Phenomenology of heavy-ion collisions
Solutions to hands-on problems

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Problem 1

Derive how the velocity of a particle depends on its rapidity (in natural units $c = 1$).

1. Show that for small rapidities $v \approx y$.
2. Plot the function $v(y)$ for y up to 5.
3. Make such a plot that the velocity can be read off the plot even for large y. Plot this for y up to 12.
4. Determine the rapidity (and the velocity) for the ion beam at the LHC (5.5 TeV per colliding nucleon pair), RHIC (200 GeV), SPS (take the centre of mass system: 17.6 GeV per colliding nucleon pair), SIS ($\sqrt{s_{NN}} = 2.42$ GeV), and also for the planned FCC ($\sqrt{s_{NN}} = 39$ TeV).

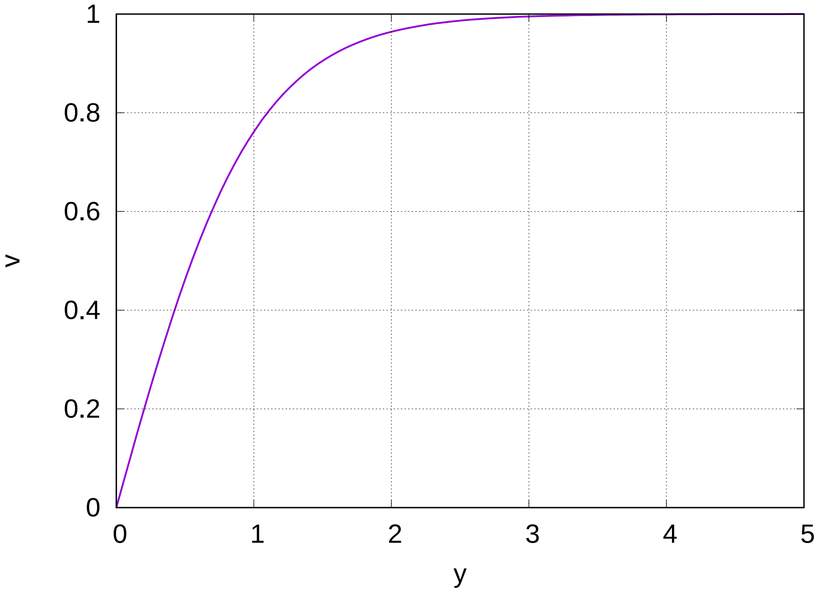
Solution 1.a

$$
\begin{aligned}
\frac{p}{E} &= v \\
y &= \frac{1}{2}\ln\frac{1+v}{1-v} \\
e^{2y} &= \frac{1+v}{1-v} \\
v &= \frac{e^{y}-e^{-y}}{e^{y}+e^{-y}} = \tanh y
\end{aligned}
$$

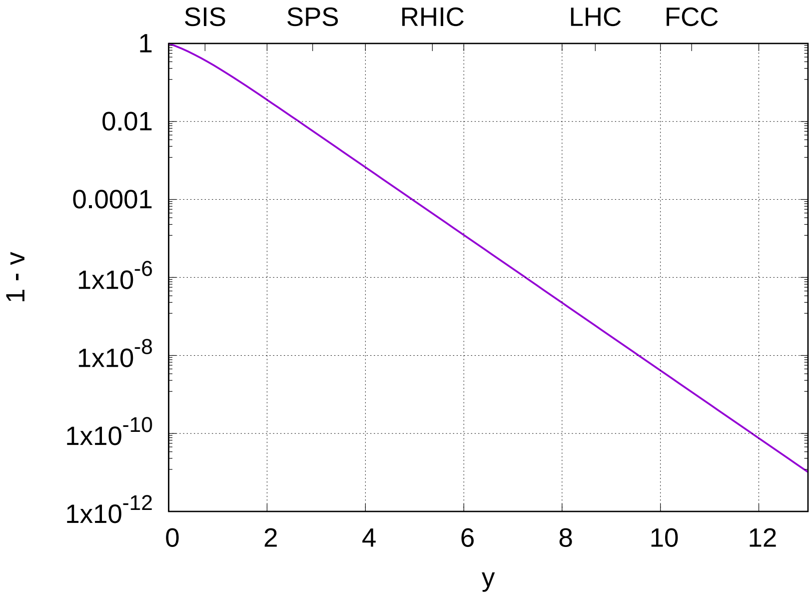
Taylor expansion for small y

$$
v \approx \frac{1+y-1+y}{1+y+1-y} \approx y + O(y^2)
$$

Solution 1.b



Solution 1.c



SIS
SPS
RHIC
LHC
FCC
1
0.01
0.0001
1x10-6
1x10-8
1x10-10
1x10-12
1 - v
0
2
4
6
8
10
12
y

