Applications of renormalization group to few-body problems in effective field theory

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Outline

- Demonstrating RG;
- RG trajectory of a toy model potential;
- Chiral EFT for P-wave halo states and RG;
- Subtractive renormalization;
- Summary;

Talk based on

E. Epelbaum, J. Gegelia, H. P. Huesmann, U.-G. Meißner and X.-L. Ren, Few Body Syst. **62**, no.3, 51 (2021)

Demonstrating RG

The main idea of using the RG to our advantage is simple and can be shown in a demonstrating example as follows:

Let a physical quantity be given in some theory by

$$f(x) = \frac{x}{1 - \hbar x},$$

where x is a parameter and $\hbar$ controls the quantum corrections.

Suppose for whatever reason we calculate this quantity as a power series expansion, order-by-order with better and better accuracy.

If $|x| < 1$ we can expand $f(x)$ in Taylor series around $x = 0$, approximating the exact function by the sum of first N terms

$$f(x) \approx S_N = x + \hbar x^2 + \hbar^2 x^3 + \cdots + \hbar^{N-1} x^N.$$

For $|x| > 1$ our expansion leads to a divergent series. In this case we can try an alternative way by rewriting the function $f(x)$ identically and expanding in a different way:

$$\begin{aligned} f(x) &= \frac{x}{1-\hbar x} \equiv \frac{x}{1-\hbar\mu x - \hbar x(1-\mu)} \\ &= \frac{1}{1-\mu}\,\frac{x_\mu}{1-\hbar x_\mu} = \frac{x_\mu}{1-\mu}\left(1+\hbar x_\mu + \hbar^2 x_\mu^2 + \cdots\right), \end{aligned}$$

where $x_\mu = x(1-\mu)/(1-\hbar\mu x)$.

The exact expression of $f(x)$ is μ-independent, however the sum of any finite number of terms depends on μ.

While formally this dependence is of higher order, i.e. $\sim \hbar^{N+1}$ for the sum of the first N terms, the convergence properties of the series crucially depends on the choice of μ.

For example, for $x = 2$ this series converges only if $\mu > 3/4$, the convergence being best close to $\mu = 1$.

The advantage of using RG in perturbative calculations is based on the μ-dependence of the sum of any finite number of terms.

This example demonstrates essential features of RG applied to perturbative calculations:

Exploiting the scale dependence of finite sums of the perturbative series one chooses such values of the scale parameter which leads to optimal convergence of perturbative series.

Numerous applications ... pQCD being the best known example.

A nice example using similarity RG has been presented by Maria Gomez Rocha on Tuesday.

RG is also applied to integral equations in chiral EFT.

For whatever reason some practitioners of few-body sector in chiral EFT decided that the main virtue is (approximate) order-by-order RG invariance of perturbative results ...

Consider a Lippmann-Schwinger equation in PW basis:

$$T(p',p) = V(p',p) + \int_0^\infty dk V(p',k) G(k) T(k,p)\,, \tag{1}$$

where $G(k)$ is the Green's function.

Modify the equation and the potential without changing $T(p',p)$:

$$T(p',p) = V(p',p,\Lambda) + \int_0^\infty dk\, V(p',k,\Lambda) G(k) f(\Lambda,k) T(k,p)\,. \tag{2}$$

There is freedom of doing this in different ways.

T is Λ-independent provided that $V(p',p,\Lambda)$ satisfies Wilsonian RG equation.

Calculating $V(p',p,\Lambda)$ order-by-order in some expansion, and solving for corresponding approximate amplitudes for different choices of Λ may lead to dramatically different convergence properties!

RG trajectory of a toy model potential

Starting with an "underlying" potential, we construct the LO EFT approximation and compare it to the exact Wilsonian RG trajectory.

$$V(r) = \frac{\alpha\left(e^{-m_1 r} - e^{-Mr}\right)}{r^3} + \frac{\alpha\,(m_1 - M)\,e^{-m_1 r}}{r^2} + \frac{\alpha\,(M - m_1)^2\,e^{-m_2 r}}{2r} - \frac{\alpha e^{-m_1 r}}{6}\,(2m_1 - 3m_2 + M)\,(M - m_1)^2\,,$$

where $\alpha = 50 \times 10^{-6}\ \mathrm{MeV}^{-2}$, $M = 138.5$ MeV, $m_1 = 750$ MeV and $m_2 = 1150$ MeV.

