Determination of the compositeness of a bound state from a and r_0

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Formalism of scattering amplitudes

Two channels, compositeness of one channel

Relationship to a and r_0

$$f = \frac{1}{k\cot\delta - ik} \approx \frac{1}{-\frac{1}{a} + \frac{1}{2}r_0 k^2 - ik}$$

$$\langle \mathbf{p}'|V|\mathbf{p}\rangle = V(\mathbf{p}', \mathbf{p}) = V\theta(q_{\max} - p')\theta(q_{\max} - p)$$

Bethe
Salpeter equation

$$\langle \mathbf{p}'|T|\mathbf{p}\rangle = T(\mathbf{p}', \mathbf{p})$$
$$= V(\mathbf{p}', \mathbf{p}) + i\int \frac{d^4q}{(2\pi)^4} \frac{V(\mathbf{p}', \mathbf{q})}{q^2 - m_1^2 + i\epsilon} \frac{T(\mathbf{q}, \mathbf{p})}{(P-q)^2 - m_2^2 + i\epsilon}$$

Integral equation

$$T(\mathbf{p}', \mathbf{p}) = \theta(q_{\max} - p')\theta(q_{\max} - p)T$$

$$T = V + VGT$$

Algebraic equation

$$G(s) = \int_{|\mathbf{q}|<q_{\max}} \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{w_1(\mathbf{q}) + w_2(\mathbf{q})}{2w_1(\mathbf{q})w_2(\mathbf{q})} \times \frac{1}{s - (w_1(\mathbf{q}) + w_2(\mathbf{q}))^2 + i\epsilon}$$

$$T = [1 - VG]^{-1}V$$

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In one channel

$T = 1/(V^{-1} - G)$ close to the pole $T = g^2 / (s - s_0)$

$g^2 = \lim (s - s_0)\, T$, for $s \rightarrow s_0$

If V is energy independent

$g^2 = \lim (s - s_0) / (V^{-1} - G) = 1 / (- \partial G / \partial s)$

Hence $- g^2\, \partial G / \partial s = 1$ This is the probability that the state is made of this channel

Two channel formalism $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & 0 \end{pmatrix}$

$$T_{11} = \frac{v_{11} + v_{12}^2 G_2}{1 - (v_{11} + v_{12}^2 G_2) G_1}$$

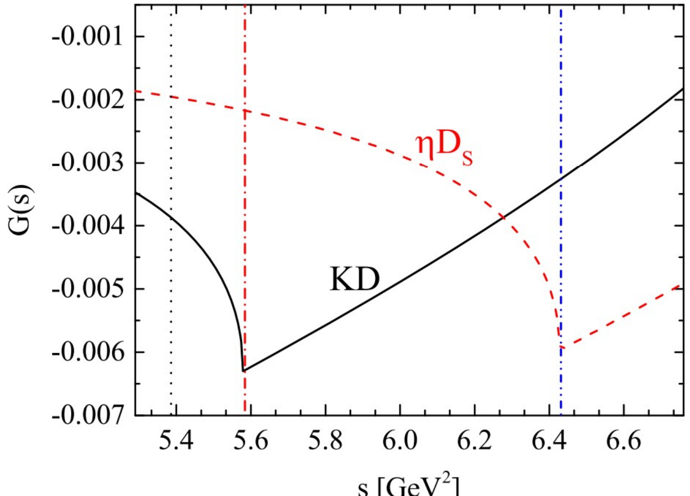
$$g_1^2 = \lim_{s - s_R} (s - s_R) T_{11}, \qquad g_2^2 = \lim_{s - s_R} (s - s_R) T_{22}$$

$$-g_1^2 \frac{\partial G}{\partial s}|_{s=s_R} - g_2^2 \frac{\partial G}{\partial s}|_{s=s_R} = P_1 + P_2 = 1$$

$$V_{\rm eff} = V_{11} + V_{12}^2 G_2$$

In the study of D_{s0}*(2317) with coupled channels

$$T_{\rm eff} = \frac{V_{\rm eff}}{1 - V_{\rm eff} G_1}$$



We have then eliminated channel 2 and work with only channel 1 with $V_{\rm eff}$

$$V_{\rm eff} = V_0 + \beta (s - s_0)$$

s_0 is the value of s for the bound state

$$T(s) = \frac{1}{[V_0 + \beta(s - s_0)]^{-1} - G(s)}$$

For s_0 T must have a pole:

