mass, momentum and energy.

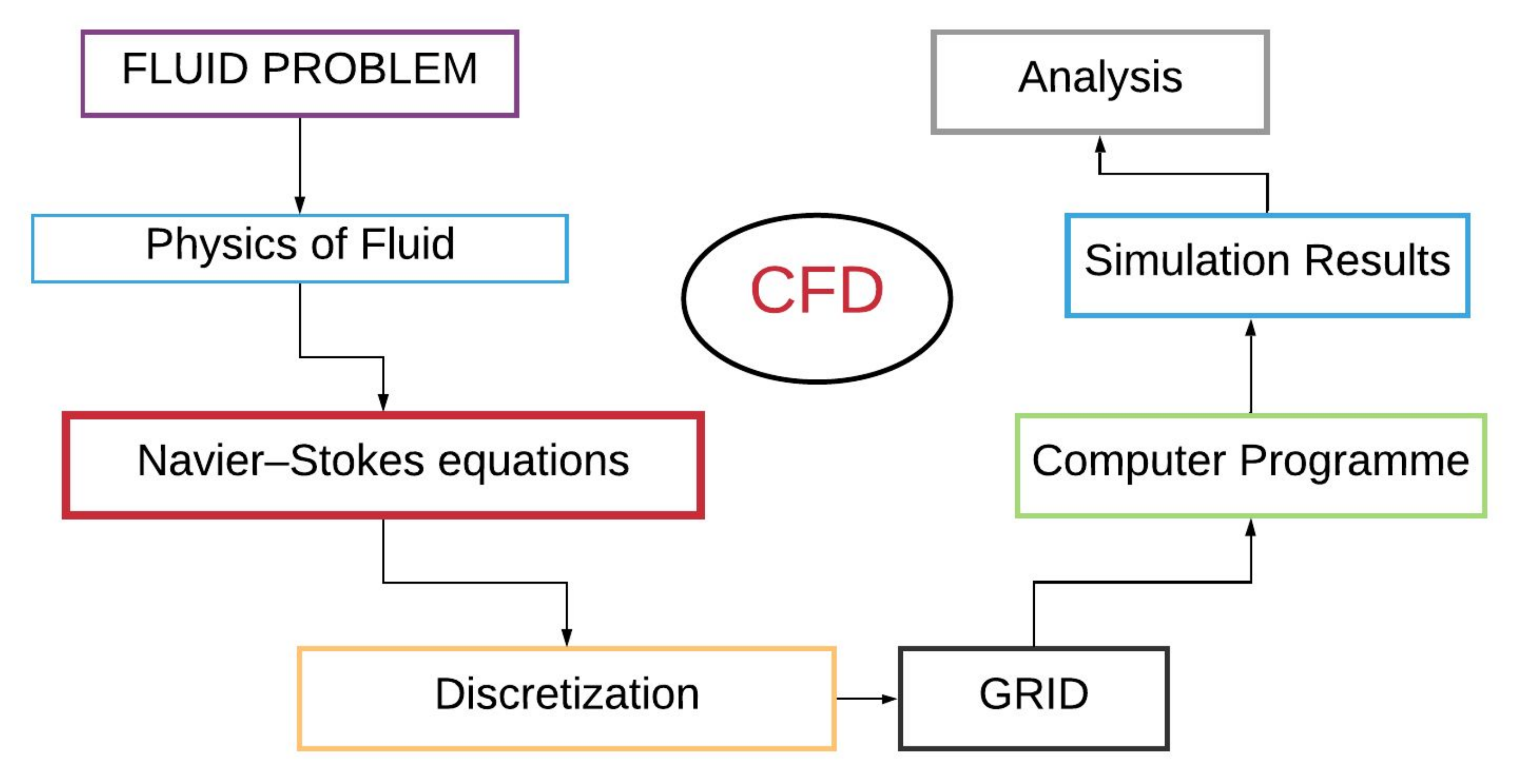


Computational fluid dynamics (CFD) solves these PDEs by replacing them with algebraic equations.

CFD APPLICATION IN DIFFERENT FIELDS



HOW DOES CFD WORK?



EQUATIONS GOVERNING FLUID FLOW

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho v)}{\partial x}$$

Mass conservation equation / continuity equation

$$\rho\left(\frac{Dv}{Dt}\right) = -\nabla P + \mu \nabla^2 v + F_x$$

mass X acceleration — Pressure gradiant — Internal Forces — External Forces

Momentum conservation equation

Navier-Stokes equation:

$$\frac{D}{Dt} = \left[\frac{\partial}{\partial t} + (v.\nabla)\right]$$

$$\frac{\partial e}{\partial t} + v.\nabla e = -\frac{P}{\rho}\nabla . v$$

Energy conservation equation

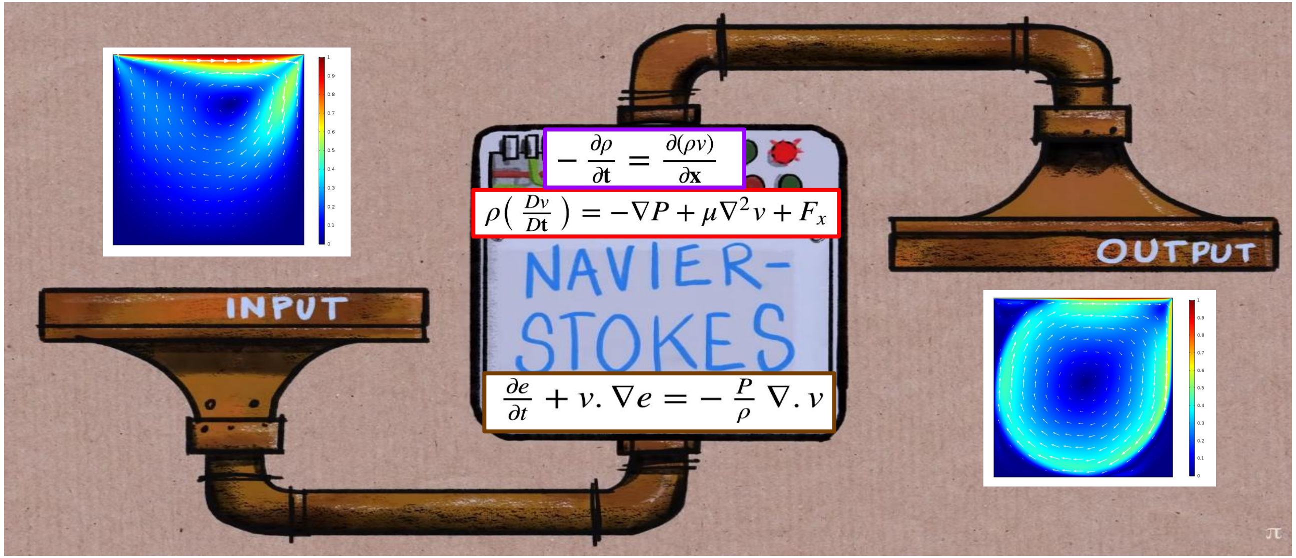
$$p = (\gamma - 1)\rho e$$

Equation of state

FROM INPUT TO OUTPUT

Initial conditions

Boundary conditions



Unknown Physical Quantities

ρ	v	p	e
Density	Velocity	Pressure	Energy

DIFFERENT APPROACHES TO MODEL FLUID IN CFD

Eulerian Approach

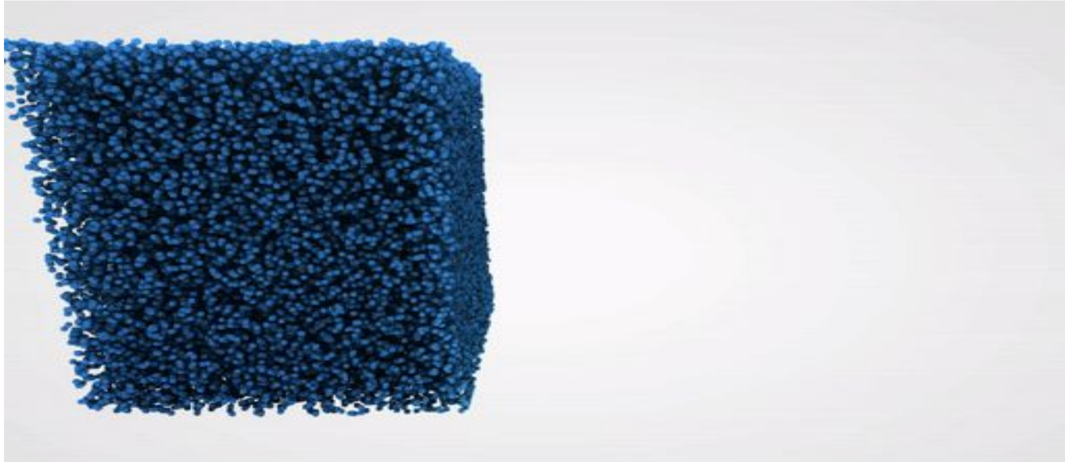
Grid based hydrodynamics

- Solves the fluid dynamics equations by calculating the flux of conserved quantities through adjacent cell boundaries

Lagrangian Approach

Smooth particle hydrodynamics(SPH)

- Calculates the properties on each particle by averaging over its nearest neighbour
- Satisfies mass conservation without extra computation as the particles themselves represent mass



*IN THIS LECTURE WE ARE GOING TO COVER ONLY THE **GRID BASED HYDRODYNAMICS***

SIMPLIFIED EQUATIONS IN CONSERVED FORM

$$\rho\left(\frac{Dv}{Dt}\right) = -\nabla P + \mu\nabla^2 v + F_x$$

mass X acceleration — Pressure gradiant — Internal Forces — External Forces

Momentum conservation equation

Navier-Stokes equations

We replace our equations by simpler ones.

