

Going to the light-front with Contour Deformations

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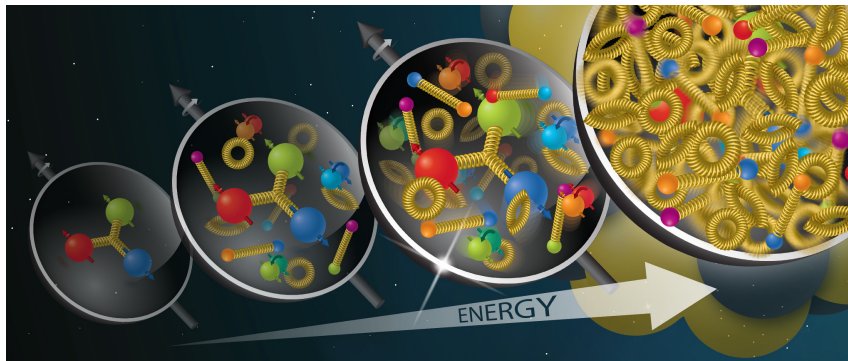
FCT

Fundação
para a Ciência
e a Tecnologia

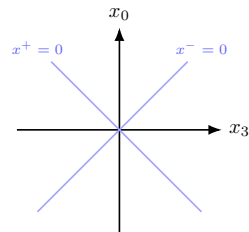


REPÚBLICA
PORTUGUESA

Hadrons on the light-front



(Source: Brookhaven National Laboratory / EIC)



$$x^+ = x^0 + x^3$$

$$x^- = x^0 - x^3$$

- ▶ Natural frame for defining parton distribution functions (PDFs, TMDs, GPDs, ...)
- ▶ Future: COMPASS/AMBER (CERN); Electron Ion Collider (BNL)

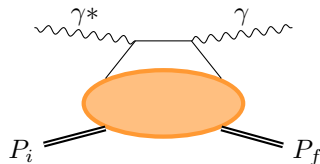
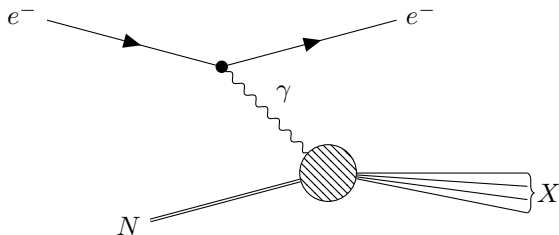
(AMBER: arXiv:1808.00848; EIC: Eur. Phys. J. A 52.9 (2016))

Factorization and hadronic matrix elements

- ▶ **Main idea:** Hadronic interactions can be written:

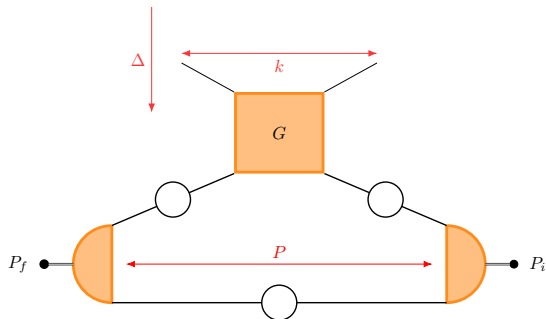
$$\text{Process} = \text{Hard Scattering} \times \text{Partonic Distribution Functions}$$

- ▶ An external current, $\gamma, W^\pm, Z^0, \dots$ couples to a parton from the hadron:
 - ▶ **Hard Scattering** - How the parton interacts with external current – perturbative regime.
 - ▶ **Partonic Distribution Functions** - Describe how the internal dynamics of the hadron give properties to the interacting parton – **Highly non-perturbative**
- ▶ **Partonic Distribution Functions** – give access to momentum/spin structure of the hadron



Writing the hadronic correlation

- ▶ **Main Goal:** Get partonic distribution functions from hadron-hadron correlations via **FUNctional Methods**



- ▶ G is the four-point quark correlation function, calculated with scattering equation.
- ▶ The quark propagator is calculated via quark DSE.
- ▶ The BSWF is calculated via the meson BSE.

