Hydrodynamics of Heavy-Ion Collisions

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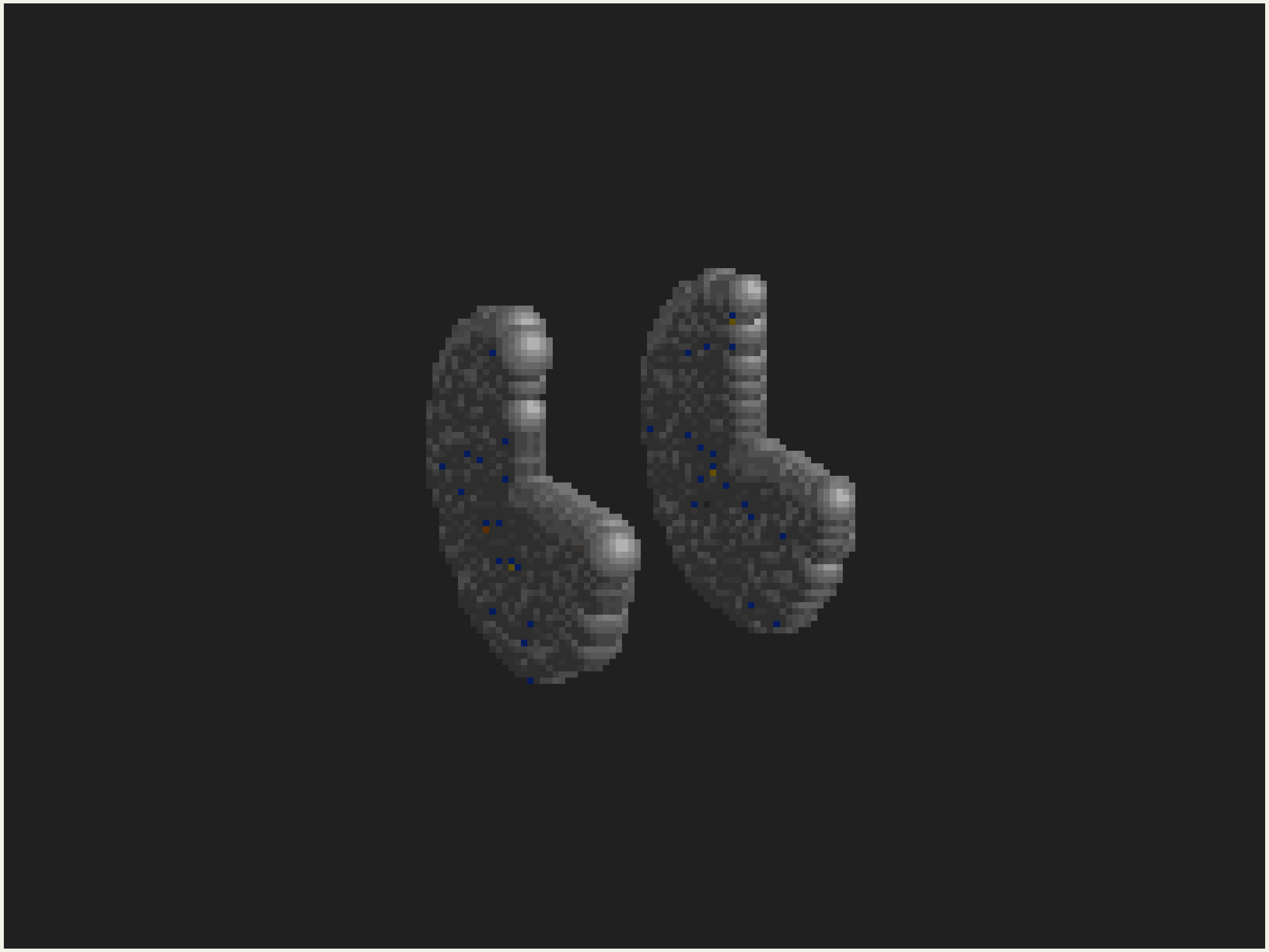