Original Navier-Stokes equation reduces to Euler equations.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + P)}{\partial x} = 0$$

$$\frac{\partial e}{\partial t} + \frac{\partial((e + P)v)}{\partial x} = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\left[\textbf{Conserved quantity}\right] + \frac{\partial}{\partial x}\left[\textbf{Flux}\right] = 0$$

How to solve it numerically ?

HOW TO SOLVE THEM NUMERICALLY?

Unknown Physical Quantities

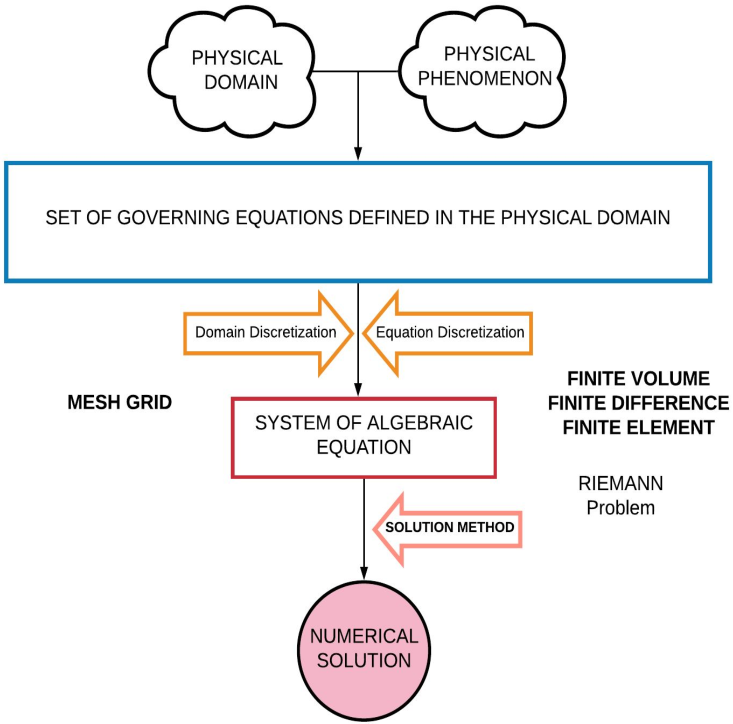
ρ	v	p	e
Density	Velocity	Pressure	Energy

$$\frac{\partial}{\partial t}\left[\textbf{Conserved quantity}\right] + \frac{\partial}{\partial x}\left[\textbf{Flux}\right] = 0$$

Physical domain: space (*x,y,z*) and time *t*
Physical quantities: $\rho\, v\, p\, e$

Since we will solve equations numerically, we have to discretize

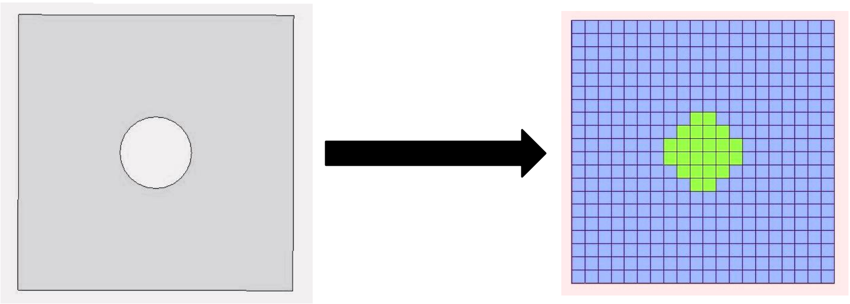
1) Physical domain
2) Physical quantities *(aka equation discretization)*

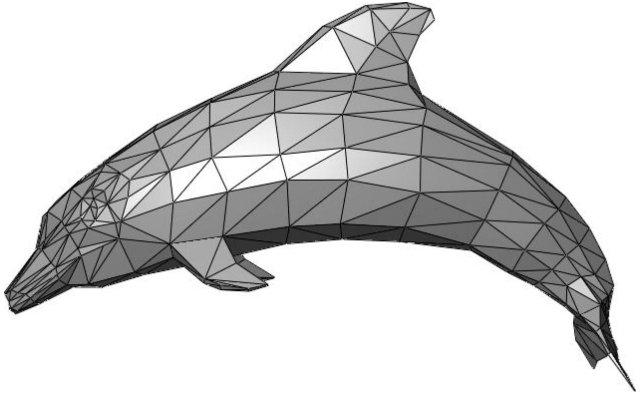


DOMAIN DISCRETIZATION

MESH GRID - Division of a continuous geometric space into discrete geometric cells

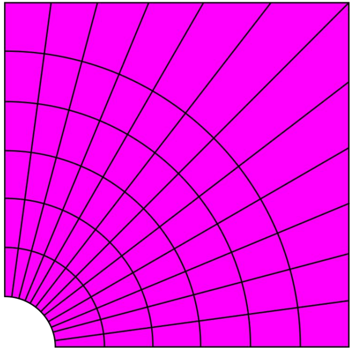
Model of flow around cylinder using **cartesian grid.**