$$V_0^{-1} - G(s_0) = 0, \qquad V_0 = \frac{1}{G(s_0)}$$

$$T(s) = \frac{1}{\left[\frac{1}{G(s_0)} + \beta(s - s_0)\right]^{-1} - G(s)}$$

$$f = \frac{1}{k\cot\delta - ik} \approx \frac{1}{-\frac{1}{a} + \frac{1}{2}r_0k^2 - ik}$$

$$\mathrm{Im}G = -\frac{1}{8\pi}\frac{k}{\sqrt{s}}$$

We have two parameters: β and q_{max}

$$8\pi\sqrt{s}\left\{\left[\frac{1}{G(s_0)} + \beta(s - s_0)\right]^{-1} - \mathrm{Re}G(s)\right\} + ik$$
$$\approx \frac{1}{a} - \frac{1}{2}r_0k^2 + ik$$

We try to get them from *a* and r_0

$$8\pi\sqrt{s_{\text{th}}}\left\{\left[\frac{1}{G(s_0)}+\beta(s_{\text{th}}-s_0)\right]^{-1}-\text{Re}G(s_{\text{th}})\right\}=\frac{1}{a}$$

$$\frac{1}{2\sqrt{s_{\text{th}}}}8\pi\left[\left[\frac{1}{G(s_0)}+\beta(s_{\text{th}}-s_0)\right]^{-1}\right.$$
$$\left.-\text{Re}G(s)_{\text{th}}\right]\frac{s}{w_1(k)w_2(k)}|_{s_{\text{th}}}$$
$$+8\pi\sqrt{s_{\text{th}}}\left[-\beta\left[\frac{1}{G(s_0)}+\beta(s_{\text{th}}-s_0)\right]^{-2}\right.$$
$$\left.-\frac{\partial\text{Re}[G(s)]}{\partial s}|_{s_{\text{th}}^{+}}\right]\frac{s}{w_1(k)w_2(k)}|_{s_{\text{th}}}=-\frac{1}{2}r_0$$

$$\beta=\frac{1}{s_{\text{th}}-s_0}\left\{\left[\frac{1}{a}\frac{1}{8\pi}\frac{1}{\sqrt{s_{\text{th}}}}+\text{Re}G(s_{\text{th}})\right]^{-1}-\frac{1}{G(s_0)}\right\}$$

$$g^2 = \lim_{s-s_0} (s - s_0) T(s)$$

$$= \lim_{s-s_0} \frac{s - s_0}{[\frac{1}{G(s_0)} + \beta(s - s_0)]^{-1} - G(s)}$$

$$= \frac{1}{-(\frac{1}{G(s_0)})^{-2}\beta - \frac{\partial G}{\partial s}|_{s_0}} = \frac{1}{-G(s_0)^2\beta - \frac{\partial G}{\partial s}|_{s_0}}$$

$$R_0 \equiv -\frac{1}{2\sqrt{s_{\rm th}}} 16\pi \left[\left[\frac{1}{G(s_0)} + \beta(s_{\rm th} - s_0) \right]^{-1} - {\rm Re}G(s_{\rm th}) \right] \frac{s}{w_1(k) w_2(k)} |_{s_{\rm th}} + 16\pi\sqrt{s_{\rm th}} \left[\beta \left[\frac{1}{G(s_0)} + \beta(s_{\rm th} - s_0) \right]^{-2} + \frac{\partial {\rm Re}[G(s)]}{\partial s} |_{s_{\rm th}^+} \right] \frac{s}{w_1(k) w_2(k)} |_{s_{\rm th}} = r_0$$

$$T_{11} = [1 - V_{\rm eff} G_1]^{-1} V_{\rm eff} = \frac{1}{V_{\rm eff}^{-1} - G_1}$$

$$V_{\rm eff} = \frac{1}{G(s_0)} + \beta(s - s_0)$$

L'Hôpital rule

$$g_1^2 = \lim_{s - s_R} (s - s_R) T_{11} = \frac{1}{\frac{\partial V_{\rm eff}^{-1}}{\partial s}|_{s_R} - \frac{\partial G}{\partial s}|_{s_R}}$$

and hence

$$-g_1^2 \frac{\partial G_1}{\partial s}|_{s_R} = P_1,$$

$$g_1^2 \frac{\partial V_{\rm eff}^{-1}}{\partial s}|_{s_R} = -g_1^2 \frac{1}{V_{\rm eff}^2} \frac{\partial V_{\rm eff}}{\partial s} = 1 - P_1 \equiv P_2 \equiv Z$$