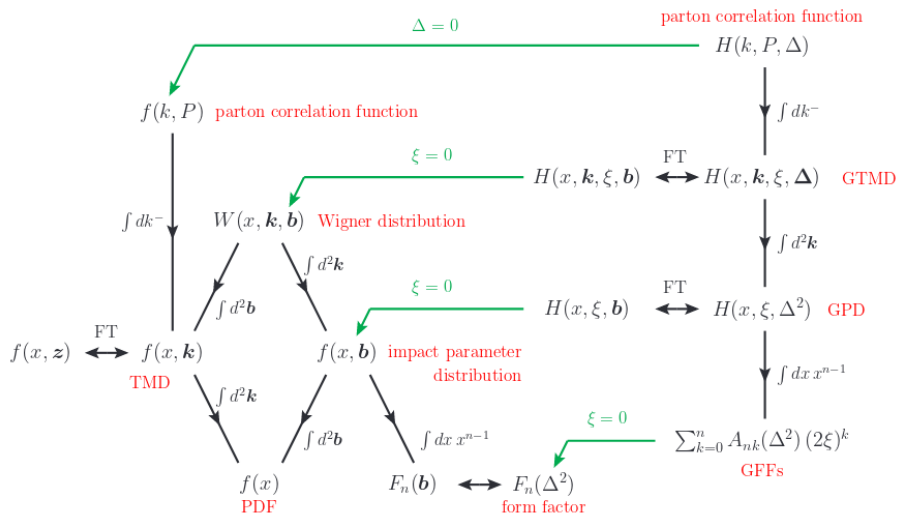
(Mezrag, arXiv:1507.05824); (Diehl, Gousset, 1998); (Tiburzi, Miller, 2003); (Mezrag, Chang, Moutarde, Roberts, Rodriguez-Quintero, Sabatié, Schmidt, 2015); (Cloët, Roberts, 2018), many many others, ...

$$\mathcal{G}^{[\Gamma]}(P, k, \Delta) = \frac{1}{2} \text{Tr} \left[\int dk^- \int \frac{d^4 z}{2\pi^4} e^{ik \cdot z} \langle P_f | \bar{\psi}(z) \mathcal{W} \Gamma \psi(0) | P_i \rangle \right]$$

- ▶ Partonic distributions are calculated by integrating the correlator in k^- and taking appropriate traces.

The full picture of the hadron

- ▶ LF projection of the hadronic correlator uncovers the internal dynamics of the hadron

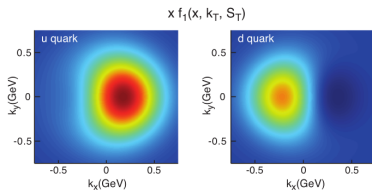


- ▶ Two different limits of the GTMDs (the *mother distributions*)

- ▶ **Transverse momentum distributions (TMDs):**

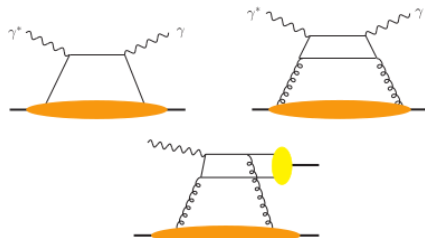
- Limit $\Delta = 0$ of GTMDs
- Info on confined motion of partons
- Semi-inclusive DIS experiments
- Transverse motion of the partons in the hadron
- Generalization of the well-known PDFs:

$$\int d^2\mathbf{k}_T f(x, k_T, Q^2) = f(x, Q^2)$$



- ▶ **Generalized Parton Distributions (GPDs):**

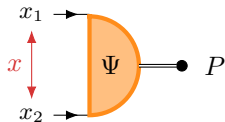
- GTMDs after integration in k_T .
- Non-zero momentum transfer to the hadron
- Exclusive processes: Deeply virtual Compton scattering, meson production, ...
- Transverse and spin structure
- Path to form factors



Going to the light-front

- ▶ Bethe-Salpeter Wavefunction

$$\langle 0 | T \Phi(x) \Phi(0) | P \rangle$$

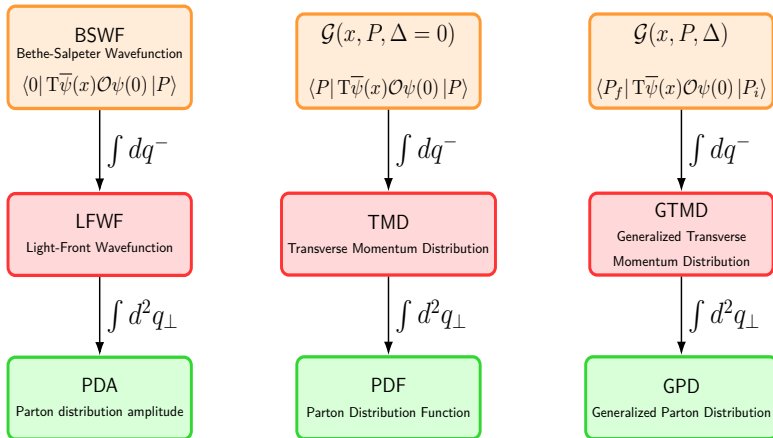


- ▶ Determined from the Bethe-Salpeter Equation (for the amplitude):

$$\Psi = G_0 \psi$$

$$\psi = K G_0 \psi$$

- ▶ With $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$, $\vec{x}_\perp = \{x^1, x^2\}$.