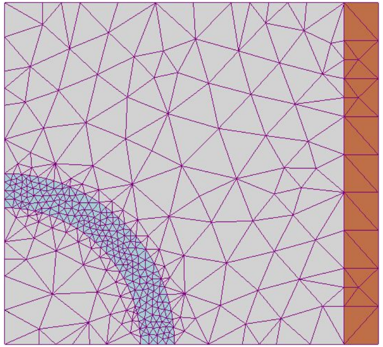


Example of triangle mesh representing a dolphin

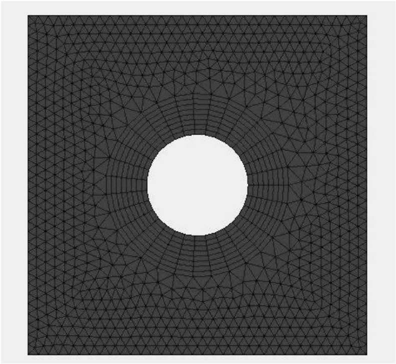
Structured curvilinear grid



Unstructured curvilinear grid

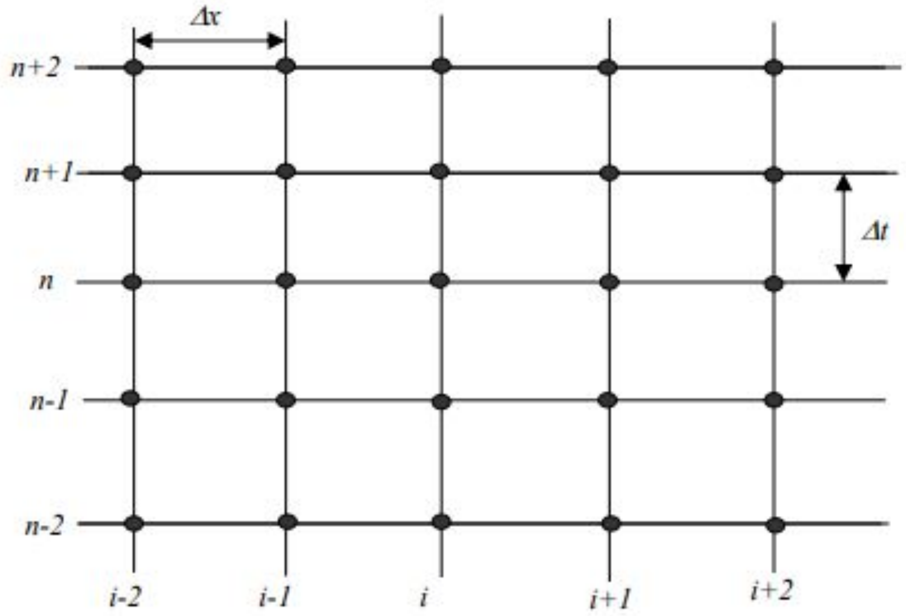


Hybrid grid



THE DISCRETIZED PHYSICAL DOMAIN

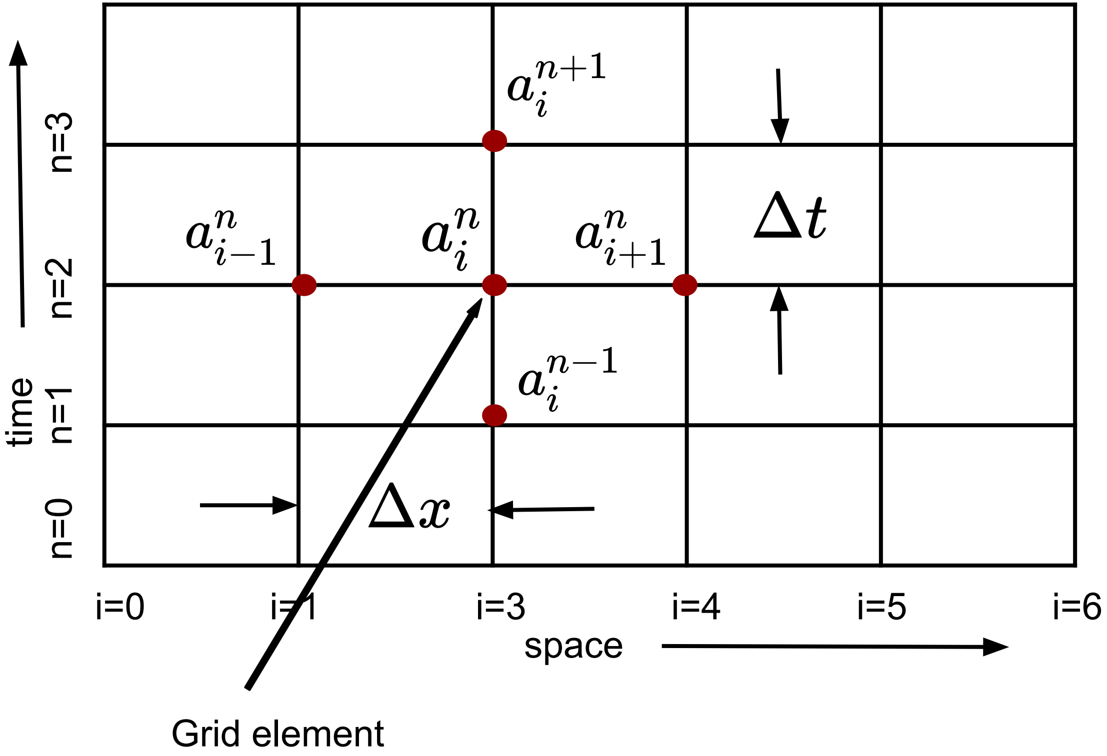
Grid generation



- A simple method of placing points in the domain
- Each point is labeled using i for spatial discretization and n for time discretization
- The spacing can be of variable size

DISCRETIZATION OF PHYSICAL QUANTITIES

Equation discretization



Backward difference

$$\frac{\partial a_i^n}{\partial x} \simeq \frac{a_i^n - a_{i-1}^n}{\Delta x}$$

Forward difference

$$\frac{\partial a_i^n}{\partial x} \simeq \frac{a_i^{n+1} - a_i^n}{\Delta x}$$

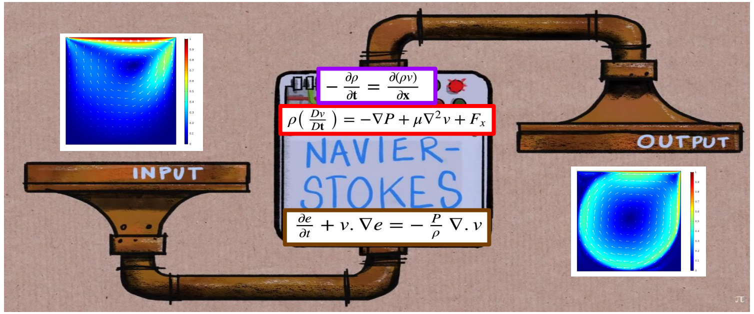
Central difference

$$\frac{\partial a_i^n}{\partial x} \simeq \frac{a_{i+1}^n - a_{i-1}^n}{2\Delta x}$$

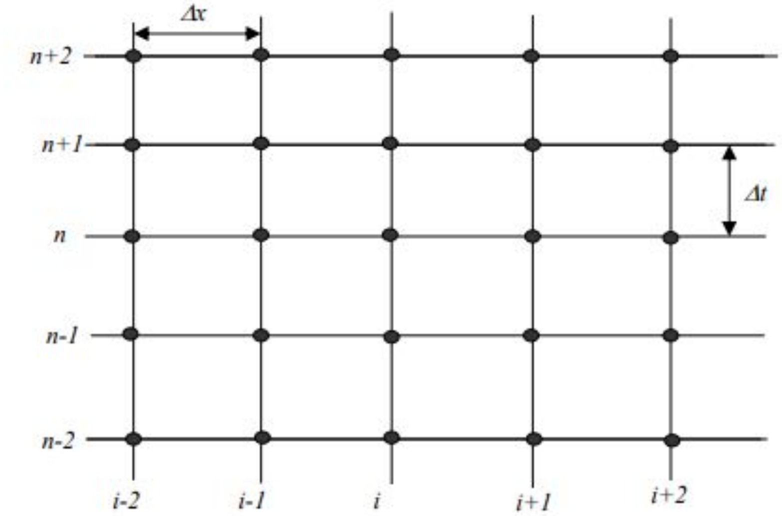
SPECIFYING INPUT THROUGH INITIAL AND BOUNDARY CONDITIONS

Initial conditions

Boundary conditions



$\rho \quad v \quad p \quad e$

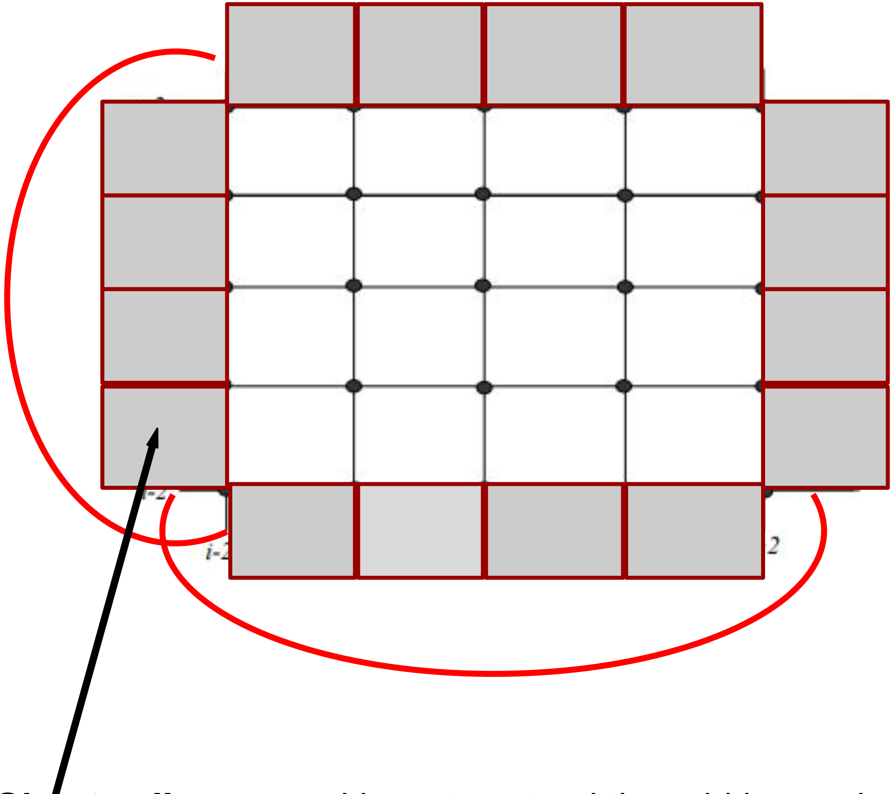


Some of the boundary conditions used in CFD

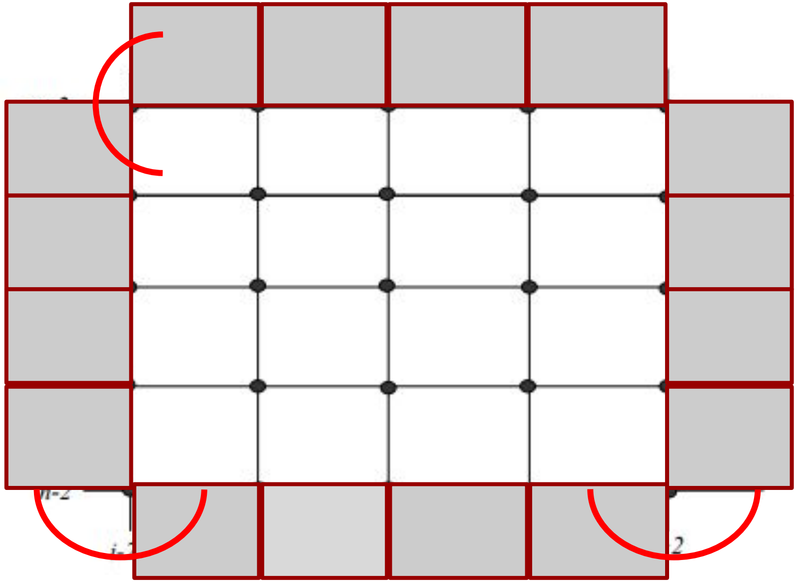
1- inlet condition
2- symmetric condition
3- periodic boundary condition
4- reflective boundary condition
5- outlet condition