$V(r)$ vanishes for $r \to 0$, and behaves as $-\alpha\, e^{-Mr}/r^3$ for large r.

E. Epelbaum, A. M. Gasparyan, J. Gegelia and U. G. Meißner, Eur. Phys. J. A **54**, no.11, 186 (2018).

We consider the LS equation for the S-wave K-matrix

$$K(p',p) = V(p',p) + mP.V. \int_0^\infty \frac{dl\, l^2}{2\pi^2}\, V(p',l)\, \frac{1}{k^2 - l^2}\, K(l,p)\,.$$

At low energies, we can integrate out the high-energy modes and obtain the scattering amplitude by solving the regularized equation

$$K(p',p) = V(p',p,\Lambda) + mP.V. \int_0^\Lambda \frac{dl\, l^2}{2\pi^2}\, V(p',l,\Lambda)\, \frac{1}{k^2 - l^2}\, K(l,p)\,,$$

where the potential $V(p',p,\Lambda)$ satisfies the RG equation

$$V(p',l,\Lambda) = V(p',p) + mP.V. \int_\Lambda^\infty \frac{dl\, l^2}{2\pi^2}\, V(p',l)\, \frac{1}{k^2 - l^2}\, V(l,p,\Lambda)\,.$$

The potential $V(p',p,\Lambda)$ is the exact Wilsonian RG trajectory of the underlying potential.

For low energies the potential $V(p', p, \Lambda)$ can be approximated by LO EFT potential $V_{\mathrm{LO}} = C\,\delta^3(\vec{r}) - \alpha\, e^{-Mr}/r^3$.

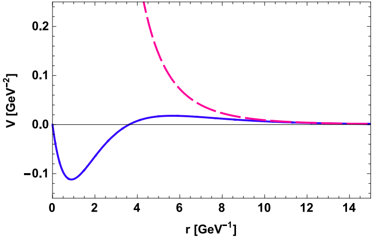

Figure: "Underlying" potential (solid line) and the LO long-range approximation (dashed line).

We adjust $C(\Lambda)$ such that at low energies, the phase shifts are well described by the solution to the equation:

$$K_{\rm LO}(p',p) = V_{\rm LO}(p',p) + mP.V. \int_0^\Lambda \frac{dl\, l^2}{2\pi^2}\, V_{\rm LO}(p',l)\, \frac{1}{k^2 - l^2}\, K_{\rm LO}(l,p)\,.$$

The resulting phase shifts are plotted as a function of k together with the phase shifts corresponding to the underlying model.

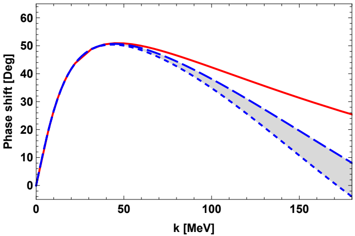

Figure: The S-wave phase shift for the underlying toy model and the LO approximation shown by the solid red and dashed blue lines, respectively. Short- and long-dashed lines correspond to $\Lambda = 300$ and $\Lambda = 450$ MeV, respectively.

Following the lcRG-invariant approach, by adjusting $C(\Lambda)$, (almost) cutoff-independent results for phase shifts at low energies can be obtained for arbitrarily large Λ.

The RG trajectories of the LO and underlying potentials are plotted in next page.