(Lorce, Pasquini, Vanderhaeghen; 2011)

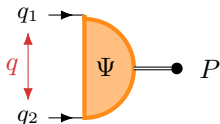
Light-front wavefunction

- ▶ **Light-Front Wavefunction (LFWF):**
Fourier transform of the BSWF at
 $x = \lambda n + \vec{x}_\perp$, with n along the light front.

$$\begin{aligned}\Psi_{LF}(q^+, \vec{q}_\perp, P) &= \\ &= \mathcal{N} \int dq^- \Psi(q^-, q^+, \vec{q}_\perp, P),\end{aligned}$$

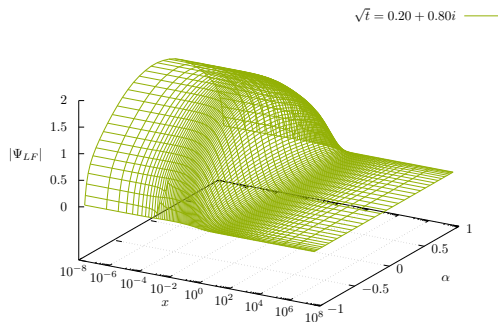
- ▶ **Parton Distribution Amplitude (PDA):**
Integration of the Ψ_{LF} over q_\perp .

$$\phi(\alpha, P) = \int d^2 q_\perp \Psi_{LF}\left(\frac{\alpha}{2} P^+, \vec{q}_\perp, P\right).$$



$\xi = \frac{q_1^+}{P^+} = \frac{1+\alpha}{2}$ is the longitudinal momentum fraction

$q = k + \frac{\alpha}{2} P$ is the relative momentum (with $k^+ = 0$).

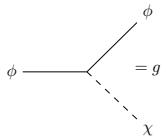


Scalar toy model

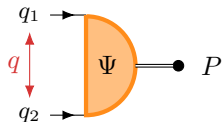
Scalar toy model:

ϕ of mass m

χ of mass μ



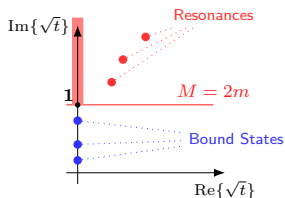
- **Objective:** Calculate BSWF in Euclidean metric for two ϕ particles,



- Need to solve the BSE:

$$\Psi = \mathbf{G}_0 \psi$$

$$\psi = \mathbf{K} \mathbf{G}_0 \psi$$



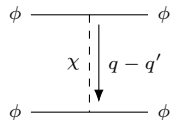
$$\psi(x, \omega, t, \alpha) = \frac{m^4}{(2\pi)^3} \frac{1}{2} \int_0^\infty dx' x' \int_{-1}^1 d\omega' \sqrt{1 - \omega'^2} \mathbf{G}_0(x', \omega', t, \alpha) \times \int_{-1}^1 dy \mathbf{K}(x, \omega, x', \omega', y) \psi(x', \omega', t, \alpha)$$

- \mathbf{G}_0 — We use tree level propagators

$$\mathbf{G}_0 = \frac{1}{q_1^2 + m^2} \frac{1}{q_2^2 + m^2}$$

- \mathbf{K} — We assume a single scalar exchange

$$\mathbf{K} = \frac{g^2}{(q - q')^2 + \mu^2}$$



- The BSWF is a function of the kinematic invariants:

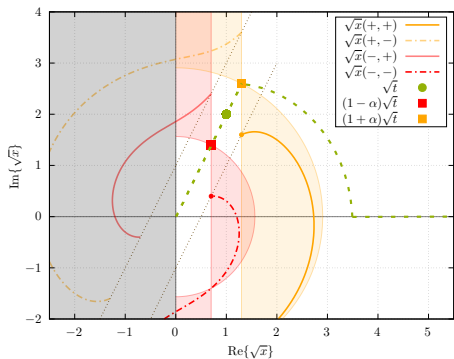
$$-M^2 = \frac{P^2}{4m^2} = t \quad \frac{k^2}{m^2} = x \quad \omega = \hat{k} \cdot \hat{P}$$

Analytic Structure

$$\mathbf{G}_0 = \frac{1}{q_1^2 + m^2} \frac{1}{q_2^2 + m^2}$$

- ▶ Poles when $q_{1/2}^2 = -m^2$
- ▶ Integration in $\omega \implies$ branch cuts in complex x plane:

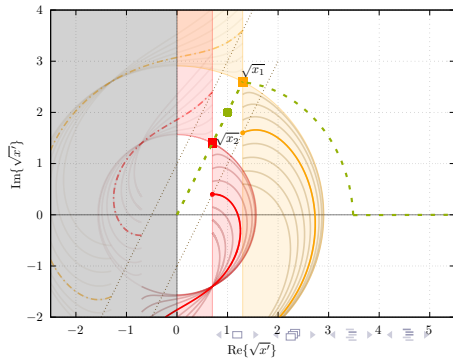
$$\sqrt{x}^\lambda = \mp (1 \pm \alpha) \sqrt{t} \left[\omega + i\lambda \sqrt{1 - \omega^2 + \frac{1}{(1 \pm \alpha)^2 t}} \right]$$



$$\mathbf{K} = \frac{g^2}{(q - q')^2 + \mu^2}$$

- ▶ Poles when $(q - q')^2 = -\mu^2$.
- ▶ Branch cuts in complex x' plane:

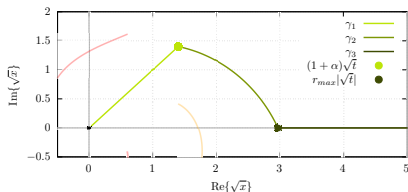
$$\sqrt{x'} = \sqrt{x} \left(\Omega \pm i \sqrt{1 - \Omega^2 + \frac{\beta^2}{x}} \right)$$



Integration path

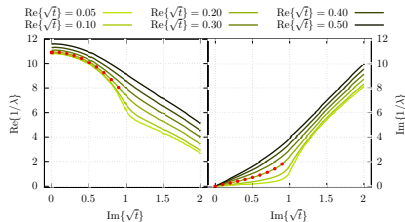
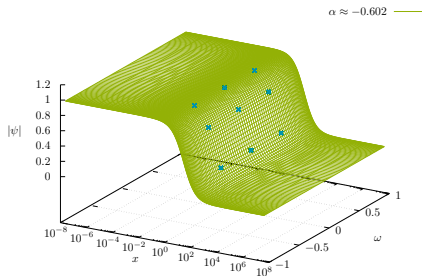
► Constraints:

1. Must go through $(1 + |\alpha|)\sqrt{t}$.
2. $\text{Re}\{\sqrt{x}\}$ and $|\sqrt{x}|$ must always increase.



- γ_1 — Line from the origin to $(1 + |\alpha|)\sqrt{t}$.
- γ_2 — Return to the real axis, with increasing radius.
- γ_3 — $\sqrt{x} \rightarrow \infty$ on the real axis.

- If the cuts don't cross the real axis \implies no deformation needed.



(Kusaka, Williams; 1995); (Sauli, Adam, Jr; 2003)
 (Karmanov, Carbonell; 2006); (Frederico, Salme, Viviani; 2014)

Light-front wavefunction

- ▶ The LFWF is defined as:

$$\begin{aligned}\Psi_{LF}(\alpha, k_{\perp}, P) \\ = \mathcal{N} \int dq^- \Psi(q, P)|_{q^+ = \frac{\alpha}{2} P^+, q_{\perp} = k_{\perp}}\end{aligned}$$

- ▶ In our kinematic variables:

$$q^- = -\frac{2m^2}{P^+} \left(2\sqrt{x}\sqrt{t}\omega + \alpha t \right)$$

α and $x = \frac{k^2}{m^2} = \frac{k_{\perp}^2}{m^2}$ and t are external variables

- ▶ Need the BSWF in $\omega \in (-\infty, \infty)$.
- ▶ We use the Schlessinger method for analytic continuation:

$$R(\omega) = \frac{f(\omega_1)}{1 + \frac{a_1(\omega - \omega_1)}{1 + \frac{a_2(\omega - \omega_2)}{1 + \frac{a_3(\omega - \omega_3)}{\dots}}}}$$

(L. Schlessinger, 1968) (Tripolt et al., 2019) (D. Binosi, R-A. Tripolt; 2019)

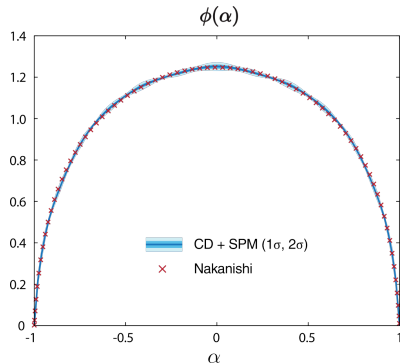
- ▶ $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$

Definition of the LFWF

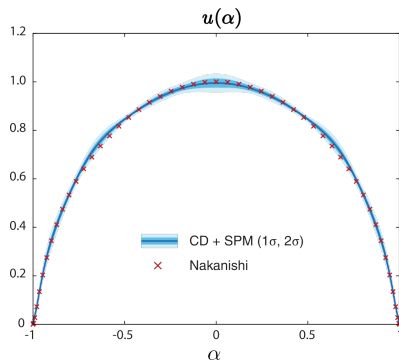
$$\Psi_{LF}(\alpha, x, t) = \mathcal{N} \frac{2\sqrt{x}\sqrt{t}}{i\pi} \int_{-\infty}^{\infty} d\omega \Psi(x, \omega, t, \alpha)$$

