NANJING UNIVERSITY

EUROPEAN CENTRE FOR THEORETICAL STUDIES IN NUCLEAR PHYSICS AND RELATED AREAS

Semileptonic decays of heavy+light and heavy+heavy mesons

Zhao-Qian Yao

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Nov 8, 2022
in Seville

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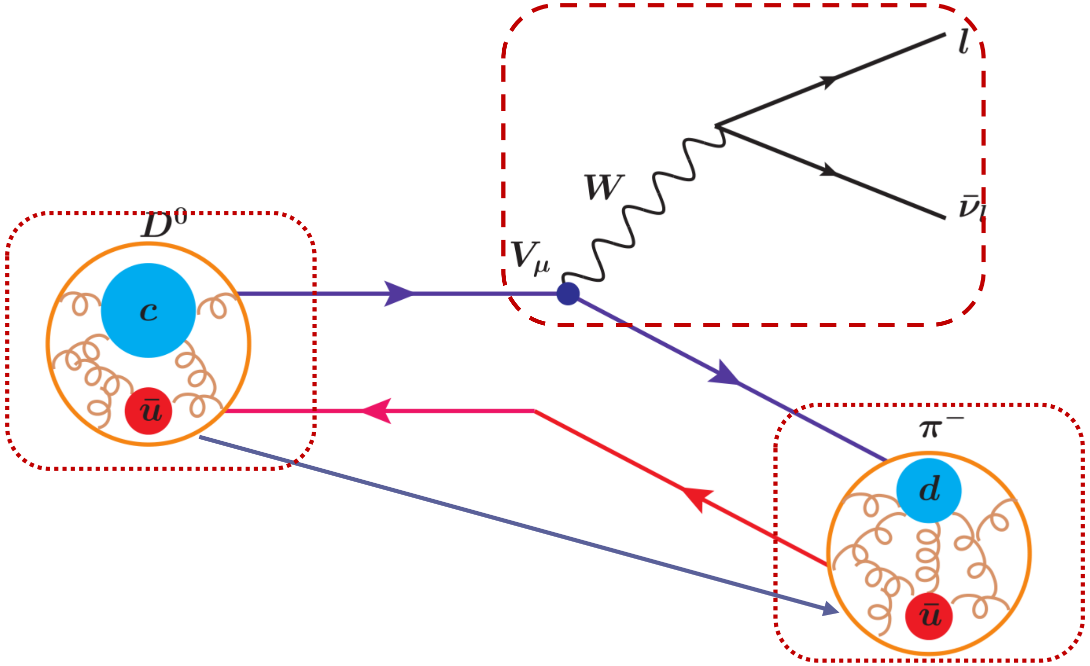
Semileptonic decays

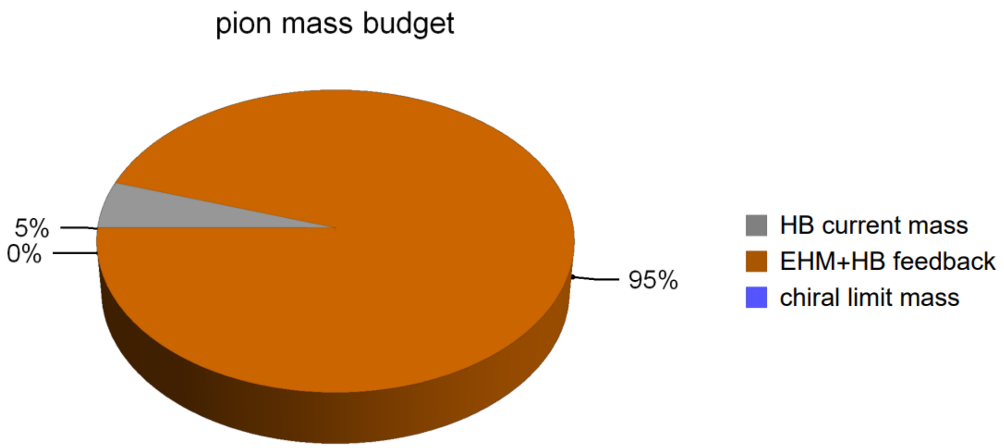


Studies of semileptonic decays

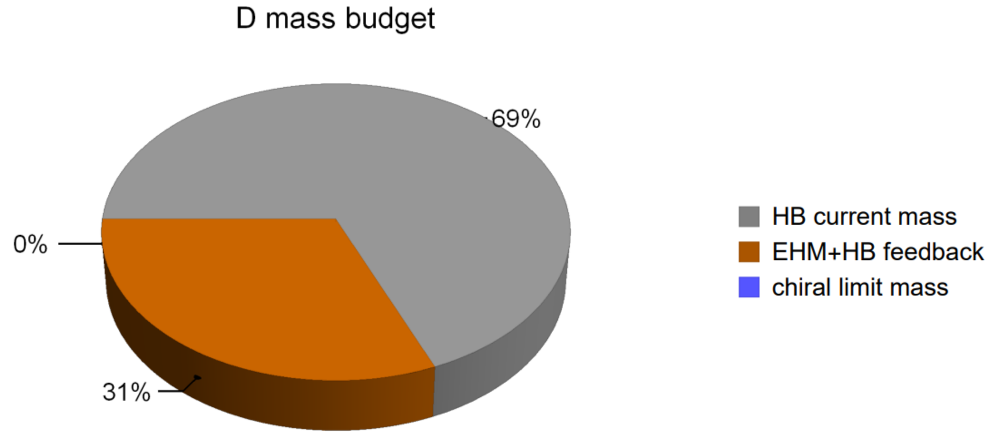
- are exploited to determine the magnitudes of fundamental parameters of the Standard Model, e.g., the Cabibbo-Kobayashi-Maskawa (CKM) matrix.
- give people a platform to observe the internal structure of hadrons.
- proceeding from a heavy-light meson to a (pseudo)-Goldstone mode, analysis of these processes will provide fresh ways of revealing the interplay between Higgs-boson (HB) effects and emergent hadronic mass (EHM).

EHM with Semileptonic decay

- Pion - (pseudo)-Goldstone boson
  - The grey wedge shows that the sum of the valence-quark and -antiquark (HB) current-masses accounts for 5% of its measured mass. The orange part expresses the EHM+HB interference, which is responsible for 95% of the pion mass.

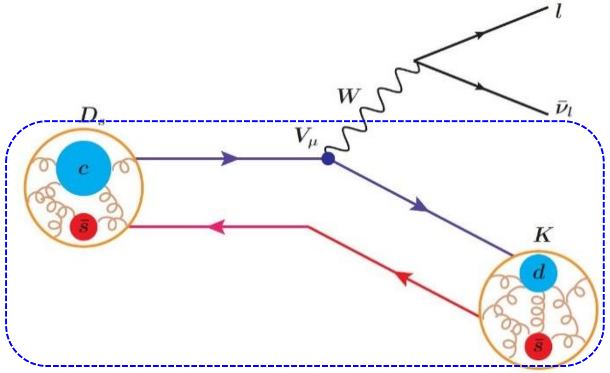


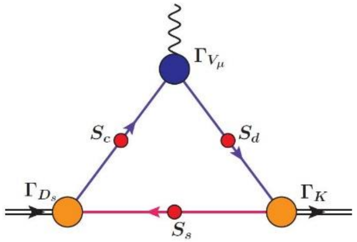
- D meson - heavy+light meson
  - HB current-masses accounts for 69% of its measured mass, which is 14 times more than in the pion.



- The decay of heavy+light meson to (pseudo)-Goldstone boson embodies the change of the dominated mechanism from the HB effects to the EHM.