$$P_2 = 1 - P_1 = Z = -g^2 G(s_0)^2 \beta$$

Weinberg formulas assuming zero range interaction and small binding

X_W is P_1 for p n scattering

$$a = R\left[\frac{2X_W}{1+X_W} + O\left(\frac{R_{\rm typ}}{R}\right)\right]$$

$$r_0 = R\left[-\frac{1-X_W}{X_W} + O\left(\frac{R_{\rm typ}}{R}\right)\right]$$

$$X_W = \frac{a}{2R-a} + O\left(\frac{R_{\rm typ}}{R}\right)$$

$$X_W = \frac{R}{R-r_0} + O\left(\frac{R_{\rm typ}}{R}\right)$$

Deuteron case:

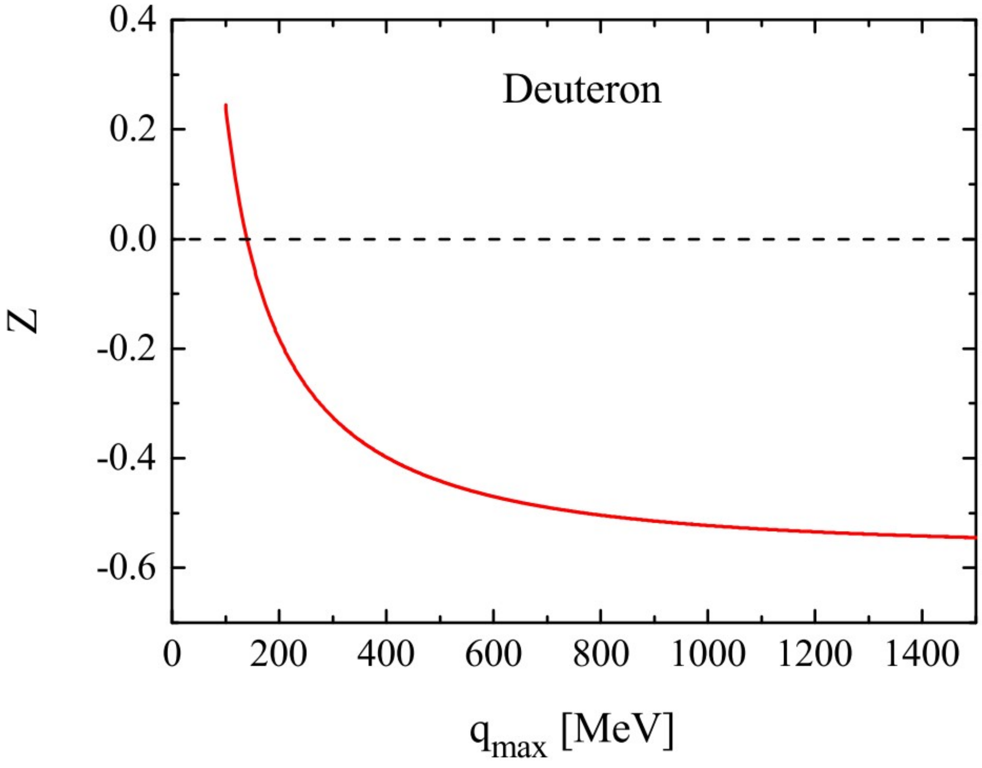
$$a = 5.419(7)\,\text{fm},$$
$$r_0 = 1.766(8)\,\text{fm},$$
$$B = 2.224575(9)\,\text{MeV}$$