PDA: on the \sqrt{t} imaginary axis

- ▶ Use SPM to get to $\sqrt{t} = bi$, $b \in \mathbb{R}$ (cuts prevent direct evaluation!).
- ▶ Expand PDA in Chebyshev-U: $\phi(\alpha) = (1 - \alpha^2) \sum \phi_n U_n(\alpha)$.
- ▶ Analytic continuation of ϕ_n with SPM.



$$\phi(\alpha) \propto \int dx \Psi_{LF}(x, \alpha)$$



$$u(\alpha) \propto \int dx |\Psi_{LF}(x, \alpha)|^2$$

- ▶ **Input data:** $N \in [10, 50]$ different points starting with $\text{Re} \sqrt{t} = 0.1$, with steps in N of 2.

G. Eichmann, EF, A. Stadler; Phys. Rev. D 105, 034009 (2022)

Unequal masses

- ▶ Consider two ϕ of different masses:

$$m_1 = m(1 + \varepsilon) \quad m_2 = m(1 - \varepsilon)$$

$$\frac{m_1}{m_2} = \frac{1 + \varepsilon}{1 - \varepsilon} \quad 2m = m_1 + m_2$$

- ▶ $\varepsilon \in [-1, 1]$ sets the ratio of the masses

- ▶ \mathbf{G}_0 is now:

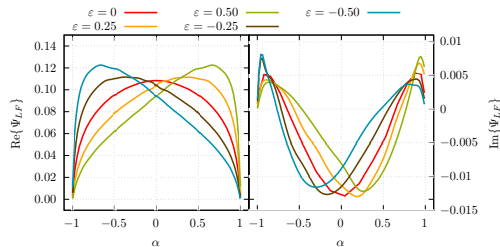
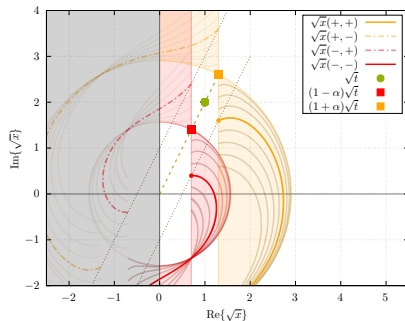
$$\mathbf{G}_0 = \frac{1}{q_1^2 + m_1^2} \frac{1}{q_2^2 + m_2^2}$$

- ▶ Cuts in x :

$$\sqrt{x}_\pm^\lambda = \mp(1 \pm \alpha)\sqrt{t} \times \left[\omega + i\lambda \sqrt{1 - \omega^2} + \frac{1}{t} \left(\frac{1 \pm \varepsilon}{1 \pm \alpha} \right)^2 \right]$$

- ▶ Integration path still works

- ▶ ε adds skewness



Complex conjugate masses

- ▶ Also consider complex conjugate mass poles:

$$D_\phi(q, m) = \frac{1}{2} \left(\frac{1}{q^2 + m^2} + \frac{1}{q^2 + (m^*)^2} \right)$$

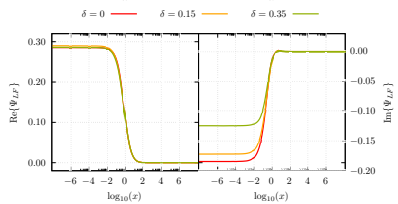
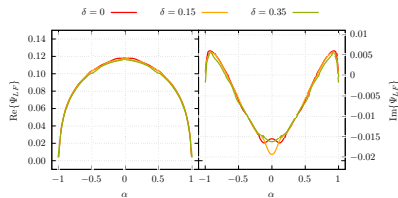
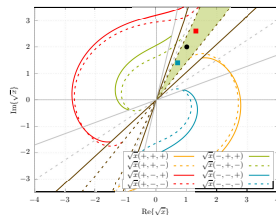
- ▶ $m^2 \rightarrow m^2(1 + i\delta)$, with $m^2, \delta \in \mathbb{R}_+$
- ▶ \mathbf{G}_0 becomes:

$$\mathbf{G}_0 = D_\phi(q_1, m) D_\phi(q_2, m)$$

- ▶ There are now 8 cuts:

$$\sqrt{x}_\pm^{\{\lambda, \nu\}} = \mp(1 \pm \alpha)\sqrt{t} \times \left[\omega + i\lambda \sqrt{1 - \omega^2} + \frac{1}{t} \frac{1 + \nu i \delta}{(1 \pm \alpha)^2} \right]$$

- ▶ For $\delta < \delta_{crit}$, contour deformation always possible.