BOUNDARY CONDITIONS

Periodic Boundary Condition

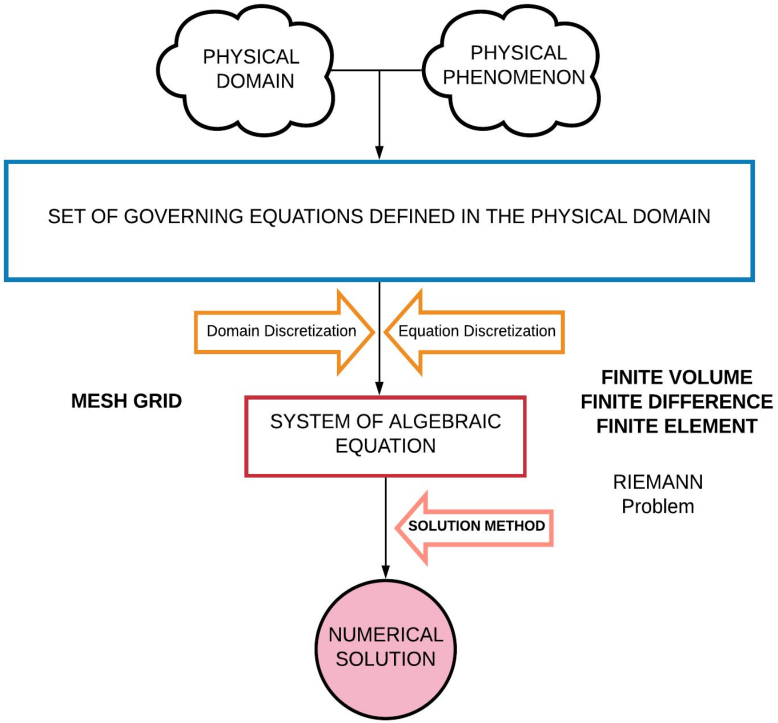


Symmetric Boundary Condition

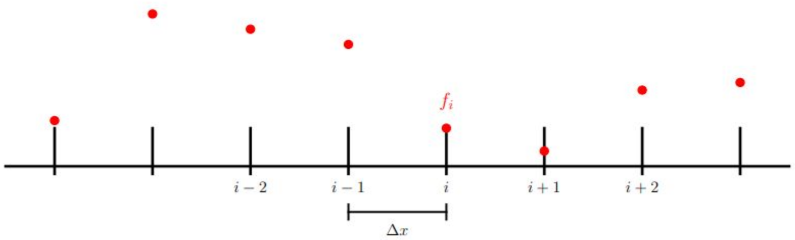


Ghost cells are used here to extend the grid beyond physical boundary to accommodate boundary condition

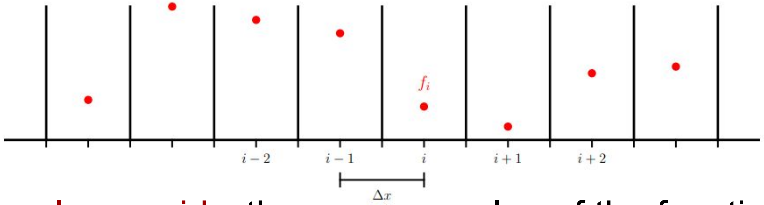
STORING DATA IN GRIDS



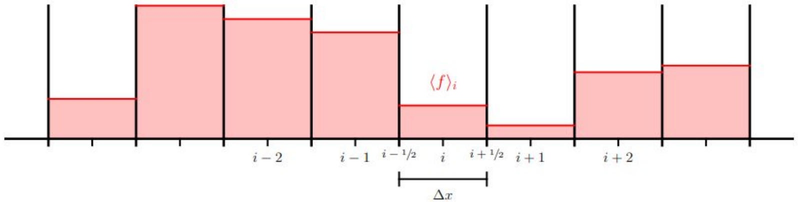
Finite-difference grid : Data is stored at grid edges



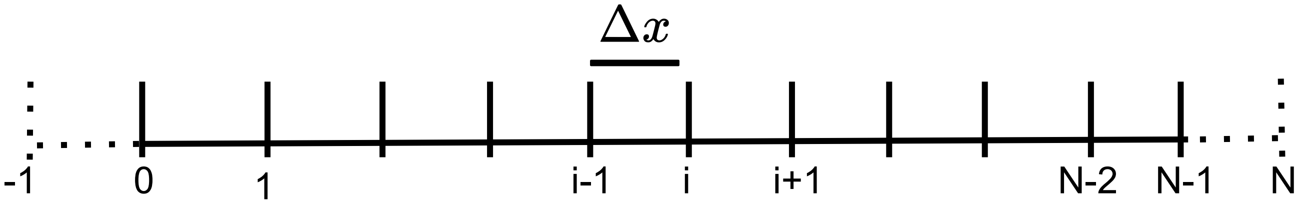
Cell-centered finite-difference grid : Data is stored at cell centers



Finite-volume grid—the average value of the function is stored within each zone.



EQUATION DISCRETIZATION USING FINITE DIFFERENCE METHOD



Finite difference grid with ghost cell at each end

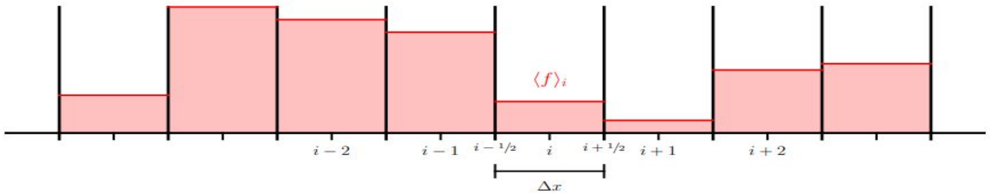
$$\frac{\partial U}{\partial t} = -\frac{\partial F}{\partial x}$$

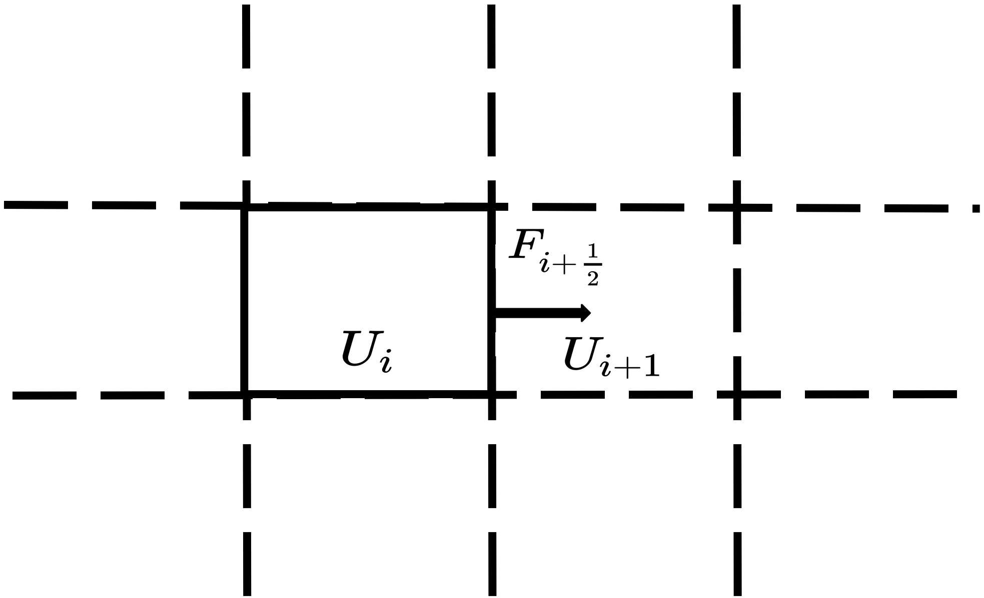
$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{-1}{\Delta x}\left[F_i^n - F_{i-1}^n\right]$$