Heavy-ion collision



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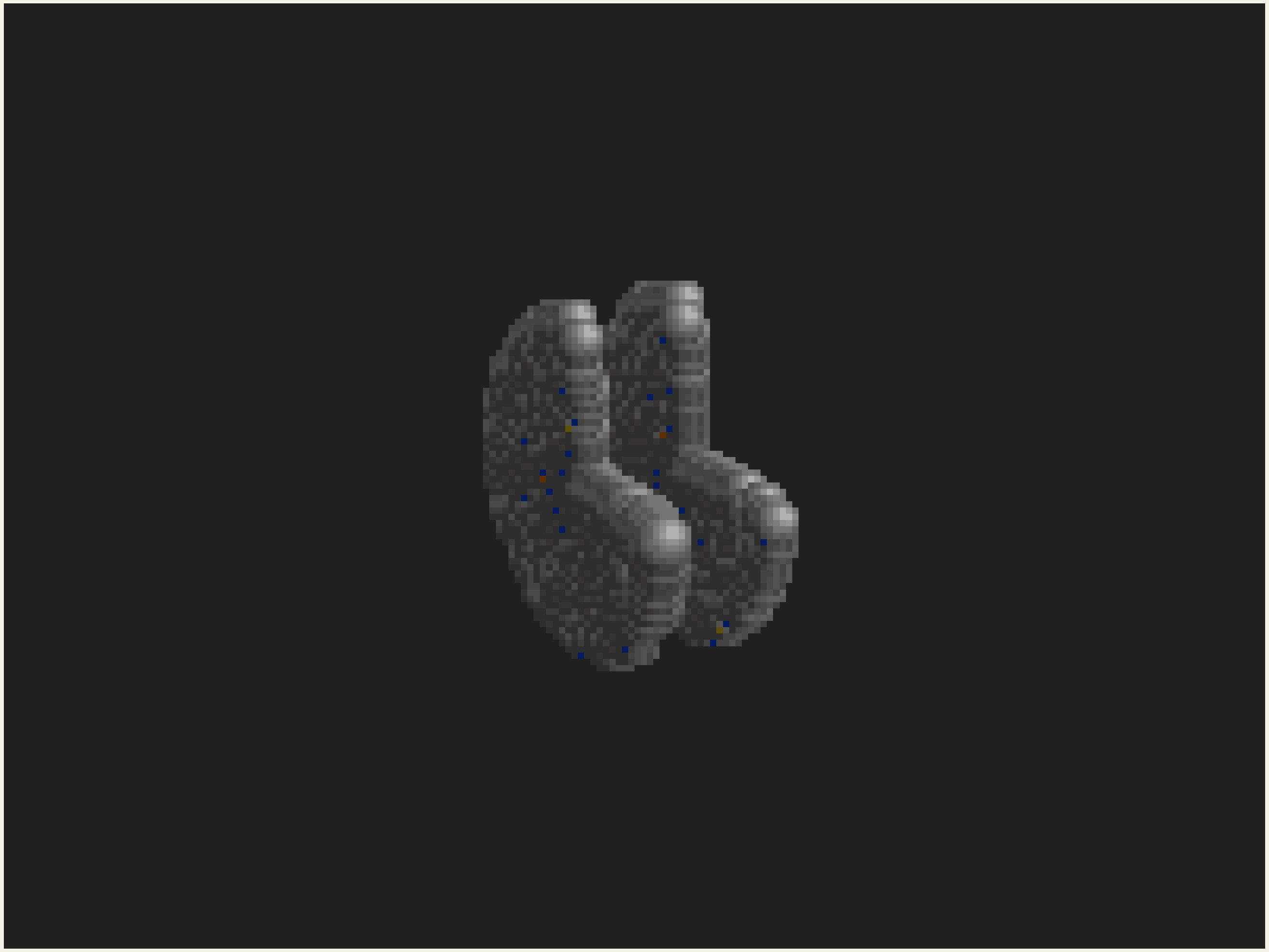
Heavy-ion collision