Conclusions

- ▶ We explored a new way to calculate the LFWFs and PDAs.
- ▶ Very good agreement with the established Nakanishi method.
- ▶ We can also tackle extensions to the scalar toy model: unequal masses and complex conjugate mass poles in the propagators — features of many QCD calculations.
- ▶ We can calculate beyond the $M^2 = 4m^2$ threshold.
- ▶ The contour deformation method can also be applied to other correlation functions — just need a region free of singularities where the path can be deformed.
- ▶ We have also explored a way forward for the calculation of partonic structure functions.

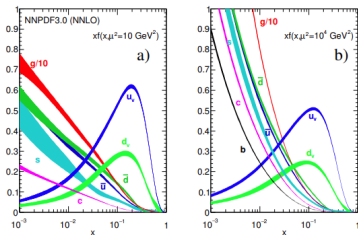
Thank you!

This work is supported by national funds through FCT - Fundação para a Ciência e a Tecnologia, I.P.,
under project CERN/FIS-PAR/0023/2021

Hadrons on the Light Front

Focus: Hadrons on the light front, $x^+ = 0$.

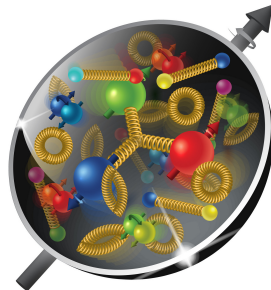
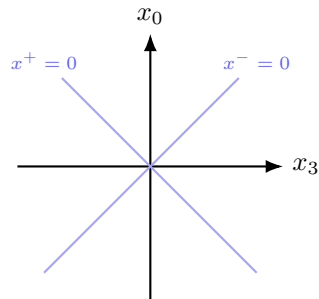
- ▶ Natural frame for defining parton distribution functions: PDFs, TMDs, ...



- ▶ Future: COMPASS/AMBER @ CERN
EIC @ Brookhaven National Laboratory.

(AMBER: [arXiv:1808.00848](https://arxiv.org/abs/1808.00848))

(EIC: [Eur. Phys. J. A 52.9 \(2016\)](https://arxiv.org/abs/1605.08016))



Bethe-Salpeter Wavefunction

- ▶ The Bethe-Salpeter Wavefunction (BSWF) appears as the residue of a correlation function $G(p)$:

$$\Psi(x, P) = \langle 0 | T \phi(0) \phi(x) | P \rangle$$

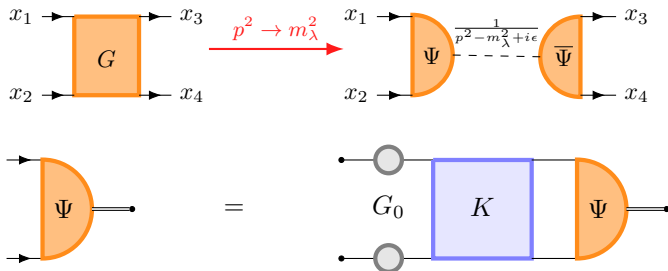
$$\Psi(k, P) = \int d^4 x e^{-ik \cdot x} \Psi(x, P)$$

- ▶ Determined by the Bethe-Salpeter Equation:

$$\Psi = \mathbf{G}_0 \mathbf{K} \Psi$$

\mathbf{G}_0 product of the dressed propagators;

\mathbf{K} interaction kernel between the two particles.



- ▶ BSWF defined from a smooth weight function $g(x, \alpha)$.

$$\Psi(q, P) = \frac{1}{m^4} \int_0^\infty dx' \int_{-1}^1 d\alpha' \frac{g(x', \alpha')}{[\kappa + 1 + x' + (1 - \alpha'^2)t]^3}, \quad \kappa = \frac{1}{m^2} \left(q - \frac{\alpha'}{2} P \right)^2.$$

- ▶ Light front quantities obtained from the weight function g , for example LFWF:

$$\Psi_{LF} = \frac{\mathcal{N}}{m^2} \int_0^\infty dx' \frac{g(x', \alpha)}{[x' + 1 + x + (1 - \alpha^2)t]^2}$$

- ▶ The BSE can be rewritten for g :