Time evolution

$$U_i^{n+1} = \frac{-\Delta t}{\Delta x}\left[F_i^n - F_{i-1}^n\right] + U_i^n$$

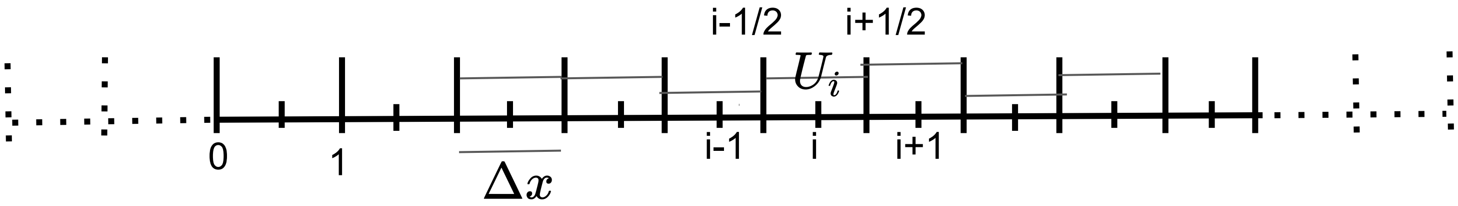
FINITE VOLUME METHOD





Fluxes are calculated at cell edges (i ± 1/2)

EQUATION DISCRETIZATION USING FINITE DIFFERENCE METHOD



Finite volume grid with two ghost cells at both ends

$$\frac{\partial U}{\partial t} = -\frac{\partial F}{\partial x}$$

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{-1}{\Delta x}\left[F_{i+\frac{1}{2}}^{n+\frac{1}{2}} - F_{i-\frac{1}{2}}^{n-\frac{1}{2}}\right]$$

Time evolution

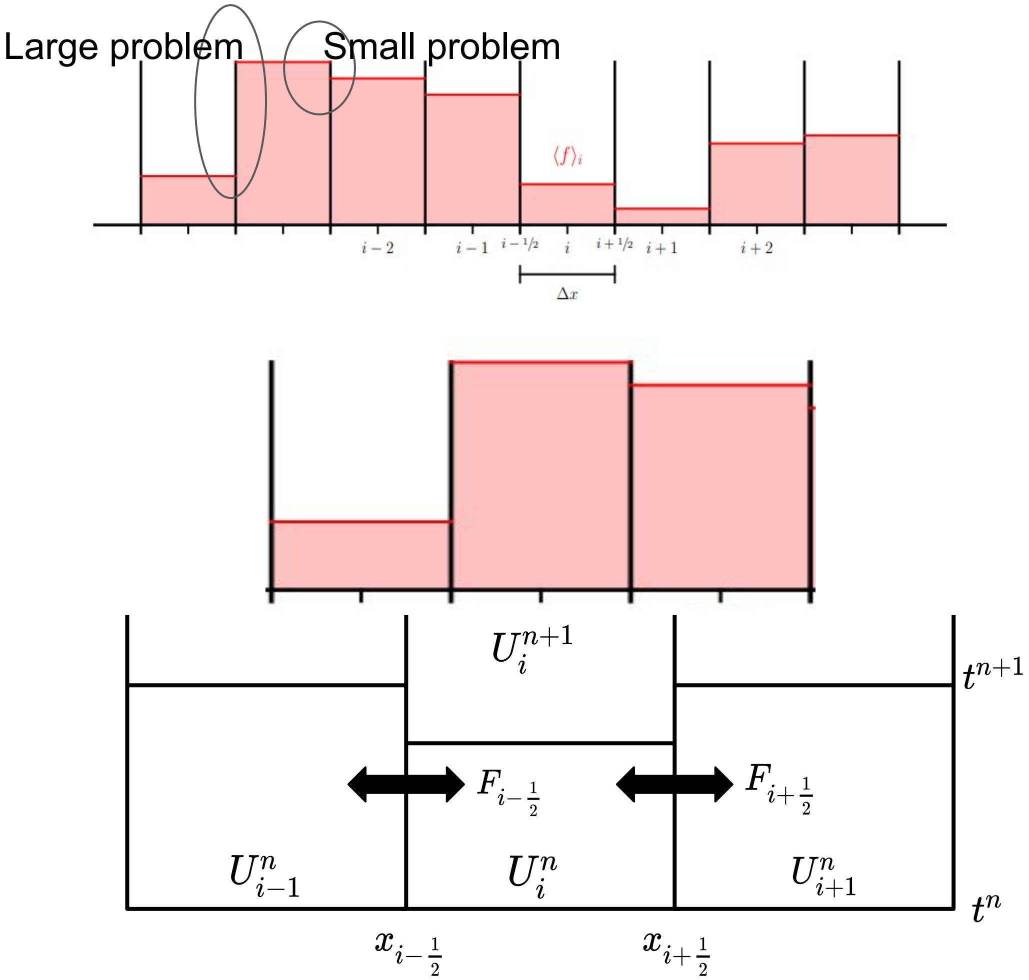
$$U_i^{n+1} = \frac{-\Delta t}{\Delta x}\left[F_{i+\frac{1}{2}}^{n+\frac{1}{2}} - F_{i-\frac{1}{2}}^{n-\frac{1}{2}}\right] + U_i^n$$

THE RIEMANN PROBLEM

At the interface there will be a jump.
How do we calculate flux at the interface ?

For flux evaluation at half time, we need information of state U at half time

$$[F_{i+\frac{1}{2}}^{n+\frac{1}{2}}] = f(U_{i+\frac{1}{2}}^{n+\frac{1}{2}})$$

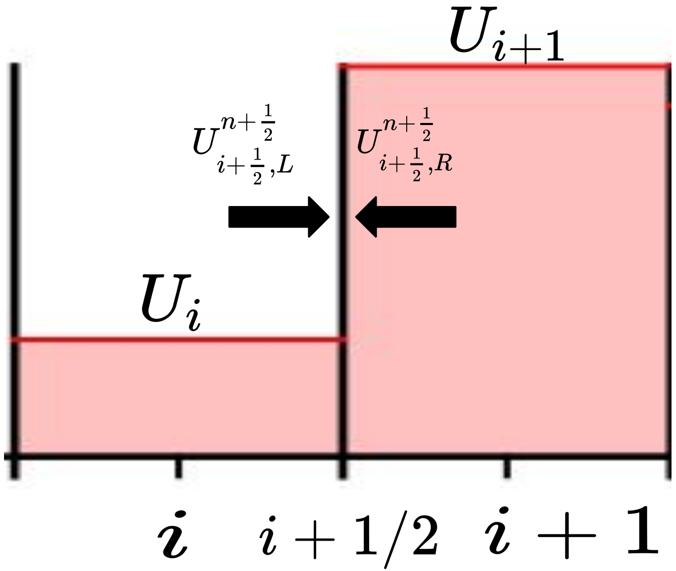


THE RIEMANN PROBLEM

Two states separated by a discontinuity.

This is called a Riemann problem.

Solution to Riemann problem results in single state at interface.

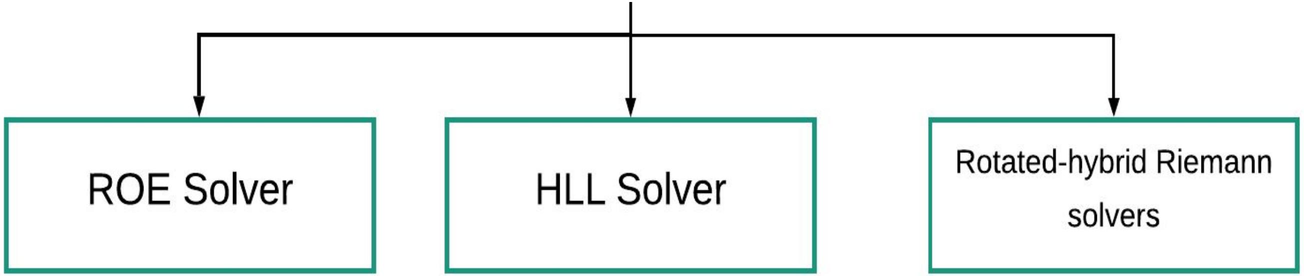


$$U^{n+\frac{1}{2}}_{i+\frac{1}{2}} = R\left(U^{n+\frac{1}{2}}_{i+\frac{1}{2},L}, U^{n+\frac{1}{2}}_{i+\frac{1}{2},R}\right)$$

APPROXIMATE RIEMANN SOLVERS

$$U_{i+\frac{1}{2}}^{n+\frac{1}{2}} = R\left(U_{i+\frac{1}{2},L}^{n+\frac{1}{2}}, U_{i+\frac{1}{2},R}^{n+\frac{1}{2}}\right)$$

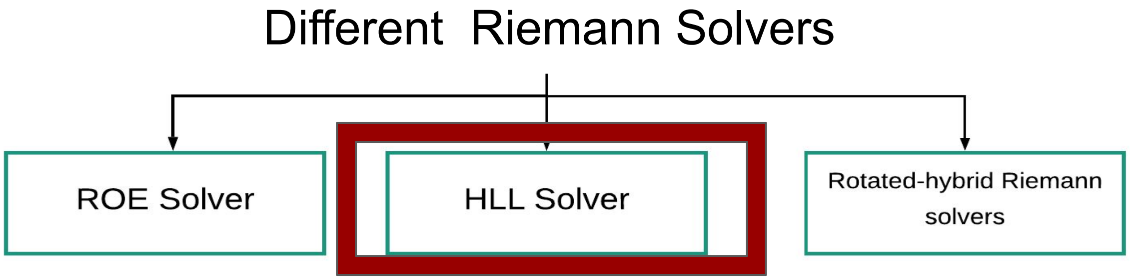
Different Riemann Solvers (Approximate solvers)



The exact solution of Riemann problem at every interface is very expensive!