Semileptonic decays of Mesons





$$
\begin{aligned}
M_\mu^{D_s^+}(P,Q) &= N_c \mathrm{tr}\int \frac{d^4q}{(2\pi)^4}\Gamma_{D_s}(q+p/2;p)S_c(q+p) \\
&\times i\Gamma_\mu^{cd}(q+p,q-k)S_d(q-k)\Gamma_K(q-k/2;-k)S_s(q)
\end{aligned}
$$

$S_{c,d,s}$: the dressed propagators of *d, s, c* quarks;

$\Gamma_{Ds,K}$: the amplitudes for the D_s and *K* mesons;

Γ_{V_μ} : the dressed-quark-W-boson vertex

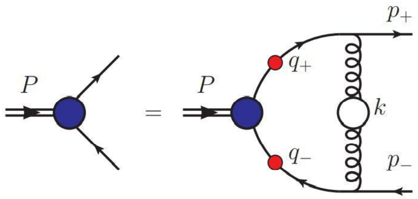
Continuum Schwinger function methods (CSMs):

- The dressed quark propagator is determined from the quark gap equation

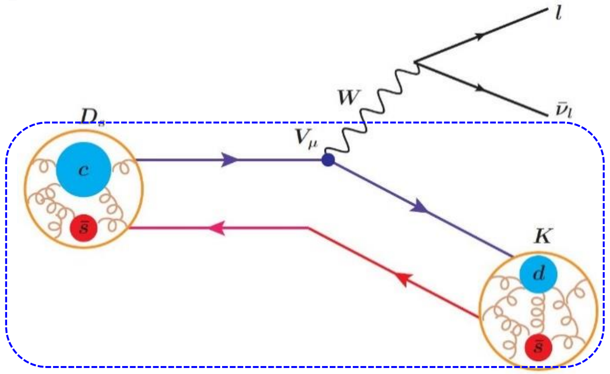
$$S^{-1}(p) = Z_2(i\gamma\cdot p + m_f^{bm}(\Lambda^2)) + \int_q^\Lambda g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu S(q)\Gamma_\nu^a(q,p)$$

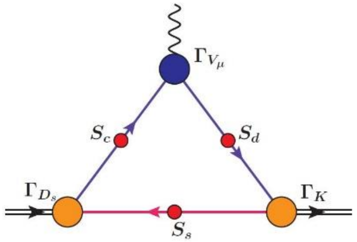
- Bound-State amplitude is determined from the homogeneous Bethe-Salpeter equation (BSE):

$$\Gamma_\pi(p;P) = \int_q^\Lambda K(p,q;P)S(q_+)\Gamma^{f\bar{g}}(q;P)S(q_-)$$



Semileptonic decays of Mesons





$$\begin{aligned} M_\mu^{D_s^+}(P,Q) &= N_c \mathrm{tr} \int \frac{d^4 q}{(2\pi)^4} \Gamma_{D_s}(q+p/2;p) S_c(q+p) \\ &\times i\Gamma_\mu^{cd}(q+p, q-k) S_d(q-k) \Gamma_K(q-k/2;-k) S_s(q) \end{aligned}$$

$S_{c,d,s}$: the dressed propagators of *d, s, c* quarks;

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A	m_π	f_π	m_K	f_K	m_ρ	m_ρ
herein	0.138	0.093	0.494	0.110	0.751	0.15
PDG	0.138	0.092	0.494	0.110	0.78	0.15
mean	0.138	0.092	0.494	0.110	0.766	0.15
B	m_{η_c}	f_{η_c}	$m_{J/\psi}$	$f_{J/\psi}$	m_{B_c}	f_{B_c}
herein	2.98	0.28	3.12	0.30	6.27	0.43
PDG	2.98	0.24	3.10	0.29	6.27	
mean	2.98	0.26	3.11	0.29	6.27	

M. Tanabashi et al., Phys. Rev. D 98, 030001 (2018). (All quantities in GeV.)

Semileptonic decays of heavy-light mesons

Phys.Lett.B 824 (2022) 136793 • e-Print: 2111.06473

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PHYSICS LETTERS B

Semileptonic transitions: $B_{(s)} \to \pi(K)$; $D_s \to K$; $D \to \pi, K$; and $K \to \pi$

Check for updates

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A R T I C L E   I N F O

Article history:
Received 16 November 2021
Accepted 22 November 2021
Available online 25 November 2021
Editor: A. Ringwald

Keywords:
Heavy-quark mesons
Semileptonic decays
CKM matrix elements
Emergence of hadron mass
Nambu-Goldstone bosons
Schwinger function methods

A B S T R A C T

Continuum Schwinger function methods for the strong-interaction bound-state problem are used to arrive at a unified set of parameter-free predictions for the semileptonic $K \to \pi$, $D \to \pi, K$ and $D_s \to K$, $B_{(s)} \to \pi(K)$ transition form factors and the associated branching fractions. The form factors are a leading source of uncertainty in all such calculations: our results agree quantitatively with available data and provide benchmarks for the hitherto unmeasured $D_s \to K^0$, $\bar{B}_s \to K^+$ form factors. The analysis delivers a value of $|V_{cs}| = 0.974(10)$ and also predictions for all branching fraction ratios in the pseudoscalar meson sector that can be used to test lepton flavour universality. Quantitative comparisons are provided between extant theory and the recent measurement of $\mathcal{B}_{B_s^0 \to K^- \mu^+ \nu_\mu}$. Here, further, refined measurements would be useful in moving toward a more accurate value of $|V_{ub}|$.

Semileptonic decays of $D \to \pi$ and $D_s \to K$

- $D \to \pi$:
  - The $D \to \pi$ transition form factors:

  data: cyan squares-Belle; green diamonds-CLEO; black stars-Babar; dark-blue up-triangles and indigo down-triangles - BESIII

  our prediction at $t = 0$: $f_+^{D_u^d}(0) = 0.673(09)$

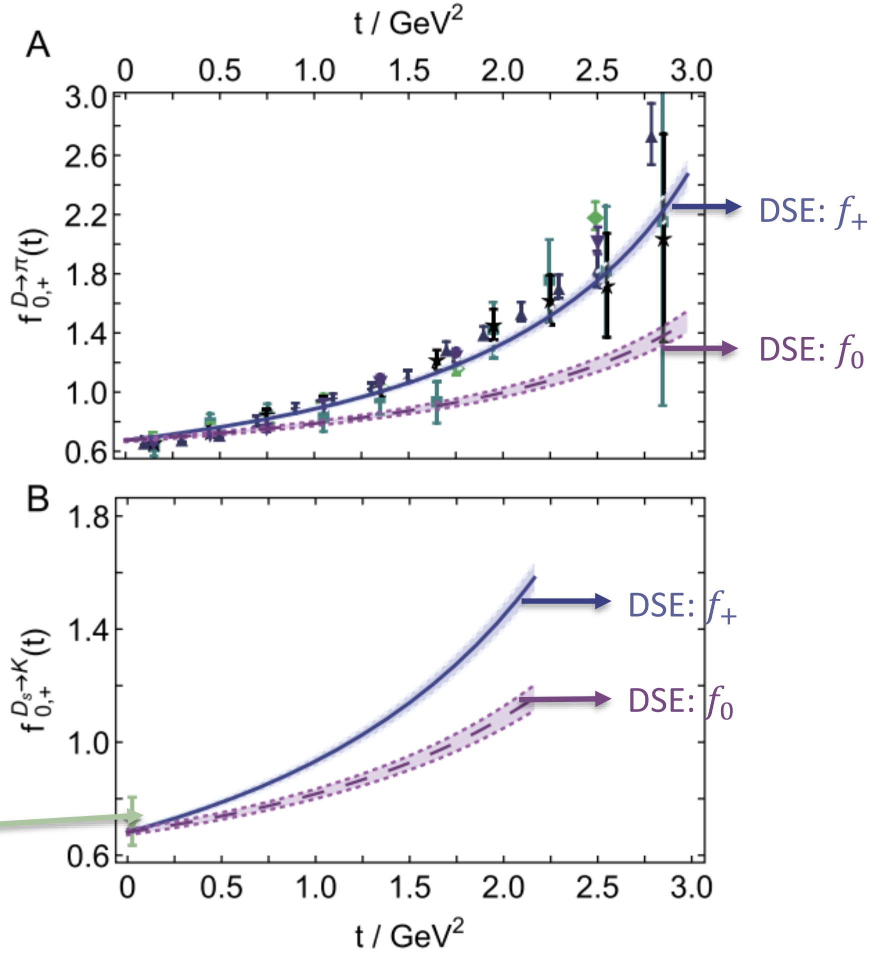
- $D_s \to K$:
  - The $D_s \to K$ transition form factors:

  our prediction at $t = 0$: $f_+^{D_s^d}(0) = 0.681(10)$

  No data are available for these form factors, except for the t = 0 data from BESIII

$$f_+^{D_s^d}(0) = 0.720 \pm 0.084_{\text{stat}} \pm 0.013_{\text{syst}}$$

  This value agrees with our prediction.