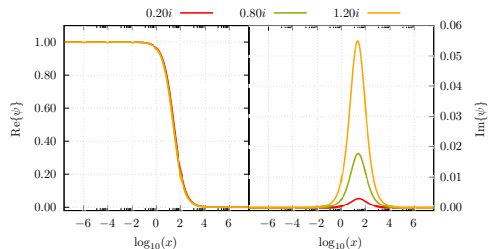
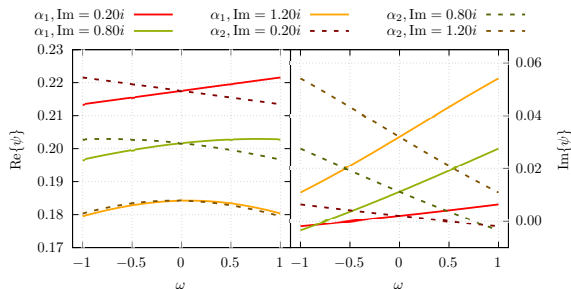
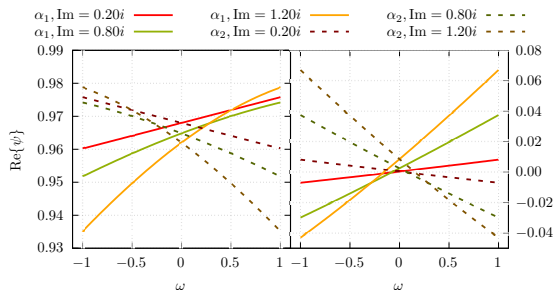
$$\int_0^\infty dx' \frac{g(x', \alpha)}{[x' + 1 + x + (1 - \alpha^2)t]^2} = c \int_0^\infty dx' \int_{-1}^1 d\alpha' V(x, x', \alpha, \alpha') g(x', \alpha')$$

$$V(x, x', \alpha, \alpha') = \frac{K(x, x', \alpha, \alpha') + K(x, x', -\alpha, -\alpha')}{2[x + 1 + (1 - \alpha^2)t]}$$

$$K(x, x', \alpha, \alpha') = \int_0^1 dv \frac{\theta(\alpha - \alpha')(1 - \alpha)^2}{[v(1 - \alpha)(x' + 1 + (1 - \alpha'^2)t) + (1 - v)C]^2}$$

$$C = (1 - \alpha')(1 + x + (1 - \alpha^2)t) + (1 - \alpha) \left(\frac{\beta}{v} + x' \right)$$

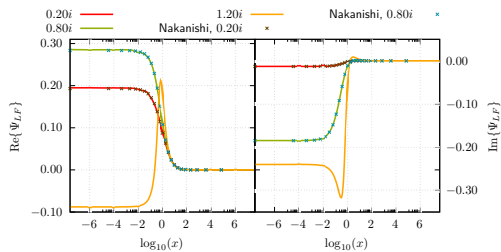
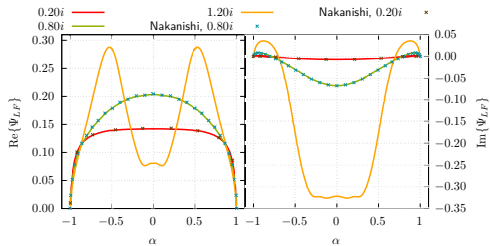
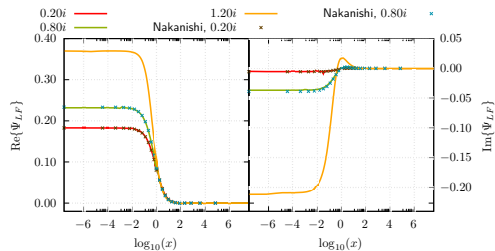
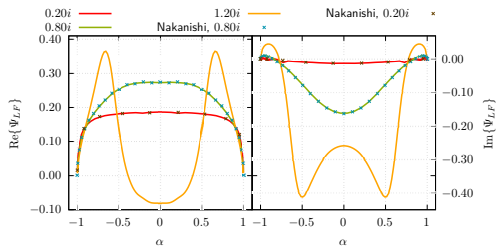
BSWF Results



- ▶ Small ω and α dependence.
- ▶ Symmetry for the combined transformation $\alpha \rightarrow -\alpha$ and $\omega \rightarrow -\omega$.
- ▶ Approximately a monopole:

$$\psi \approx \frac{1}{q^2 + \gamma}$$

LFWF: α and x dependence

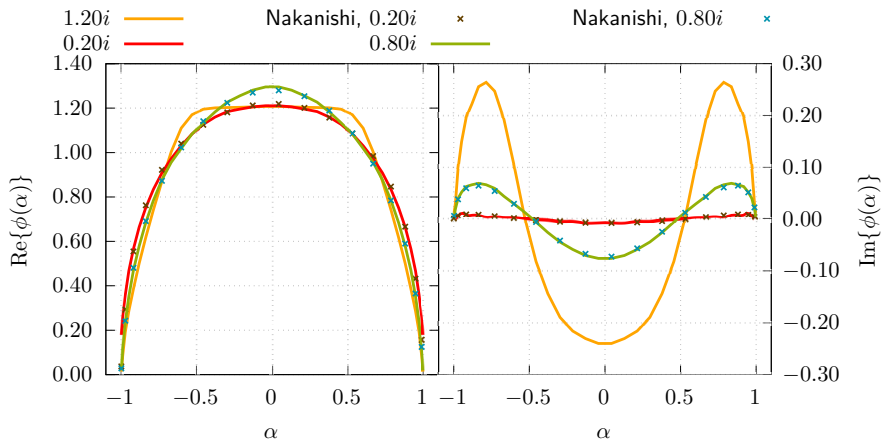


- Symmetric in α .
- Vanishes at $\alpha = \pm 1$.