Solution 1.d

beam energy, velocity, rapidity

$$E = \frac{\sqrt{s_{NN}}}{2}, \quad \gamma = \frac{E}{m}, \quad v = \sqrt{1 - \frac{m^2}{E^2}}, \quad y = \ln\left(\frac{E}{m}\left(1 + \sqrt{1 - \frac{m^2}{E^2}}\right)\right)$$

	$\sqrt{s_{NN}}$/GeV	γ	v	y
SIS	2.42	1.289	0.630692233281	0.743
SPS	17.6	9.372	0,994290772349	2.928
RHIC	200	106.49	0,999955912978	5,361
LHC	5500	2928.6	0.999999941704	8.675
FCC	39000	20767	0.999999998841	10.634

Problem 2.a

From $\eta = \frac{1}{2}\ln\frac{|p|+p_z}{|p|-p_z}$ show that $\eta = -\ln\tan\frac{\theta}{2}$. (Where θ is the angle between $\vec{p}$ and the beam axis.)

$$\eta = \frac{1}{2}\ln\frac{p+p_l}{p-p_l} = \frac{1}{2}\ln\frac{(p+p_l)^2}{p_t^2} = \ln\frac{p+p_l}{p_t}$$

$$= \ln\frac{p(1+\cos\theta)}{p\sin\theta} = \ln\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\ln\tan\frac{\theta}{2}$$

Problem 2.b

Prove that $|p| = p_t \cosh\eta$ and $p_z = p_t \sinh\eta$.

$$
\begin{aligned}
e^{\eta} &= \frac{p - p_l}{p_t} \\
e^{\eta} p_t - p_l &= p = \sqrt{p_l^2 + p_t^2} \\
p_l &= p_t \frac{e^{\eta} - e^{-\eta}}{2} = p_t \sinh\eta
\end{aligned}
$$

$$
p = \sqrt{p_l^2 + p_t^2} = \sqrt{p_t^2 \sinh^2\eta + p_t^2} = p_t \sqrt{1 + \sinh^2\eta} = p_t \cosh\eta
$$

Problem 2.c

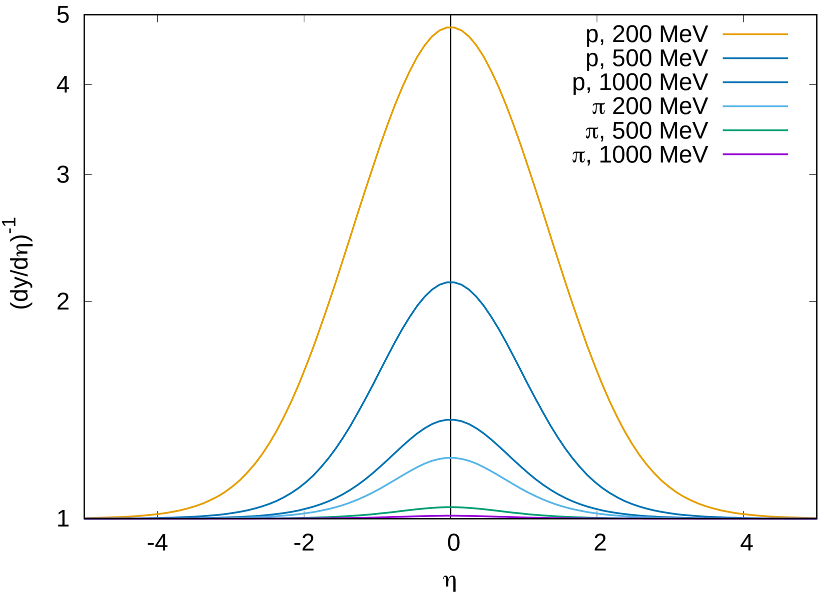
Provided that pion spectrum is flat in rapidity, plot it in pseudorapidity for $p_t = 200$ MeV, 500 MeV, and 1000 MeV? Do the same for proton spectrum. (This exercise should show that the pseudorapidity is good approximation to rapidity for $m \ll p_t$.)