Semileptonic decays of $B \to \pi$ and $B_s \to K$

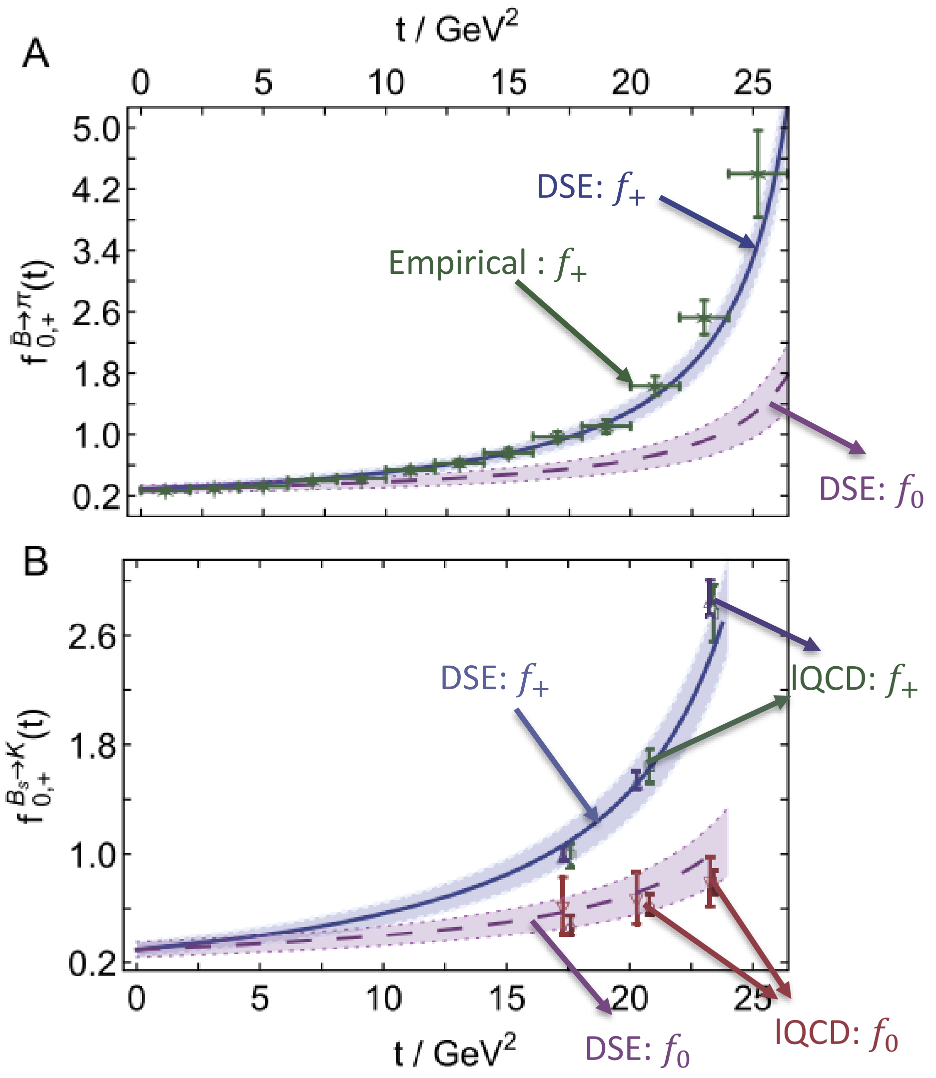
- $B \to \pi$:
  - The $B \to \pi$ transition form factors:

our prediction at $t = 0 : f_+^{\bar{B}_u^d}(0) = 0.29(5)$

Empirical result: $f_+^{\bar{B}_u^d}(0) = 0.27(2)$

- $B_s \to K$:
  - The $B_s \to K$ transition form factors:

our prediction at $t = 0 : f_+^{\bar{B}_s^d}(0) = 0.293(55)$



Branching fractions

- Branching fractions:

	Our work			Experimental results		
	$e^+\nu_e$	$\mu^+\nu_\mu$	ratio	$e^+\nu_e$	$\mu^+\nu_\mu$	ratio
$K^+ \to \pi^0$	50.0(9)	33.0(6)	0.665	50.7(6)	33.5(3)	0.661(07)
$D^0 \to \pi^-$	2.70(12)	2.66(12)	0.987(02)	2.91(4)	2.67(12)	0.918(40)
$D_s^+ \to K^0$	2.73(12)	2.68(12)	0.982(01)	3.25(36)		
$D^0 \to K^-$	39.0(1.7)	38.1(1.7)	0.977(01)	35.41(34)	34.1(4)	0.963(10)
		$\tau^-\bar{\nu}_\tau$	ratio	$\mu^-\bar{\nu}_\mu$	$\tau^-\bar{\nu}_\tau$	ratio
$\bar{B}^0 \to \pi^+$	0.162(44)	0.120(35)	0.733(02)	0.150(06)		
$\bar{B}_s^0 \to K^+$	0.186(53)	0.125(37)	0.667(09)			

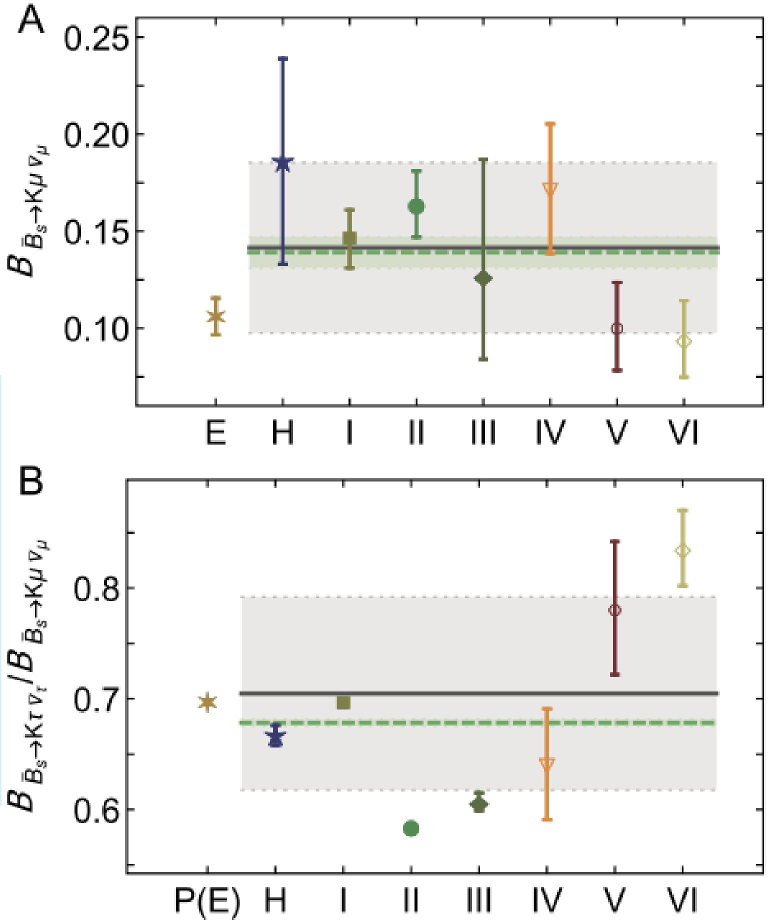
- For D $\to$ K: Using the value of $\boxed{|V_{cs}|=0.987(11)}$ in PDG, both fractions exceed PDG values; but the ratio agrees within 1.4σ. The value $\boxed{|V_{cs}| = 0.937(17)}$ combined with our form factors delivers branching fractions in agreement with the PDG values.
- Our inferred value is both a match for and more precise than one of the two used by the PDG to arrive at the average 0.987(11). Using $\boxed{|V_{cs}| = 0.937(17)}$ instead to compute this average, one finds

$$\boxed{|V_{cs}| = 0.974(10)}$$

Empirical results:

R. Aaij et al. (LHCb Collaboration)
Phys. Rev. Lett. 120 (2018) 12, 121801

$$\mathcal{B}_{B_s^0\to K^-\mu^+\nu_\mu} = [0.106 \pm 0.005_{\text{stat}} \pm 0.008_{\text{syst}}] \times 10^{-3}$$



E empirical
H herein
I sum rules
II quark model
III sum rules
IV lQCD
V lQCD
VI lQCD

Semileptonic decays of heavy-light mesons

Phys.Lett.B 818 (2021) 136344 • e-Print: 2104.10261

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PHYSICS LETTERS B

Semileptonic $B_c \to \eta_c$, J/ψ transitions

Check for updates

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A R T I C L E I N F O

Article history:
Received 21 April 2021
Received in revised form 30 April 2021
Accepted 3 May 2021
Available online 12 May 2021
Editor: A. Ringwald

Keywords:
Heavy-quark mesons
Semileptonic decays
Charmonia
CKM matrix elements
Emergence of hadron mass
Schwinger function methods

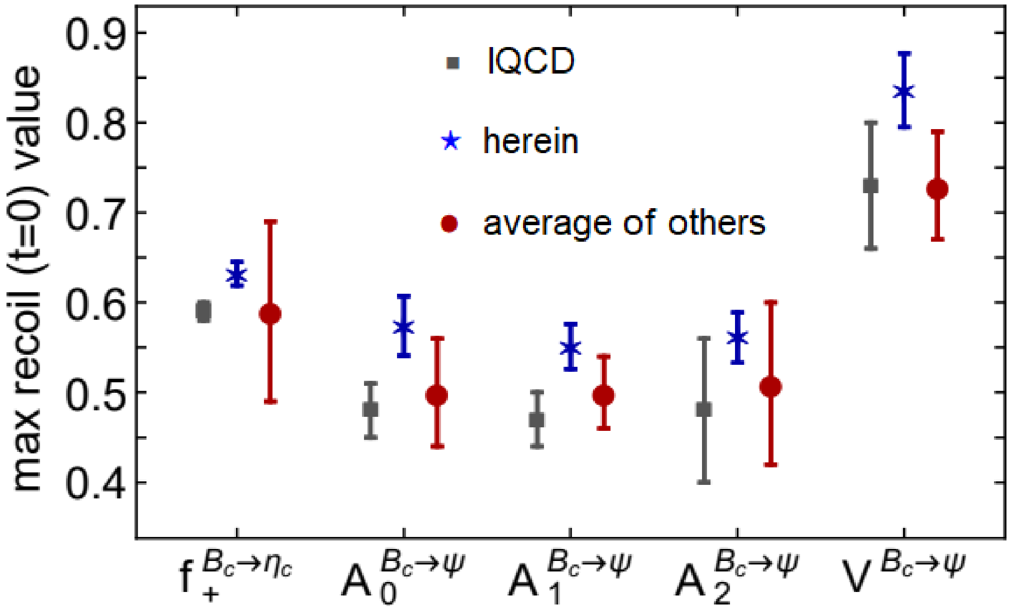
A B S T R A C T

Using a systematic, symmetry-preserving continuum approach to the Standard Model strong-interaction bound-state problem, we deliver parameter-free predictions for all semileptonic $B_c \to \eta_c$, J/ψ transition form factors on the complete domains of empirically accessible momentum transfers. Working with branching fractions calculated therefrom, the following values of the ratios for τ over μ final states are obtained: $R_{\eta_c} = 0.313(22)$ and $R_{J/\psi} = 0.242(47)$. Combined with other recent results, our analysis confirms a 2σ discrepancy between the Standard Model prediction for $R_{J/\psi}$ and the single available experimental result.

Semileptonic decay of B_c

Form factors for $B_c \to \eta_c$ and $B_c \to J/\psi$ at t = 0

	$f_+^{B_c\to\eta_c}$	$A_0^{B_c\to J/\psi}$	$A_1^{B_c\to J/\psi}$	$A_2^{B_c\to J/\psi}$	$V^{B_c\to J/\psi}$
herein	0.63(1)	0.57(3)	0.55(3)	0.56(3)	0.84(4)
QM	0.75	0.56	0.55	0.56	0.78
ph	0.56	0.48	0.46	0.49	0.70
SR	0.62(5)	0.54(4)	0.55(4)	0.35(3)	0.73(6)
mpQCD	0.56(7)	0.40(5)	0.47(5)	0.62(6)	0.75(9)
SE	0.41	0.46	0.48	0.54	0.63
lQCD	0.59(1)		0.49(3)		0.70(2)
lQCD		0.48(3)	0.47(3)	0.48(8)	0.73(7)
mean-e	0.58(11)	0.49(6)	0.50(4)	0.51(9)	0.72(5)
mean-i	0.59(10)	0.50(6)	0.50(4)	0.51(9)	0.73(6)



Semileptonic decay of $B_c \to \eta_c$

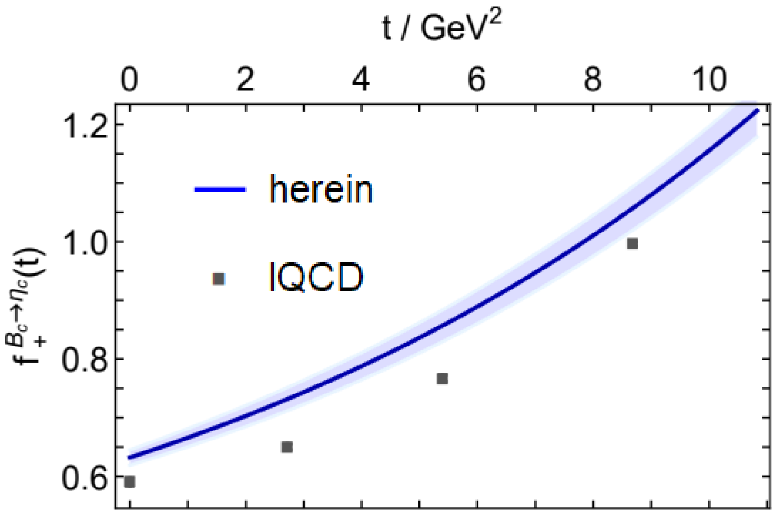
- Transition form factors

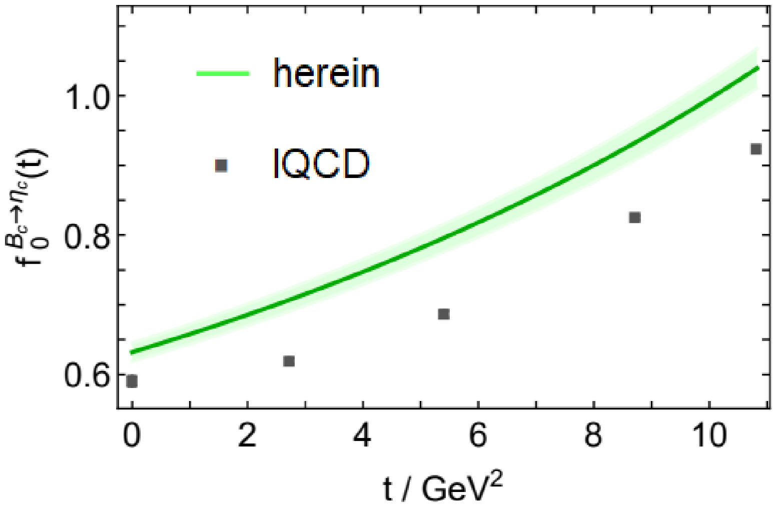
  The difference between our predictions and the lQCD results is 10(3)%.