PDA: α dependence

- In our variables:

$$\phi(\alpha) = \int_0^\infty dx \Psi_{LF}(x, \alpha, t)$$



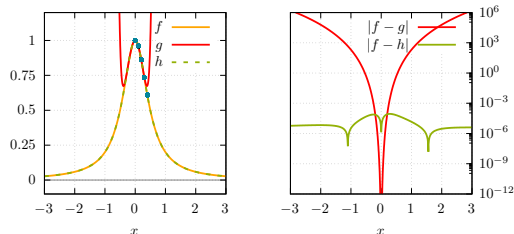
- Symmetric in α .
- Vanishes at $\alpha = \pm 1$.

Schlessinger Point Method

- ▶ Numerical analytic continuation method:

$$R(\omega) = \frac{f(\omega_1)}{1 + \frac{a_1(\omega - \omega_1)}{1 + \frac{a_2(\omega - \omega_2)}{1 + \frac{a_3(\omega - \omega_3)}{\dots}}}}$$

- ▶ $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$



- ▶ Recurrence relations:

$$R(\omega) = \frac{f(\omega_1)}{1 + \mathcal{Z}_1} = \frac{f(\omega_1)}{1 + \frac{a_1(\omega - \omega_1)}{1 + \mathcal{Z}_2}} = \dots$$

$$\mathcal{Z}_k = \frac{a_k(\omega - \omega_k)}{1 + \mathcal{Z}_{k+1}} \Leftrightarrow \mathcal{Z}_{k+1} = \frac{a_k(\omega - \omega_k)}{\mathcal{Z}_k} - 1, \\ \omega = \omega_k \implies \mathcal{Z}_k = 0$$

$$f(\omega_2) = \frac{f(\omega_1)}{1 + a_1(\omega_2 - \omega_1)}, \\ \mathcal{Z}_1 = \frac{f(\omega_1)}{f(\omega_2)} - 1 \Leftrightarrow a_1 = \frac{\mathcal{Z}_1}{\omega_2 - \omega_1},$$

Why not do one more iteration?

- ▶ Do *one more iteration* for a value of $\omega = W \in \mathbb{C}$, with the obtained Ψ

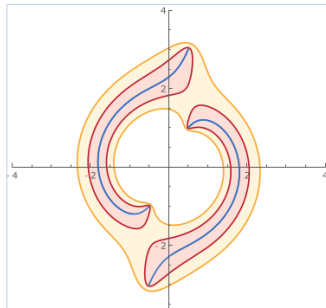
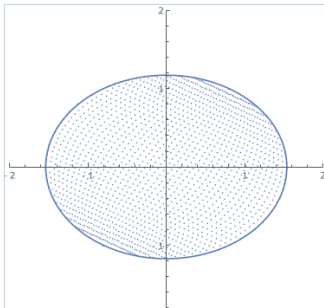
$$\Psi(x, W, t, \alpha) = \mathcal{N} \int_0^\infty dx' \int_{-1}^1 d\omega' \mathcal{K}(x, x', W, \omega') \Psi(x', \omega', t, \alpha)$$

- ▶ **Problem:** Kernel cuts will change

- ▶ For $\omega \in \mathbb{C}$, and $y, \omega' \in [-1, 1]$, Ω turns into a region bounded by the $r(\theta)$ ellipse, with $\omega = a + ib$ and $\sqrt{1 - \omega^2} = c + id$:

$$r(\theta) = \sqrt{a^2 + c^2} \sqrt{\cos^2 \theta + E^2 \sin^2 \theta} \quad E = \begin{cases} \frac{d^2}{a^2} & \alpha \neq 0 \\ \frac{b^2}{1+b^2} & \alpha = 0 \end{cases}$$

- ▶ Kernel cuts will eventually overlap



Cuts for complex conjugate mass poles

$$\text{Im}\{\sqrt{\tau}\} \text{Re}\{i\sqrt{1+i\delta}\} < \text{Im}\{i\sqrt{1+i\delta}\} \text{Re}\{\sqrt{\tau}\}.$$