$$\frac{dN}{d\eta} = \frac{dN}{dy}\frac{dy}{d\eta}$$

$$y = \ln \frac{\sqrt{m^2 + p^2 + p_l}}{m_t} = \ln \frac{\sqrt{m^2 + p_t^2 \cosh^2 \eta} + p_t \sinh \eta}{\sqrt{m^2 + p_t^2}}$$

$$\frac{dy}{d\eta} = \frac{\cosh \eta \sinh \eta p_t^2}{m^2 + p_t^2 \cosh^2 \eta + p_t \sinh \eta \sqrt{m^2 p_t^2 \cosh^2 \eta}} + \frac{p_t \cosh \eta}{\sqrt{m^2 + p_t^2 \cosh^2 \eta} + p_t \sinh \eta}$$

Problem 2.c



p, 200 MeV
p, 500 MeV
p, 1000 MeV
π 200 MeV
π, 500 MeV
π, 1000 MeV
$(dy/d\eta)^{-1}$
η
5
4
3
2
1
-4
-2
0
2
4

Problem 3

Derive that

$$E_t = m_t \frac{\cosh y}{\cosh \eta} \, .$$

Use:

$$E = m_t \cosh y$$

Then

$$E_t = m_t \cosh y \sin\theta = m_t \cosh y \frac{p_t}{|p|} = m_t \cosh y \frac{p_t}{p_t \cosh \eta} = m_t \frac{\cosh y}{\cosh \eta}$$

Problem 4

Show that the mean number of binary collisions is $ABT_{AB}(\vec{s})\sigma_{inel}^{NN}$.

For brevity denote: $T_{AB}(\vec{s})\sigma_{inel}^{NN} = p$, $AB = M$
Probability to have n binary collisions out of AB is distributed according to binomial distribution.

$$P(n) = \frac{M!}{n!(M-n)!} p^n (1-p)^{M-n}$$

Problem 4 — contn'd

Calculate

$$
\begin{aligned}
\langle n \rangle = \sum_{k=0}^{M} kP(k) &= \sum_{k=0}^{M} k \frac{M!}{k!(M-k)!} p^k (1-p)^{M-k} \\
&= \sum_{k=1}^{M} \frac{M!}{(k-1)!(M-k)!} p^k (1-p)^{M-k} \\
&= \sum_{l=0}^{M-1} \frac{M!}{l!(M-l-1)!} p^{l+1} (1-p)^{M-l-1} \\
&= Mp \sum_{l=0}^{M-1} \frac{(M-1)!}{l!((M-1)-l)!} p^l (1-p)^{(M-1)-l} \\
&= Mp \left(p + (1-p)\right)^{M-1} = Mp = ABT_{AB}(\vec{s})\sigma_{inel}^{NN}
\end{aligned}
$$

Q.E.D. (quod erad demonstrandum :-))

Problem 5.a

The Δ resonance appears in four isospin states: Δ^-, Δ^0, Δ^+, Δ^{++}. It decays exclusively into one nucleon and one pion. What are the possible decay channels of Δ^+ and what are their branching ratios?

Δ is the 3/2 representation of the SU(2) isospin group, isospin of Δ^+ is 1/2

N is the 1/2 representation, π is the 1 representation

possible channels are:

- $\Delta^+ \to p\pi^0$
  i.e. $|3/2, 1/2\rangle \to |1/2, 1/2\rangle |1, 0\rangle$
  the CG coefficient is $\sqrt{2/3}$
  the BR is 2/3
- $\Delta^+ \to n\pi^+$
  i.e. $|3/2, 1/2\rangle \to |1/2, -1/2\rangle |1, 1\rangle$
  the CG coefficient is $\sqrt{1/3}$
  the BR is 1/3

$1 \times 1/2$

m_1	m_2	3/2, +3/2	3/2, +1/2	1/2, +1/2	3/2, −1/2	1/2, −1/2	3/2, −3/2
+1	+1/2	1					
+1	−1/2		1/3	2/3			
0	+1/2		2/3	−1/3			
0	−1/2				2/3	1/3	
−1	+1/2				1/3	−2/3	
−1	−1/2						1

Problem 5.b

The resonance ρ^0, which is a part of an isospin triplet decays exclusively into two pions. What are the possible decay channels and what are the branching ratios?

ρ is the 1 representation of the SU(2) isospin group, isospin of ρ^0 is 0
π is the 1 representation
possible channels are:

- $\rho^0 \to \pi^+\pi^-$
  i.e. $|1,0\rangle \to |1,1\rangle|1,-1\rangle$
  the CG coefficient is twice $\sqrt{1/2}$
  the BR is 1
- $\rho^0 \to \pi^0\pi^0$
  i.e. $|1,0\rangle \to |1,0\rangle|1,0\rangle$
  the CG coefficient is 0
  the BR is 0

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1