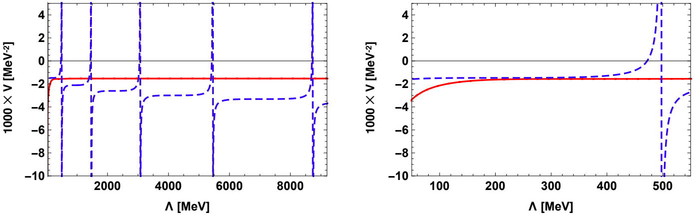

Figure: RG trajectories. Red and blue lines correspond to the underlying model and the LO approximation, respectively.
The right panel is a zoomed version of the left one.

While the LO potential does approximate well the exact RG trajectory for Λ around ~ 300 MeV, the limit-cycle behavior of the LO potential for larger values of the cutoff is just an artifact of the lcRG-invariant approach.

Chiral EFT for P-wave halo states and RG

Consider two non-relativistic particles with range of interaction $\sim 1/M_{\rm hi}$.

ERE of the amplitude with the orbital angular momentum l:

$$T(k) \propto \frac{1}{k\cot\delta - ik} \simeq \frac{k^{2l}}{\left(-1/a + 1/2\, rk^2 + v_2 k^4 + \ldots\right) - ik^{2l+1}}\,,$$

where a, r and v_i are the scattering length, effective range and the shape parameters.

We consider the EFT for P-wave scattering valid for momenta $k \sim M_{\rm lo} \ll M_{\rm hi}$.

We are interested in fine-tuned systems, for which the scattering amplitude has poles located within the validity range of the EFT.

Assume that the first two terms in the ERE are fine tuned as

$$1/a \sim M_{\rm lo}^3\,, \qquad r \sim M_{\rm lo}\,, \qquad v_n \sim M_{\rm hi}^{3-2n}\,. \tag{3}$$

In low-energy EFT with contact interactions only the two lowest-order contact interactions in the effective potential

$$V = C_2\, p' p + C_4\, p' p \left(p'^2 + p^2\right) + \ldots\,, \tag{4}$$

need to be iterated in the LS equation to all orders.

We solve

$$T(p',p) = V(p',p) + m\int_0^\Lambda \frac{l^2 dl}{2\pi^2}\,\frac{V(p,l)\;T(l,p')}{k^2 - l^2 + i\,\epsilon}\,, \tag{5}$$

for the above potential.

The on-shell amplitude $T(k) \equiv T(k,k)$ is given as:

$$\frac{k^2}{T(k)} = -I(k)\,k^2 - I_3 + \frac{(C_4 I_5 - 1)^2}{C_4\left(k^2\,(2 - C_4 I_5) + C_4 I_7\right) + C_2}\,. \tag{6}$$

where the integrals I_n and $I(k)$ are defined via

$$\begin{aligned} I_n &= -m\int_0^\Lambda \frac{l^2 dl}{2\pi^2}\, l^{n-3} = -\frac{m\Lambda^n}{2n\pi^2}, \qquad n = 1,3,5,\ldots\,, \\ I(k) &= \int_0^\Lambda \frac{l^2 dl}{2\pi^2}\,\frac{m}{k^2 - l^2 + i\epsilon} = I_1 - i\frac{mk}{4\pi} - \frac{mk}{4\pi^2}\ln\frac{\Lambda - k}{\Lambda + k}\,. \end{aligned} \tag{7}$$

Following the lcRG-invariant scheme, we express $C_2(\Lambda)$ and $C_4(\Lambda)$ in terms of a and r and take the $\Lambda \to \infty$ limit obtaining

$$T(k) = -\frac{4\pi}{m} \frac{k^2}{-\frac{1}{a} + \frac{r}{2}k^2 - ik^3} \,. \tag{8}$$

The bare LECs C_2, C_4 have the form

$$\begin{aligned} \frac{m}{10\pi^2} C_2(\Lambda) &= \frac{64a^2\Lambda^6 \pm 10\sqrt{5\alpha}\left(3\pi - 2a\Lambda^3\right) + \cdots}{7\Lambda^3\left(16a^2\Lambda^6 + 3\pi a\Lambda^3\left(3a\Lambda^2 r + 20\right) - 45\pi^2\right)}\,, \\ \frac{m}{10\pi^2} C_4(\Lambda) &= \pm\sqrt{\frac{5}{\alpha}}\frac{\left(3\pi - 2a\Lambda^3\right)}{\Lambda^5} - \frac{1}{\Lambda^5}\,, \end{aligned} \tag{9}$$

where $\alpha = -16a^2\Lambda^6 - 3\pi a\Lambda^3\left(3a\Lambda^2 r + 20\right) + 45\pi^2$.