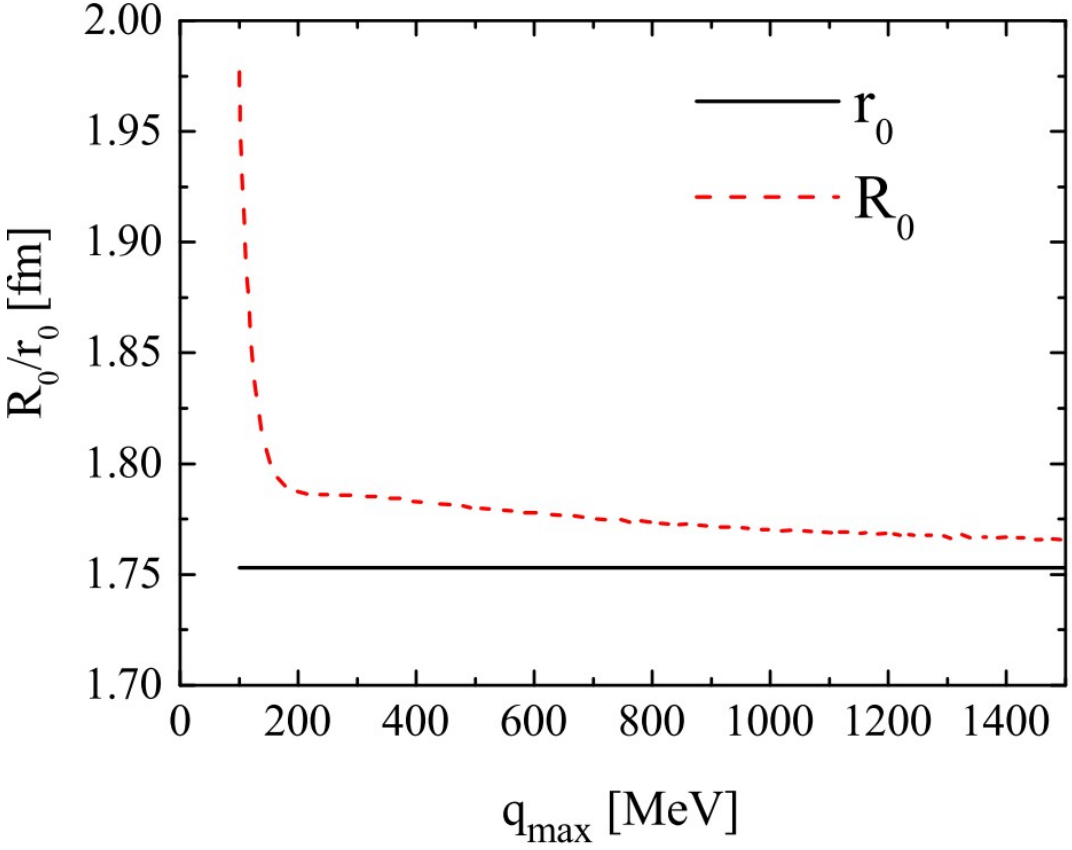
$$X_W = 1.68 \quad \text{from}$$
$$X_W = 1.69 \quad \text{from}$$

Too bad!!

"The true token
that the deuteron is composite is that r_0 is small and positive
rather than large and negative" S. Weinberg

Our results





Z must be positive and R_0 as close to r_0 as possible: q_{max} = ≈150 MeV , Z≈ 0