Use approximate Riemann solver instead!

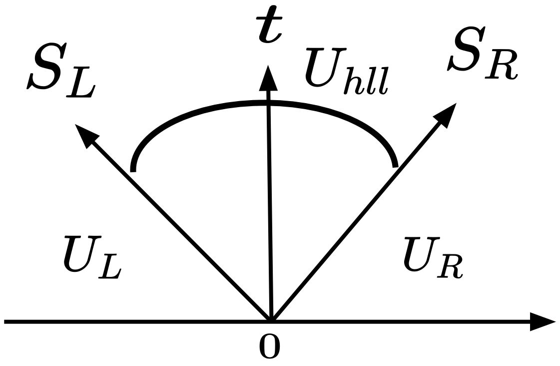
HLL RIEMANN SOLVER



Harten, **L**ax and van **L**eer

Solution is computed from two wave speeds S_L and S_R

If we have algorithm to track these wave speeds an approximate intercell flux can be calculated from it.



$$F^{hll}_{i+\frac{1}{2}} = \begin{cases} F_L & \text{if} \quad 0 \leq S_L \\ \frac{S_R F_L - S_L F_R + S_L S_R (U_R - U_L)}{S_R - S_L} & \text{if} \quad S_L \leq 0 \leq S_R \\ F_R & \text{if} \quad 0 \geq S_L \end{cases}$$

We have a new problem while discretizing the time.

CFL CONDITION

When we discretize the time, the step must be less than the time it takes for the information to propagate across a single zone.

This is called **CFL**(Courant–Friedrichs–Lewy) condition

$$\Delta t \leq \frac{\Delta x}{v}$$

$$C = \frac{v \Delta t}{\Delta x}$$

$$C \leq 1$$

Necessary condition for stability

SHOCK TUBE OR SOD PROBLEM IN 1D

Gary A.Sod (1978)

Commonly used problem to test accuracy of CFD codes using Riemann Solver.

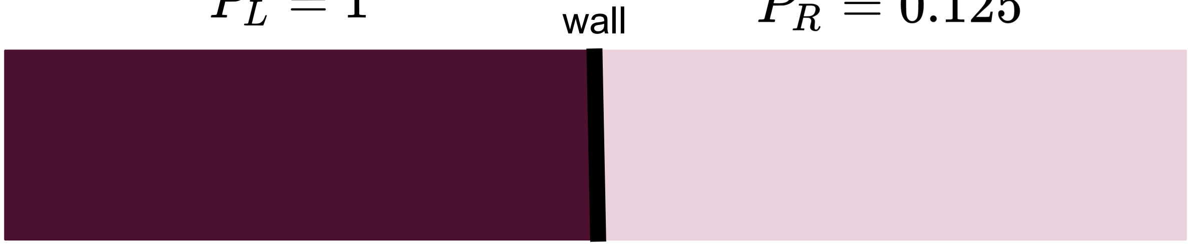
Initial Condition

$$v_L = 0 \qquad v_R = 0$$

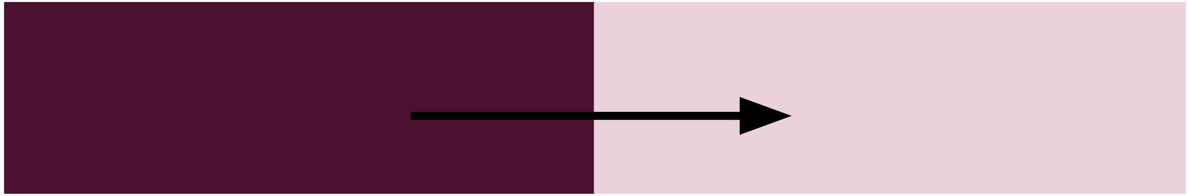
$$\rho_L = 1 \qquad \rho_R = 0.1$$

$$P_L = 1 \qquad P_R = 0.125$$

1. The fluid (gas) is initially at rest separated by a wall



2. The sudden breakdown of the wall generates a high-speed flow resulting a shock wave, which propagates to the right



THE ALGORITHM

EQUATION IN CONSERVED FORM

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + P)}{\partial x} = 0$$

$$\frac{\partial e}{\partial t} + \frac{\partial ((e + P) v)}{\partial x} = 0$$

U

$$\left\{ \begin{array}{c} \rho \\ \rho v \\ \rho e + \frac{1}{2} \rho v^2 \end{array} \right\}$$

Conserved quantities

F

$$\left\{ \begin{array}{c} \rho v \\ \rho v^2 + P \\ (\rho e + \frac{1}{2} \rho v^2 + P) v \end{array} \right\}$$

Fluxes

P

$$\left\{ \begin{array}{c} \rho \\ v \\ e \end{array} \right\}$$

Physical quantities

Discretize physical domain by setting **grid cells**

↓

Set initial conditions for physical quantities

↓

Reconstruct
Calculation left step P_L and P_R right state of physical quantity

↓

Calculate flux (from P_R and P_L , F_L and F_R are calculated)