- Branching fraction

  Working with our predictions of transition form factors, we obtain the branching fraction of $B_c \to \eta_c$

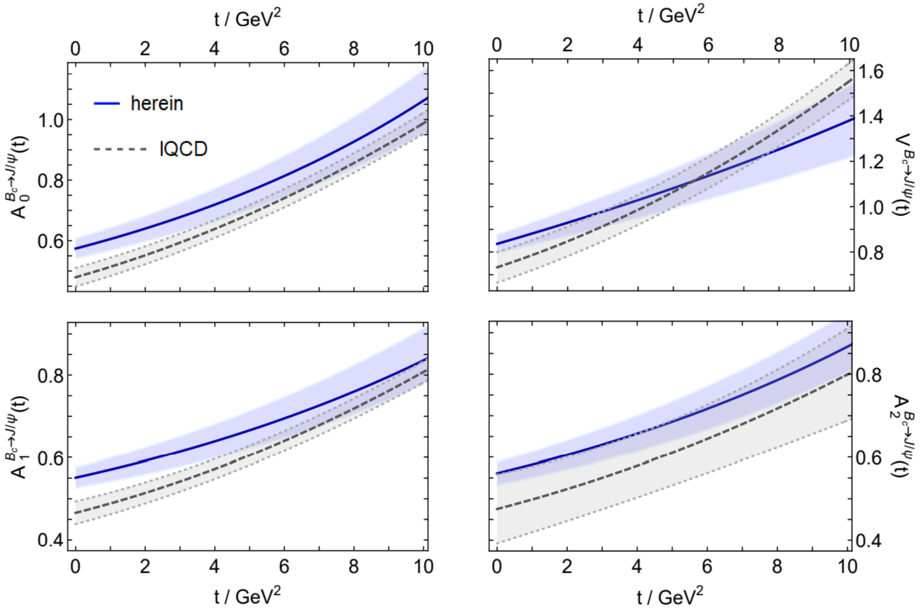
A	$\mathcal{B}_{B_c \to \eta_c \mu \nu_\mu}$	$\mathcal{B}_{B_c \to \eta_c \tau \nu_\tau}$	R_{η_c}
herein	8.10 (45) (55)	2.54(10)(17)	0.31(2)
QM [15,16]	9.5(1.9)	2.4(0.5)	0.25(7)
ph [17]	6.6(0.2)		0.31(1)
SR [18]	8.2(1.2)	2.6(0.6)	0.32(2)
mpQCD [19]	7.8(1.7)	2.4(0.4)	0.31(1)
iBS [20]	5.3(2.2)	2.2(0.7)	0.38(4)
mean-e	7.5(1.6)	2.4(0.2)	0.31(4)
mean-i	7.6(1.5)	2.4(0.2)	0.31(4)





Semileptonic decay of $B_c \to J/\psi$

- Transition form factors



- The $R_{J/\psi}$ from theory
  - Our result, $R_{J/\psi} = 0.24(5)$
  - lQCD result, $R_{J/\psi} = 0.258(4)$
  - The mean of other central values $R_{J/\psi} = 0.253(16)$

B	$\mathcal{B}_{B_c \to J/\psi \mu \nu_\mu}$	$\mathcal{B}_{B_c \to J/\psi \tau \nu_\tau}$	$R_{J/\psi}$
herein	17.2 (1.9) (1.2)	4.17(66)(28)	0.24(5)
QM [15,16]	16.7(3.3)	4.0(0.8)	0.24(7)
ph [17]	14.4(0.2)		0.26(1)
SR [18]	22.4(5.3)	5.3(1.5)	0.23(1)
mpQCD [19]	14.1(2.5)	3.8(0.6)	0.27(1)
iBS [20]	16.2(0.5)	4.3(0.1)	0.27(1)
lQCD [13,14]	15.0(1.1)(1.0)		0.258(4)
mean-e	16.5(3.1)	4.4(0.7)	0.25(2)
mean-i	16.6(2.8)	4.3(0.6)	0.25(2)

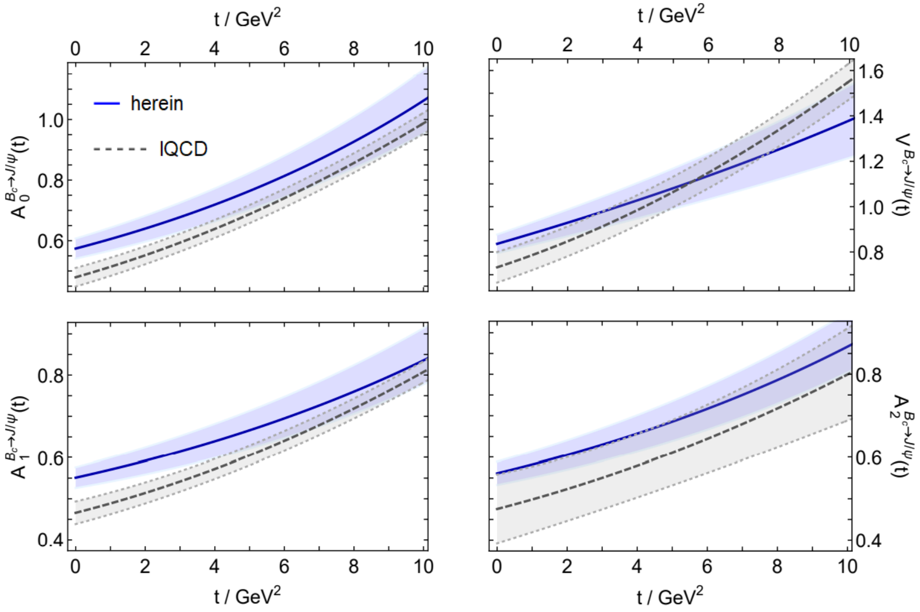
- The $R_{J/\psi}$ from LHCb collaboration

$$R_{J/\psi} := \frac{\mathcal{B}_{B_c^+ \to J/\psi \tau v}}{\mathcal{B}_{B_c^+ \to J/\psi \mu \nu}} = 0.71 \pm 0.17(\text{ stat }) \pm 0.18(\text{ syst })$$

  - This result lies approximately 2σ above the range of central values predicted by theoretical calculations.

Semileptonic decay of $B_c \to J/\psi$

- Transition form factors



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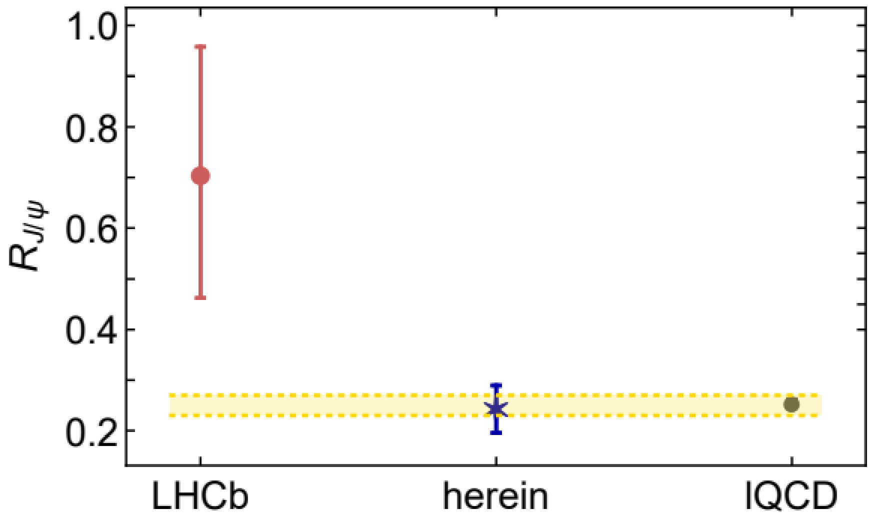


Lepton flavour universality

- Big issue in standard model: do τ(μ) leptons couple as μ(e) leptons do?