The $D_{s0}^*(2317)$

A. Martínez-Torres, E. Oset, S. Prelovsek, A. Ramos, JHEP 05, 153 (2015)

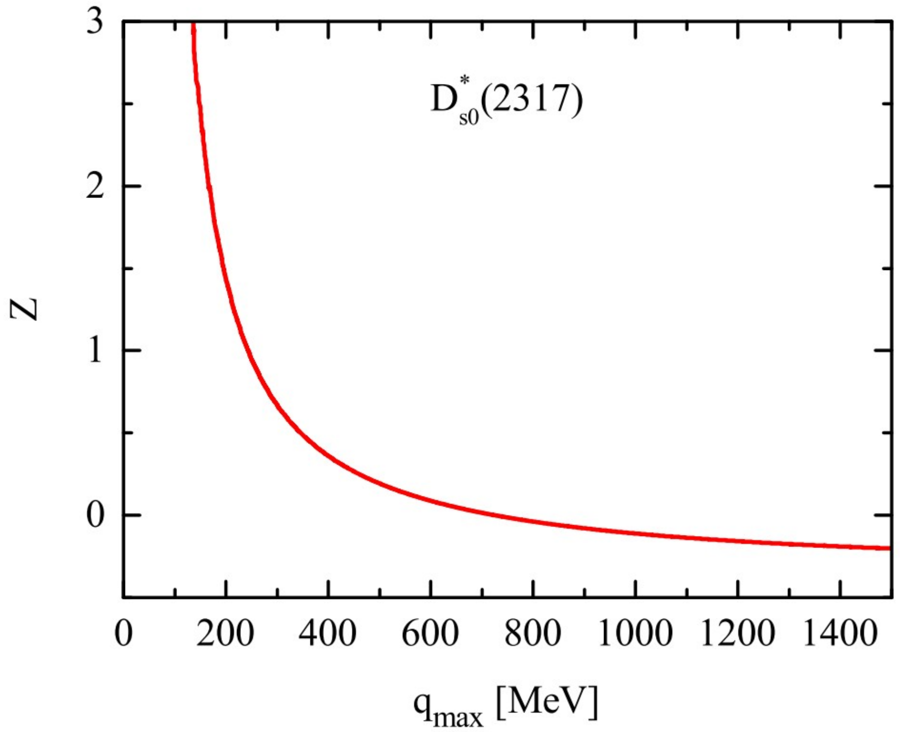
Lattice QCD

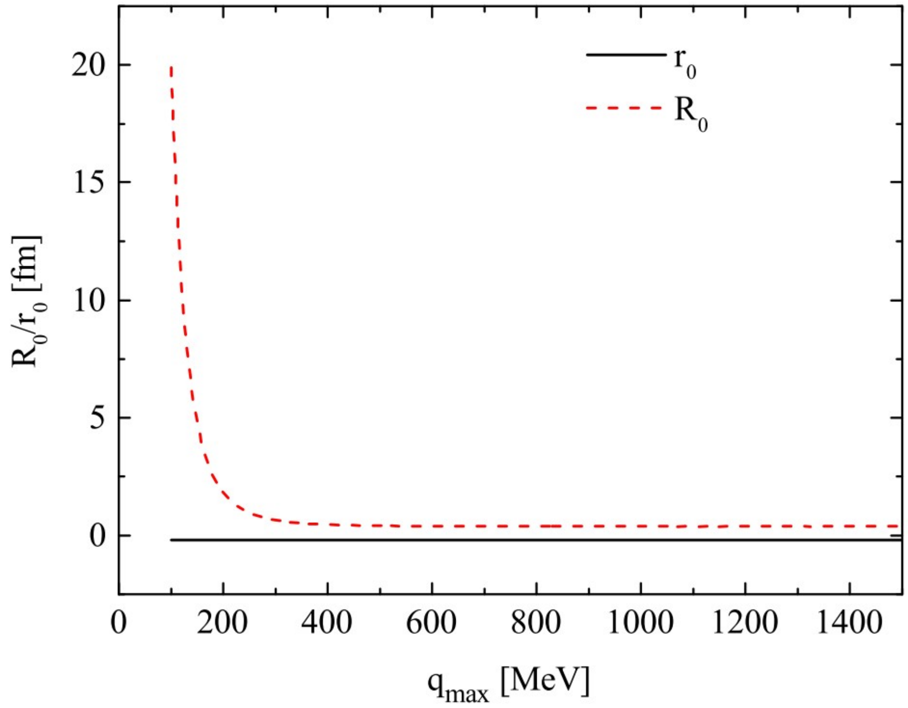
$$a(KD) = +1.3 \pm 0.5 \pm 0.1 \text{ fm}$$