Both bare couplings become complex for a sufficiently large values of the cutoff Λ.

This observation is in line with the causality bound $r \leq -2/R\,(1 + \mathcal{O}(R^3/a))$ obtained in
H. W. Hammer and D. Lee, Annals Phys. **325**, 2212-2233 (2010).
if the range of the interaction R is identified with $1/\Lambda$.

According to the lcRG-invariant approach the considered fine-tuned P-wave system cannot be described in an EFT with contact interactions only.

The problem actually lies in the procedure of the lcRG-invariant approach rather than in the EFT itself.

Subtractive renormalization

We renormalize the amplitude by applying BPHZ procedure, i.e. subtracting *all* UV divergences prior to taking the limit $\Lambda \to \infty$.

Specifically, we first separate out power-like UV divergences in the appearing integrals in the most general way via

$$
\begin{aligned}
I_n &= -m \int_0^{\mu_n} \frac{l^2 dl}{2\pi^2} l^{n-3} - m \int_{\mu_n}^{\Lambda} \frac{l^2 dl}{2\pi^2} l^{n-3} \equiv I_n^R(\mu_n) + \Delta_n(\mu_n)\,, \\
I(k) &\equiv I^R(k, \mu_1) - \Delta_1(\mu_1)\,,
\end{aligned}
$$

where μ_n denote the corresponding subtraction scales.

We renormalize the amplitude by simultaneously replacing the integrals I_n and $I(k)$ with $I_n^R(\mu_n)$ and $I^R(k, \mu_1)$ and the bare couplings C_2 and C_4 by the corresponding μ_n-dependent renormalized couplings C_2^R and C_4^R, respectively.

Since the renormalized amplitude depends only on UV-convergent integrals, we can now safely take the limit $\Lambda \to \infty$.

Fixing the renormalized LECs by the requirement to reproduce a and r leads to our final result:

$$k^3 \cot \delta = -\frac{1}{a} + \frac{1}{2} r k^2 - \frac{3k^4}{2\pi} \frac{(4\mu_1 + \pi r)^2}{6\pi a^{-1} - 4\mu_3^3 + 3k^2(4\mu_1 + \pi r)} \,. \quad (11)$$

The renormalized scattering amplitude depends on μ_1 and μ_3.

The choice of μ_i plays a key role in setting up a self-consistent power counting.

Indeed, one *must* choose $\mu_3 \sim M_{\rm hi}$ since setting $\mu_3 \sim M_{\rm lo}$ would lead to poles in the effective range function located at $k \sim M_{\rm lo}$, thereby resulting in enhanced values of the coefficients in the ERE.

A self-consistent Weinberg-like scheme with manifest power counting with all LECs scaling according to NDA emerges if we set $\mu_5 \sim \mu_7 \sim \mu_9 \sim \ldots \sim M_{\rm lo}$.

The remaining scale μ_1 can be chosen either as $\mu_1 \sim M_{\rm hi}$ or $\mu_1 \sim M_{\rm lo}$.

Summary

- Briefly discussed the idea of applying RG to perturbative calculations.
- Compared the exact RG flow for a toy-model potential to the approximate result obtained using the lcRG-invariant approach.
- The obtained limit-cycle Λ-dependence of the LO potential disagrees with the smooth RG flow of the underlying model.
- Concluded that:
  In Wilsonian approach taking $\Lambda \sim M_{\rm hi}$ or larger is not compatible with the approximate expansion of the bare potential.
- Revisited the problem of renormalization in halo EFT for P-wave scattering.
- Properly renormalized the P-wave amplitude using BPHZ scheme.