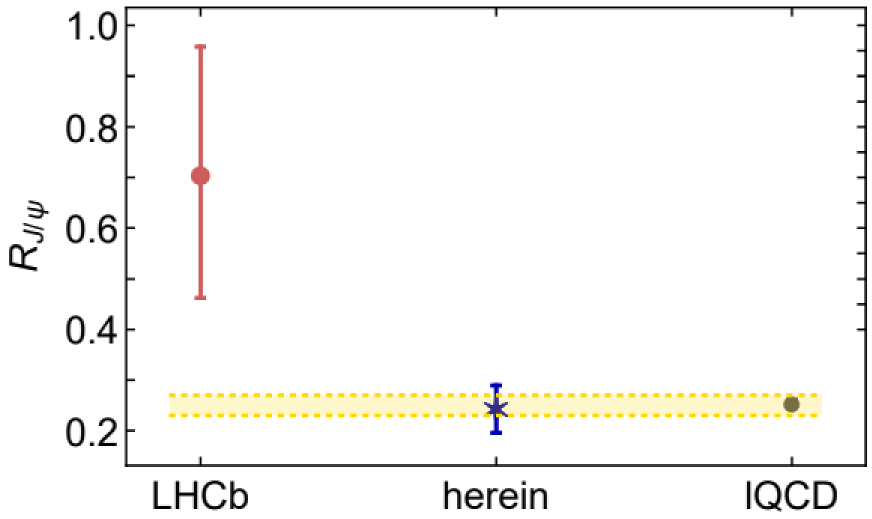
$$R_{D^{(*)}} := \frac{\mathcal{B}_{B\to D^{(*)}\tau v}}{\mathcal{B}_{B\to D^{(*)}lv}}$$

- Semileptonic decay of $B_c \to J/\psi$, the $R_{J/\psi}$ from LHCb collaboration

$$R_{J/\psi} := \frac{\mathcal{B}_{B_c^+\to J/\psi\tau v}}{\mathcal{B}_{B_c^+\to J/\psi\mu\nu}} = 0.71 \pm 0.17(\text{ stat }) \pm 0.18(\text{ syst })$$

LHCb: R. Aaij et al. Phys. Rev. Lett., 120(12):121801, 2018.
lQCD: Judd Harrison et al. *Phys.Rev.Lett.* 125 (2020) 22, 222003

- Such a discrepancy maybe a signal of the violation of lepton universality in Nature's weak interactions. However, the uncertainty of empirical information is too large to support such a claim.



Conclusions

- $K \to \pi$, $D \to \pi$, $D \to K$, $D_s \to K$, B→ π and $B_s \to$ K
  - We compare our results with extant theories and the recent measurements for these processes. Our predictions quantitatively agree with all these theories and the available measured transition form factors;
  - Provide predictions for the hitherto unmeasured $D_s \to K$, $B_s \to$ K form factors and predictions for five unmeasured branching fraction and ratios.
  - We also improve precision on the value of |Vcs|.

- $B_c \to \eta_c$, $B_c \to J/\psi$
  - Combined with other recent results, our analysis confirmed a 2σ discrepancy between the Standard Model prediction for $R_{J/\psi}$ and the single available experimental result.
  - The uncertainty of existing empirical information is too large to support the violation of lepton flavour universality.
  - Moreover, we predict R_{η_c}= 0.313(22), which agrees very well with the mean obtained from modern continuum analyses: 0.31(4).

Thank You!

CKM matrix

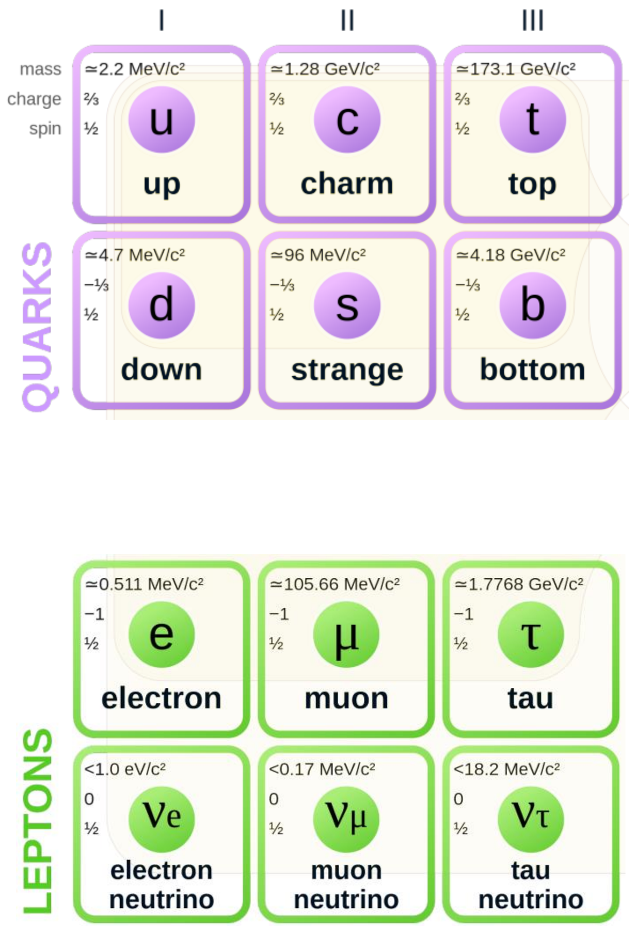
For the decay of a quark q with charge −1/3 to a quark q' with charge +2/3,

$$q \to q'W^-, W^- \to l^-\bar{\nu}_l$$

The coupling at the W vertex is proportional to $|V_{q'q}|$. The 3 × 3 matrix of these constants, which is known as the CKM matrix

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

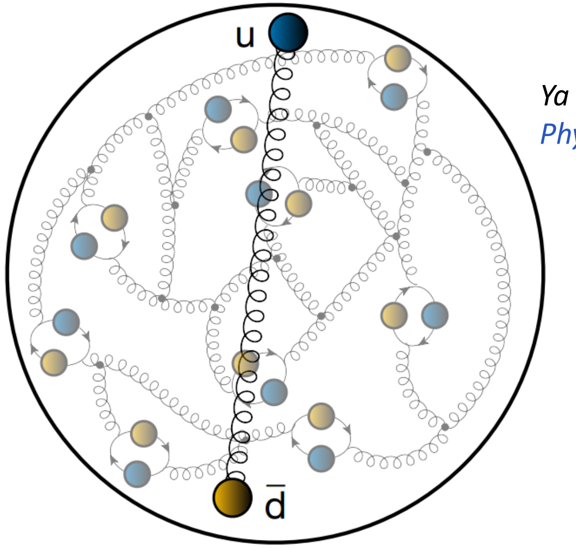
The determination of $|V_{ud}| = 0.97370(10)$, comes from β decay. For the other elements, the semileptonic decays of mesons play a crucial role to determine them.



Strong interaction

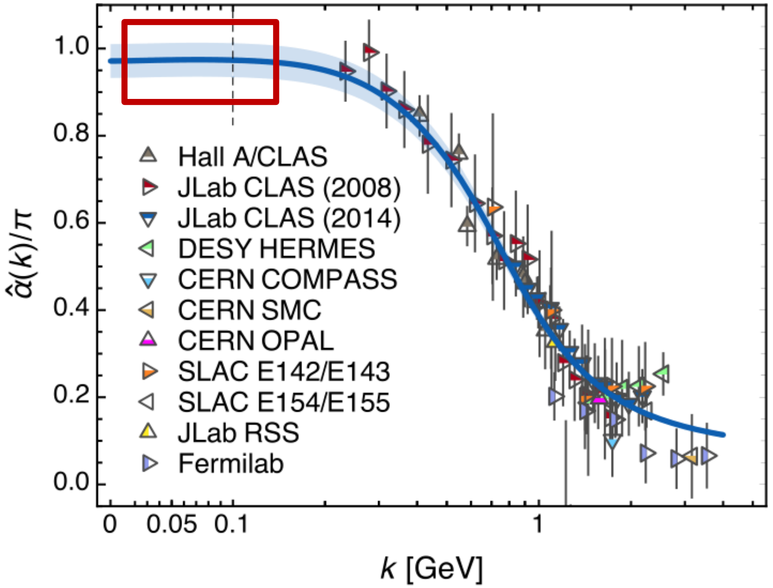
Pion contains one valence u-quark, one valence $\bar{d}$-quark, and infinitely many gluons and sea quarks. These quarks are bound together by the strong interactions, described by quantum chromodynamics (QCD).

These interactions are hard for people to make predictions with perturbative approaches due to the large coupling at the typical energy scales in low energy.



Ya Lu, et al
Phys.Lett.B 830 (2022) 137130

Z.-F. Cui *et al* 2020 *Chinese Phys. C* **44** 083102



QCD's Running Coupling

Dyson-Schwinger Equations

Continuum Schwinger function methods (CSMs) for the strong-interaction problem are used in our work.