$$r_0(KD) = -0.1 \pm 0.3 \pm 0.1 \text{ fm}$$

$$P(DK) = (72 \pm 13 \pm 5)\%$$

Less relevant channel η D_s





Z should be around or bigger than 400 MeV → Z≈0.2-0.3

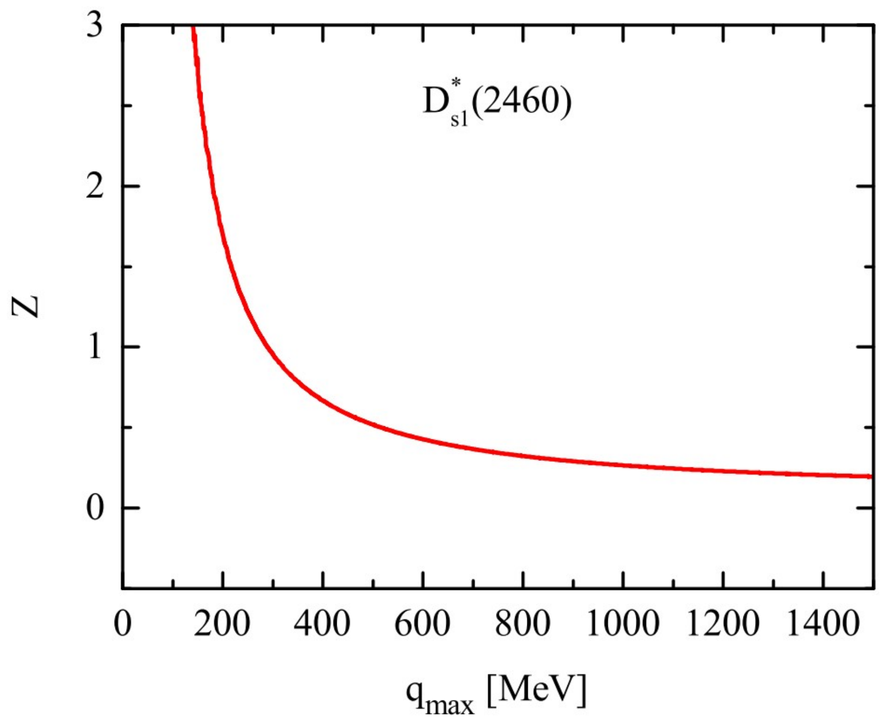
The $D_{s1}^*(2460)$

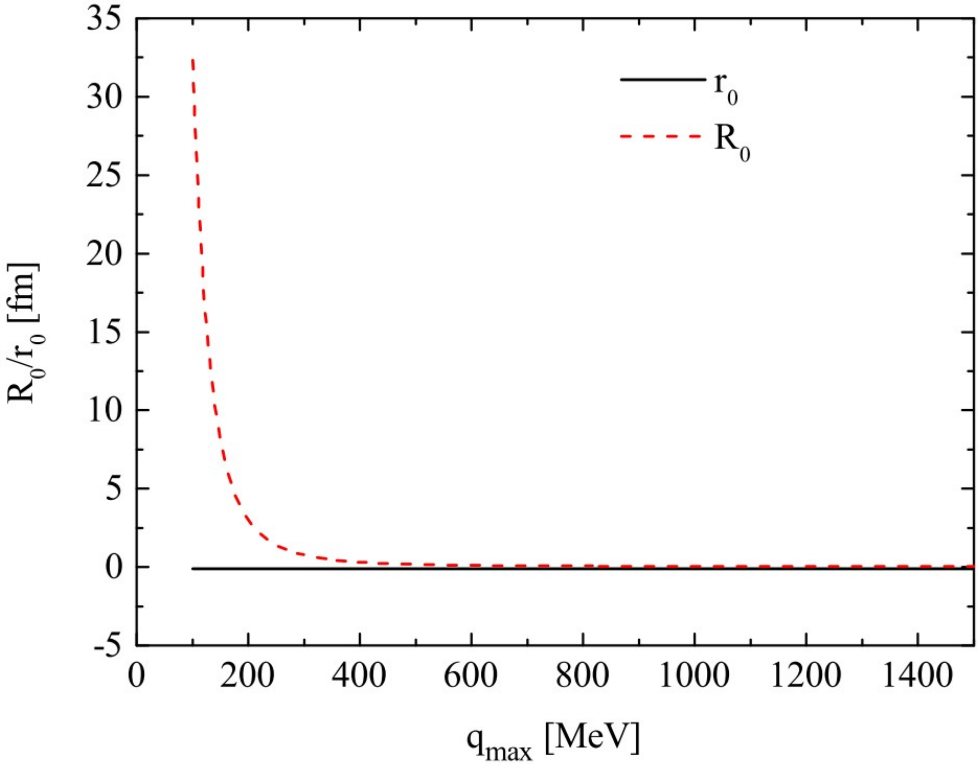
$$a(KD^*) = +1.1 \pm 0.5 \pm 0.2 \text{ fm}$$

$$r_0(KD^*) = -0.2 \pm 0.3 \pm 0.1 \text{ fm}$$

$$P(KD^*) = (57 \pm 21 \pm 6)\%$$

Less relevant channel ηD_s^*





q_{max} around or bigger than 400 MeV → Z smaller than 0.5-0.6

Conclusions

Weinberg formula for the compositeness, X=1-Z , obtained from *a* and r_0 is valid for zero range and small binding. No so good for the deuteron, even if used as a proof that the deuteron is a molecular state.

We consider the range of the interaction explicitly and find a more reliable prediction, valid for finite range and no so small binding.

From the knowledge of *a* and r_0 we determine approximately the compositeness and the range of the interaction at the same time.

This is an great improvement over the formula of Weinberg

M.~Albaladejo and J.~Nieves,
%``Compositeness of S-wave weakly-bound states from next-to-leading order Weinberg\
textquoteright{}s relations,''
Eur. Phys. J. C \textbf{82}, no.8, 724 (2022)

T.~Kinugawa and T.~Hyodo,
%``Structure of exotic hadrons by a weak-binding relation with finite-range correction,''
Phys. Rev. C \textbf{106}, no.1, 015205 (2022)