↓

Convert physical quantity to conserved quantity **(P to U)**

↓

Solve the **Reimann Problem**

↓

Check **CFL condition**

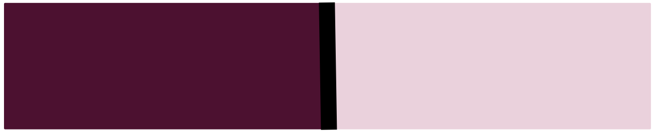
↓

UPDATE IN TIME

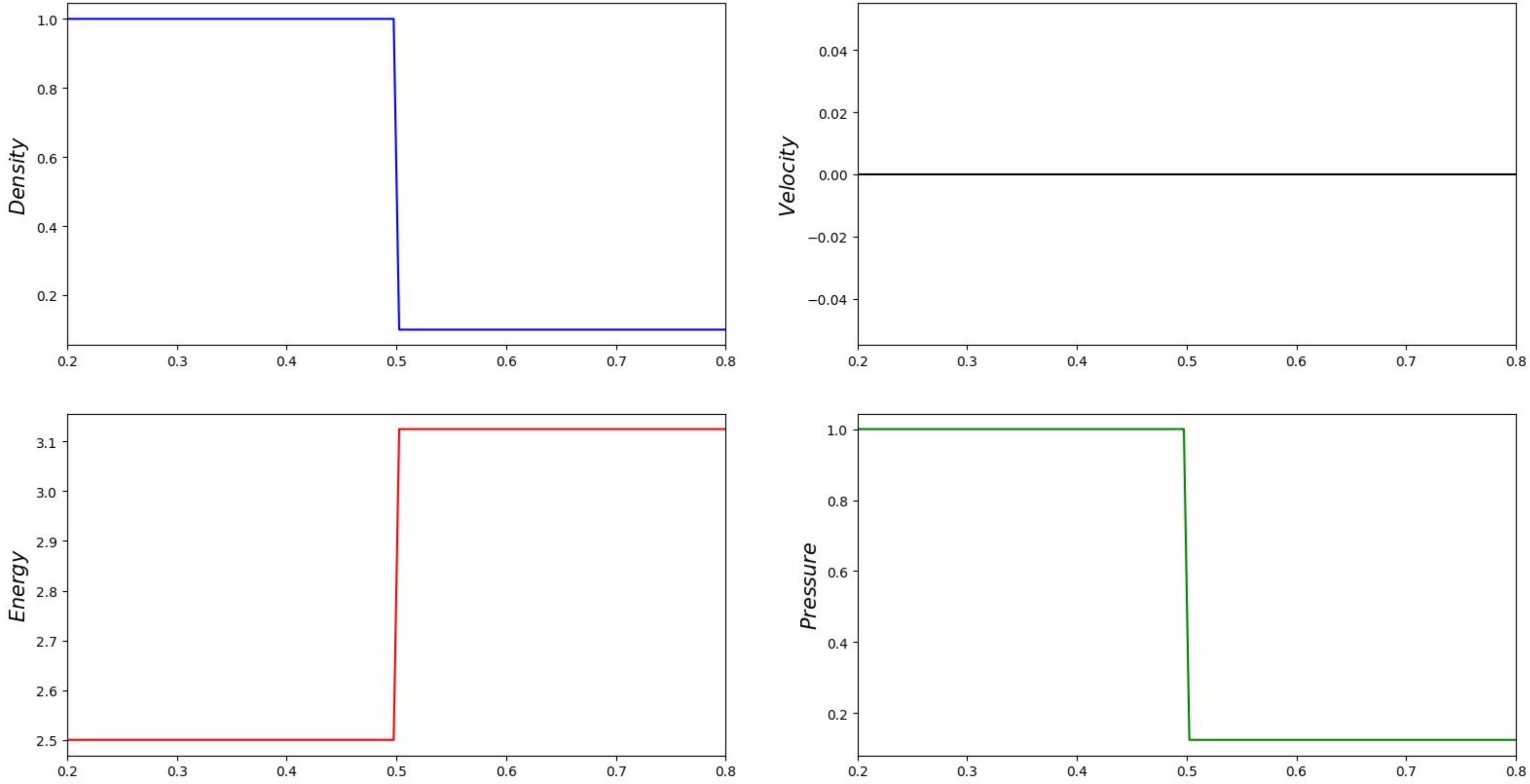
↓

Convert conserved quantity to physical quantity **(U to P)**

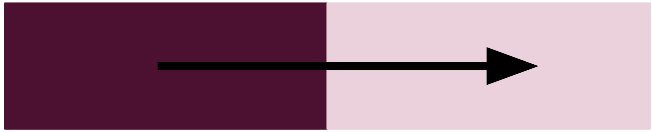
INITIAL CONDITION



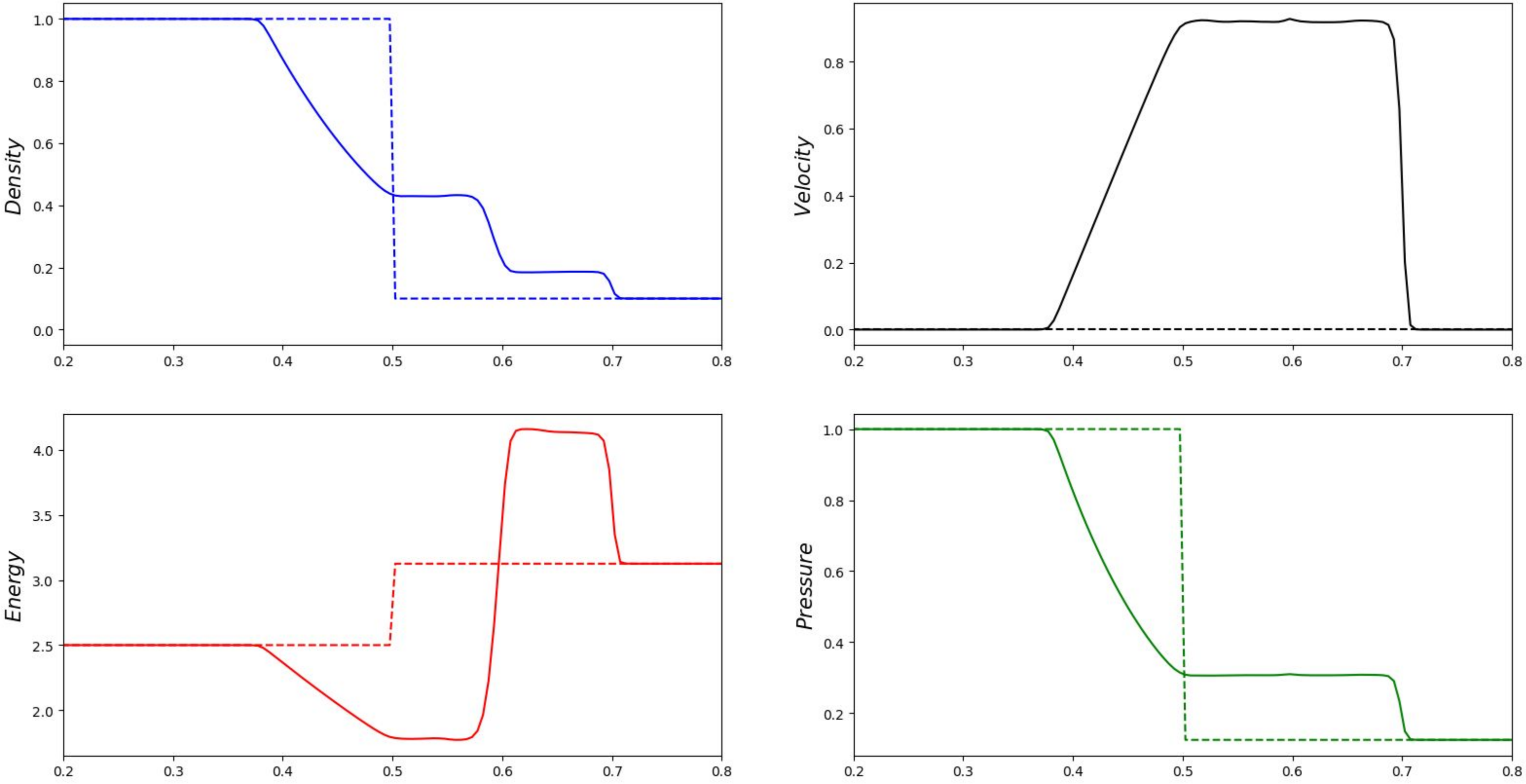
t = 0



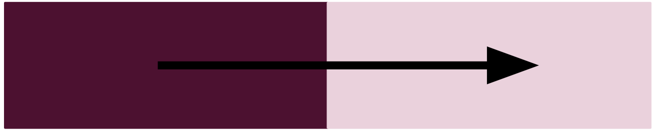
TIME EVOLUTION



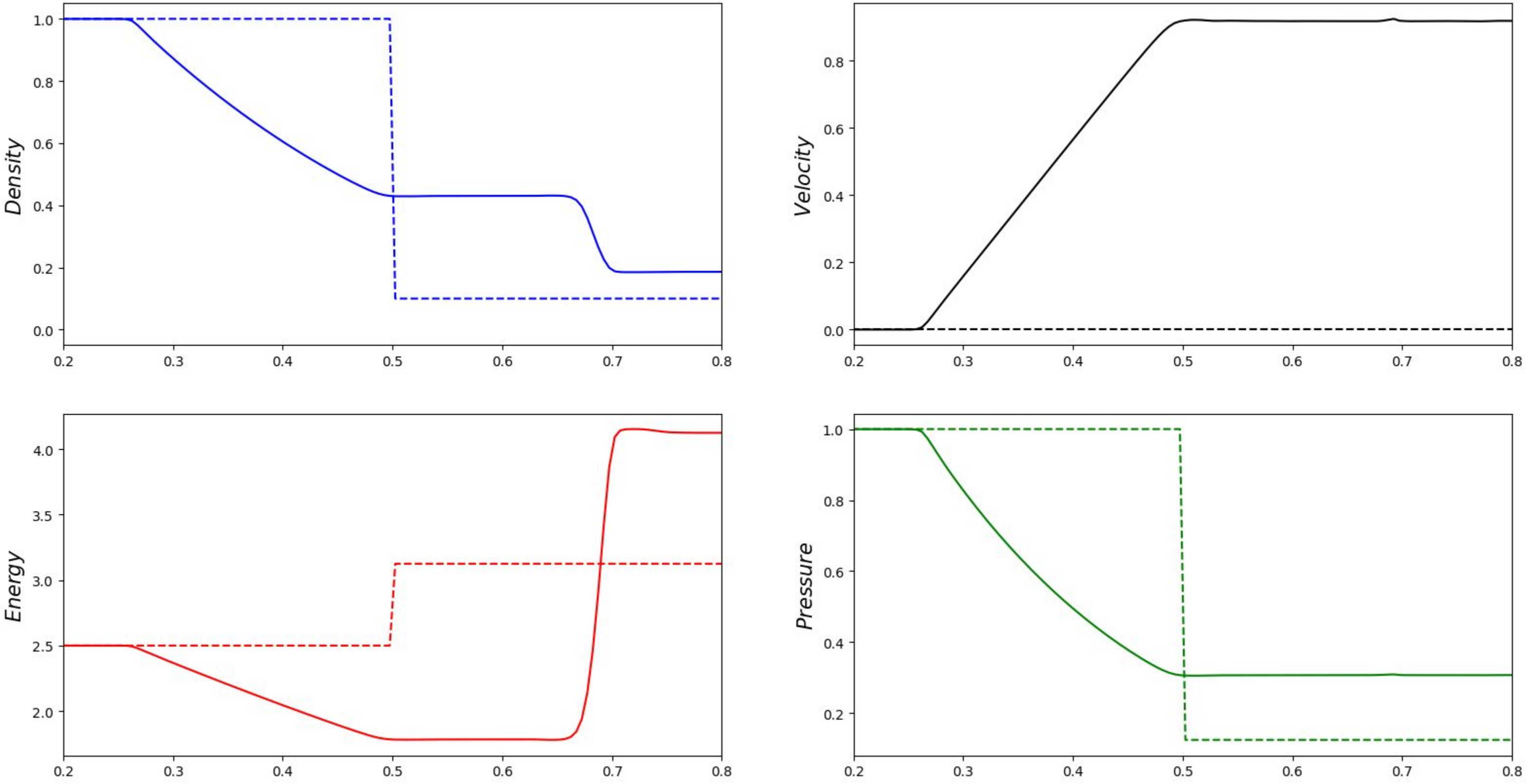
t = 0.1



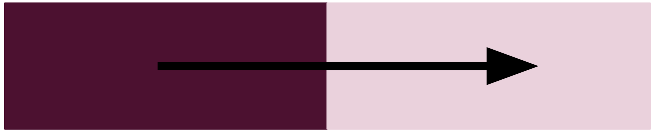
TIME EVOLUTION



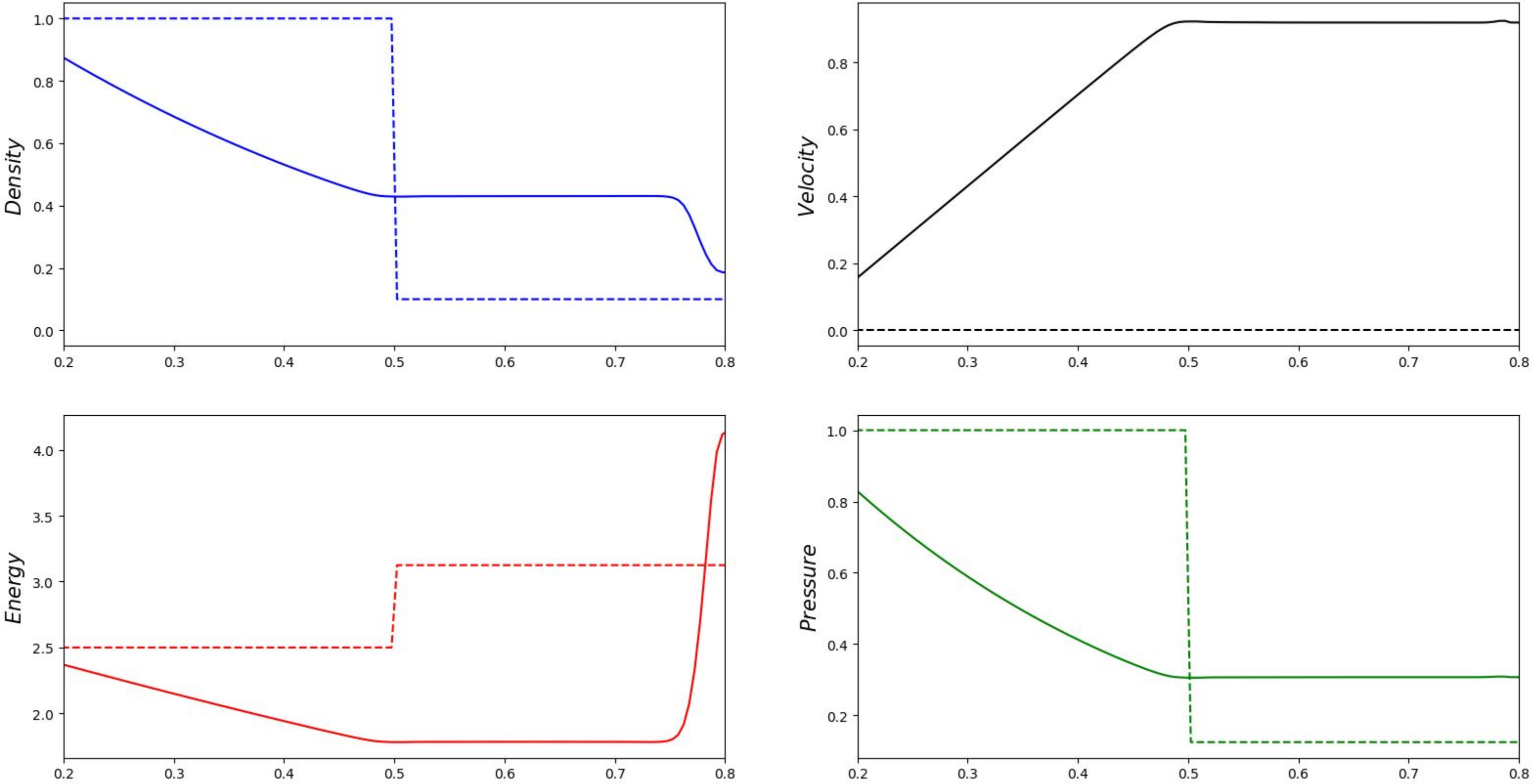
t = 0.2



TIME EVOLUTION



t = 0.3



Why do we do simulation in Astrophysics?

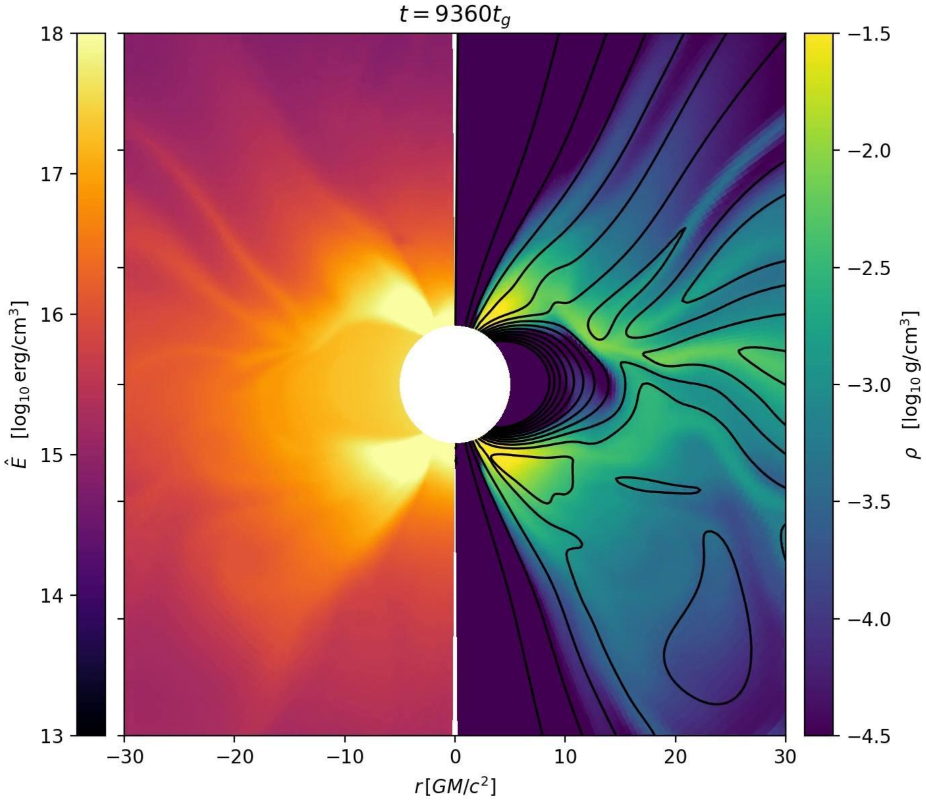
SIMULATION IN ASTROPHYSICS

Simulation enables us to build a model of a system

It allows us to do virtual experiments to understand how this system reacts to a range of conditions and assumptions

General **R**elativistic **R**adiative **M**agneto**H**ydro**D**ynamics

Using **KORAL**



TAKE HOME MESSAGE :

- CFD enables us to predict fluid flow
- The fundamentals of CFD lie in solving the set of partial differential equations that describe the fluid flow (e.g. ***Navier-Stokes equation***)
- In Eulerian grid based approach, the physical domain is discretized into large number of cells
- In each of these cells, Navier-Stokes equations can be rewritten as algebraic equations
- These equations are then solved numerically
- At the end we get the complete description of flow throughout the domain

THANK YOU