Dyson-Schwinger equations (DSEs) are a practical, predictive, unifying tool for fundamental physics.

- Dressed quark propagator:

$$S(p) = 1/[i\gamma \cdot pA(p^2) + B(p^2)]$$

- The dressed quark propagator is determined from the quark gap equation

$$S^{-1}(p) = Z_2(i\gamma \cdot p + m_f^{bm}(\Lambda^2)) + \int_q^{\Lambda} g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu S(q)\Gamma_\nu^a(q,p)$$

Bethe-Salpeter Equations

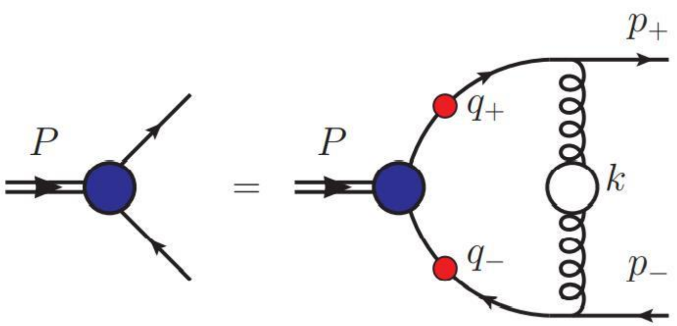
- Poincaré covariance entails that the bound-state (Bethe-Salpeter) amplitude for a pseudoscalar meson takes the form:

$$\Gamma_\pi(k;P) = \gamma_5[iE_\pi(k;P) + \gamma\cdot P F_\pi(k;P) + \gamma\cdot k G_\pi(k;P) + k_\mu P_\nu \sigma_{\mu\nu} H_\pi(k;P)]$$

  - $P^2 = -m^2$ is total momentum of system, m is the meson mass. (Euclidean space)
  - k is relative momentum between valence quark

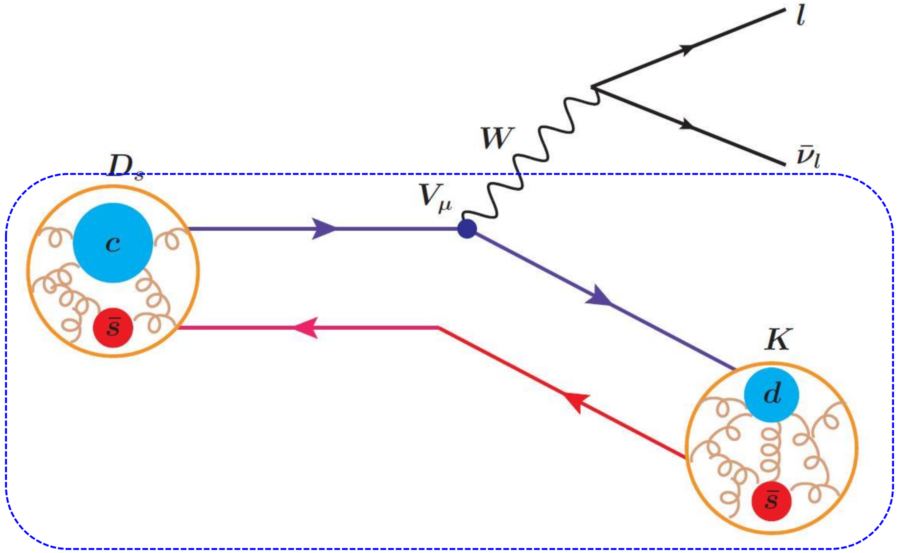
- Bound-State amplitude is determined from the homogeneous Bethe-Salpeter equation (BSE):

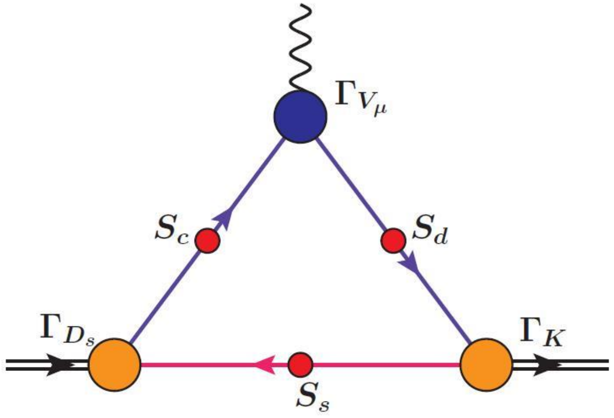
$$\Gamma_\pi(p;P) = \int_q^\Lambda K(p,q;P)S(q_+)\Gamma^{f\bar{g}}(q;P)S(q_-)$$



  - $q_+ = q + \eta P$; $q_- = q - (1-\eta)P$ are complex number.

Semileptonic decays of pseudoscalar Mesons





$S_{c,d,s}$: the propagators of dressed *d, s, c* quarks;

$\Gamma_{Ds,K}$: the BS amplitudes for the D_s and K mesons;

Γ_{V_μ} : the dressed-quark-W-boson vertex

$$M_\mu^{D_s^+}(P,Q) = N_c \mathrm{tr} \int \frac{d^4 q}{(2\pi)^4} \Gamma_{D_s}(q+p/2;p) S_c(q+p) \times i\Gamma_\mu^{cd}(q+p, q-k) S_d(q-k) \Gamma_K(q-k/2;-k) S_s(q)$$

Dyson-Schwinger Equations

- Rainbow truncation

$$\Gamma_\nu^a \to \frac{\lambda^a}{2}\gamma_\nu$$

- Interaction : Qin-Chang Model

$$Z_1 g^2 D_{\mu\nu}(k)\Gamma_\nu(p,q) = Z_2^2 D_{\mu\nu}^{\text{free}}(k) k^2 \mathcal{G}(k^2)\gamma_\nu$$

Si-xue Qin, Lei Chang, Yu-xin Liu, Craig D. Roberts, and David J. Wilson
Phys. Rev. C **84**, 042202(R)

  – the free gluon propagator

$$D_{\mu\nu}^{\text{free}}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

  – Qin-Chang Model

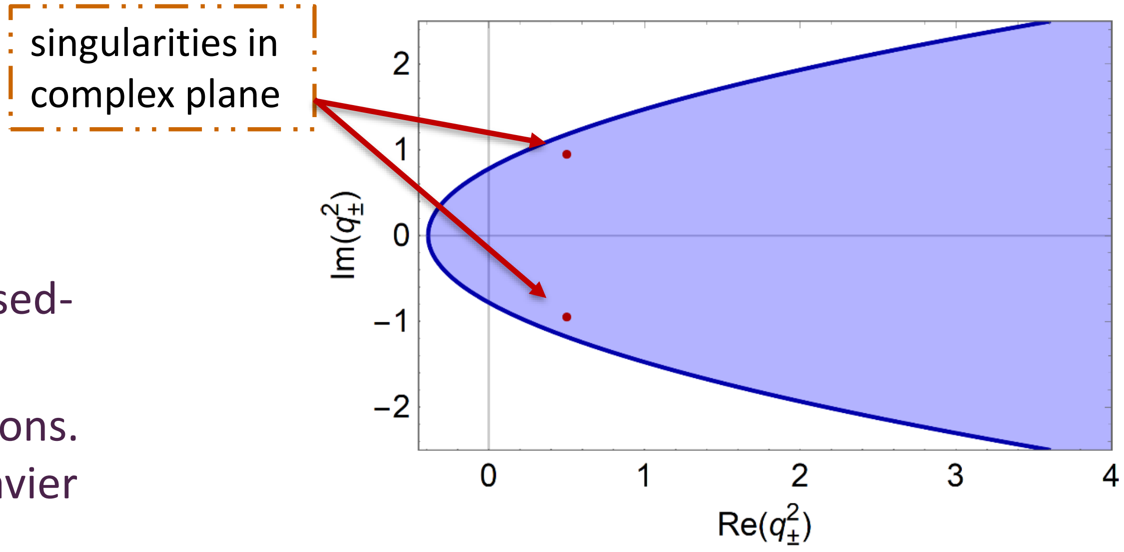
$$\mathcal{G}(s) = \frac{8\pi^2}{\omega^4} D e^{-s/\omega^2} + \frac{8\pi^2 \gamma_m \mathcal{F}(s)}{\ln[\tau + (1+s/\Lambda_{QCD}^2)^2]}$$

$$\mathcal{F}(s) = [1 - e^{-s/(4m_t^2)}]/s$$

  – Where $m_t = 0.5$ GeV, $\tau = e^2 - 1, \Lambda_{QCD} = 0.234$ GeV, $\gamma_m = 12/25$

- A mass-independent renormalization scheme

Bethe-Salpeter Equations



- The quark gap equations need to be solved in the complex plane
  - Domain of the complex plane sampled by the dressed-quark propagator.
  - It is hard to calculate properties of heavy-light mesons. The direct approach fails when the mass of the heavier quark exceeds $\hat{m}_{Q\bar{q}}$.

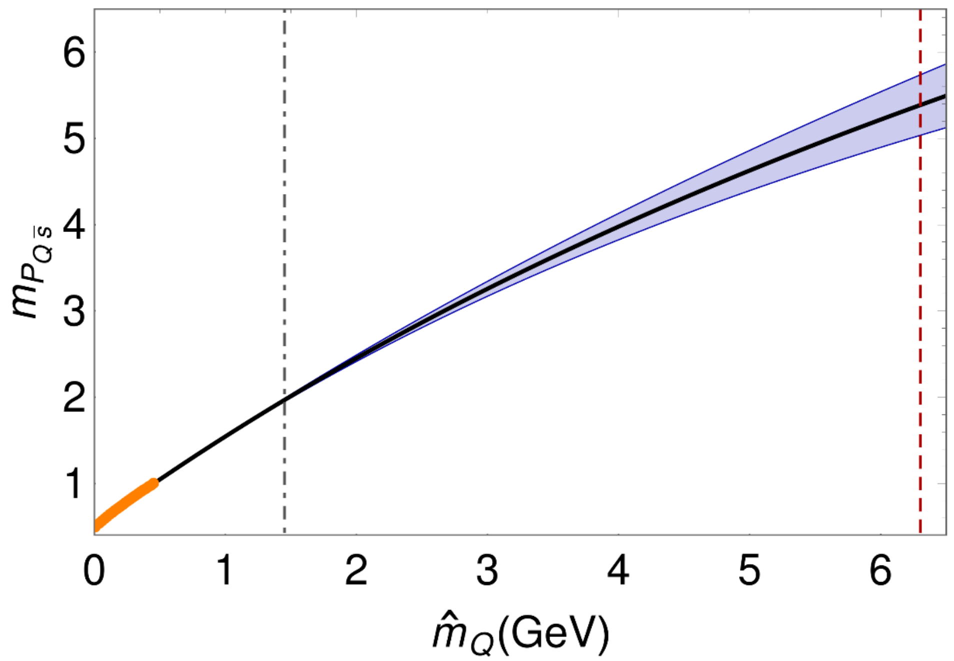
- Ladder truncation

$$K(p,q;P) \to -\mathcal{G}(k^2)k^2 D_{\mu\nu}^{\text{free}}(k)\frac{\lambda^a}{2}\gamma_\mu \otimes \frac{\lambda^a}{2}\gamma_\nu$$

  - Rainbow-Ladder (RL) truncation is the leading-order in a nonperturbative, symmetry-preserving truncation.

Heavy-light mesons

- Considering a fictitious meson $P_{Q\bar{q}}$, q=u, s. Computing its mass $m_P(\hat{m}_Q)$ as a function of current-mass $\hat{m}_Q$ up to a value $\hat{m}_{Q\bar{q}}$.
- calculated at N = 40 values of $\hat{m}_Q$ distributed evenly throughout the appropriate domain. We choose M = 20 current-mass values at random from that 40-element set, and then implement a continued fraction interpolation on this 20-element subset. Thereby, we obtain C(40,20)~100-billion interpolating functions.
- Building interpolations: $m_P(\hat{m}_Q)$ using Schlessinger point method (SPM) on the domains $\hat{m}_Q \epsilon \left[\hat{m}_q, \hat{m}_{Q\bar{q}}\right]$ to reach $D_{(s)}$, $B_{(s)}$ mesons.
- The value located at the centre of the band, within which 68% of the interpolants' values lie, is cited as the result. This 1σ band is identified as the uncertainty in the result.



	m_D	f_D	m_{D_s}	f_{D_s}	m_{D^*}	$m_{S_{c\bar{d}}}$
herein	1.87(2)	0.163(4)	1.97(2)	0.181(3)	2.08(5)	2.12(3)
PDG	1.87	0.153(7)	1.97	0.177(3)	2.01	2.30(2)

	m_B	f_B	m_{B_s}	f_{B_s}
herein	5.26(35)	0.136(17)	5.39(35)	0.160(14)
PDG	5.28	0.138(19)	5.37	0.161(2)

	$m_{D_s^*}$	$m_{S_{c\bar{s}}}$	m_{B^*}	$m_{B(0^+)}$
herein	2.16(4)	2.29(3)	5.38(36)	5.44(36)
PDG	2.11	2.32	5.33	

L. Schlessinger and C. Schwartz Phys. Rev. Lett. **16**, 1173
L. Schlessinger Phys. Rev. 167, 1411

Goldberger-Treiman relation

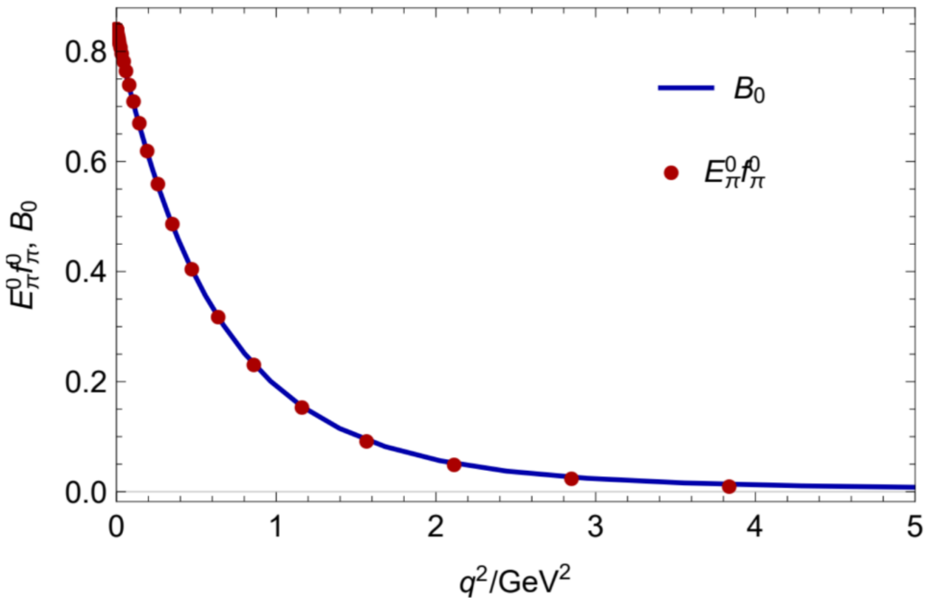
The most fundamental expression of Goldstone theories in SM.

- Dressed quark propagator:

$$S(p) = 1/[i\gamma \cdot pA(p^2) + B(p^2)]$$

- Pion's Bethe-Salpeter amplitude comes from solving the BSE

$$\Gamma_\pi(k;P) = \gamma_5[iE_\pi(k;P) + \gamma \cdot PF_\pi(k;P) + \gamma \cdot kG_\pi(k;P) + k_\mu P_\nu \sigma_{\mu\nu} H_\pi(k;P)]$$



- Axial-vector Ward-Takahashi identity

$$P_\mu \Gamma_{5\mu}^{fg}(k,P) = S_f^{-1}(k_+)i\gamma_5 + i\gamma_5 S_g^{-1}(k_-) - i(m_f + m_g)\Gamma_5^{fg}(k,P)$$

In chiral-limit, it entails

$$\boxed{f_\pi^0 E_\pi^0(p^2; P^2 = 0) = B_0(p^2)}$$

Maris, Roberts and Tandy nucl-th/9707003, Phys.Lett. B420 (1998) 267-273