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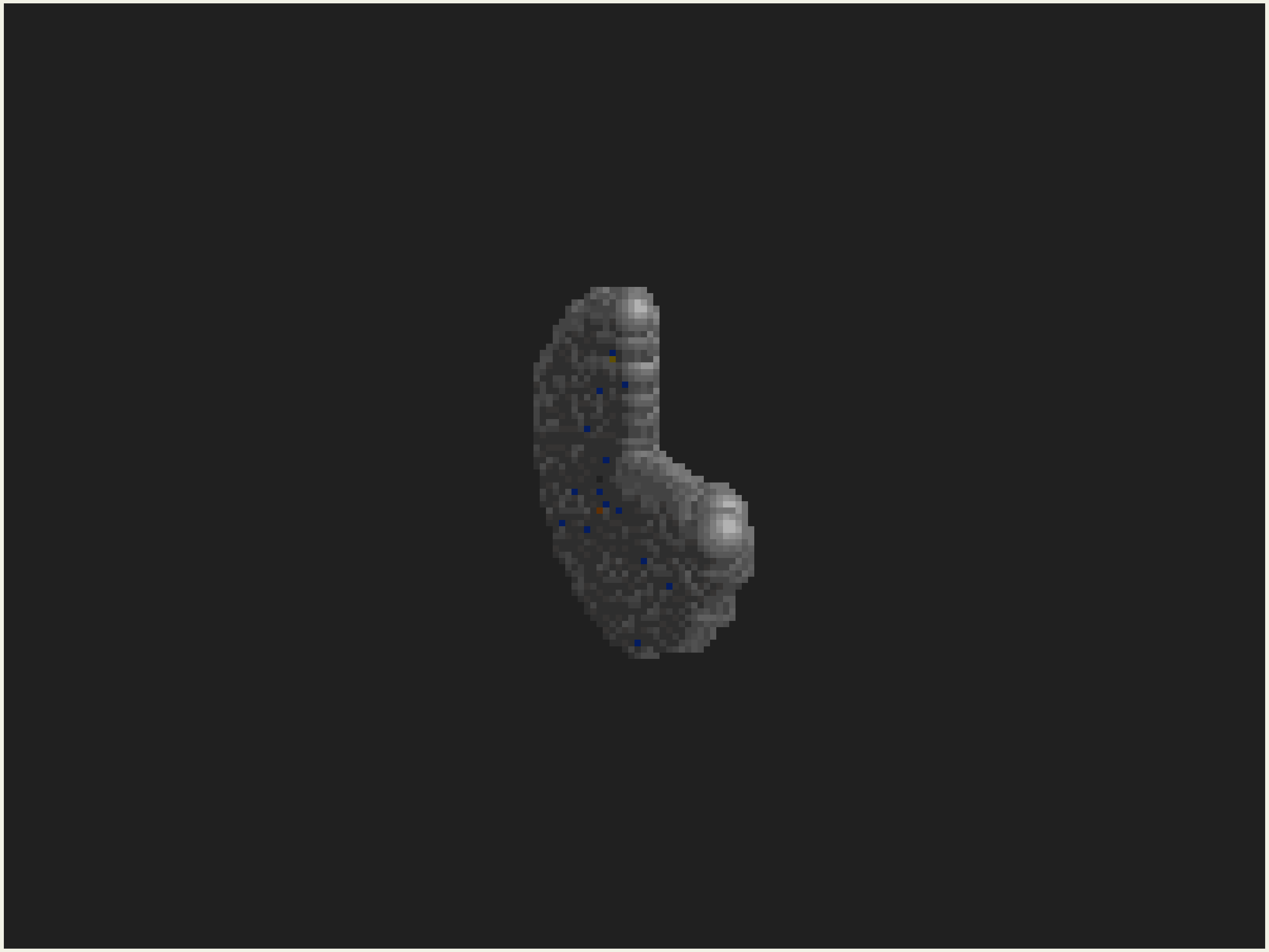

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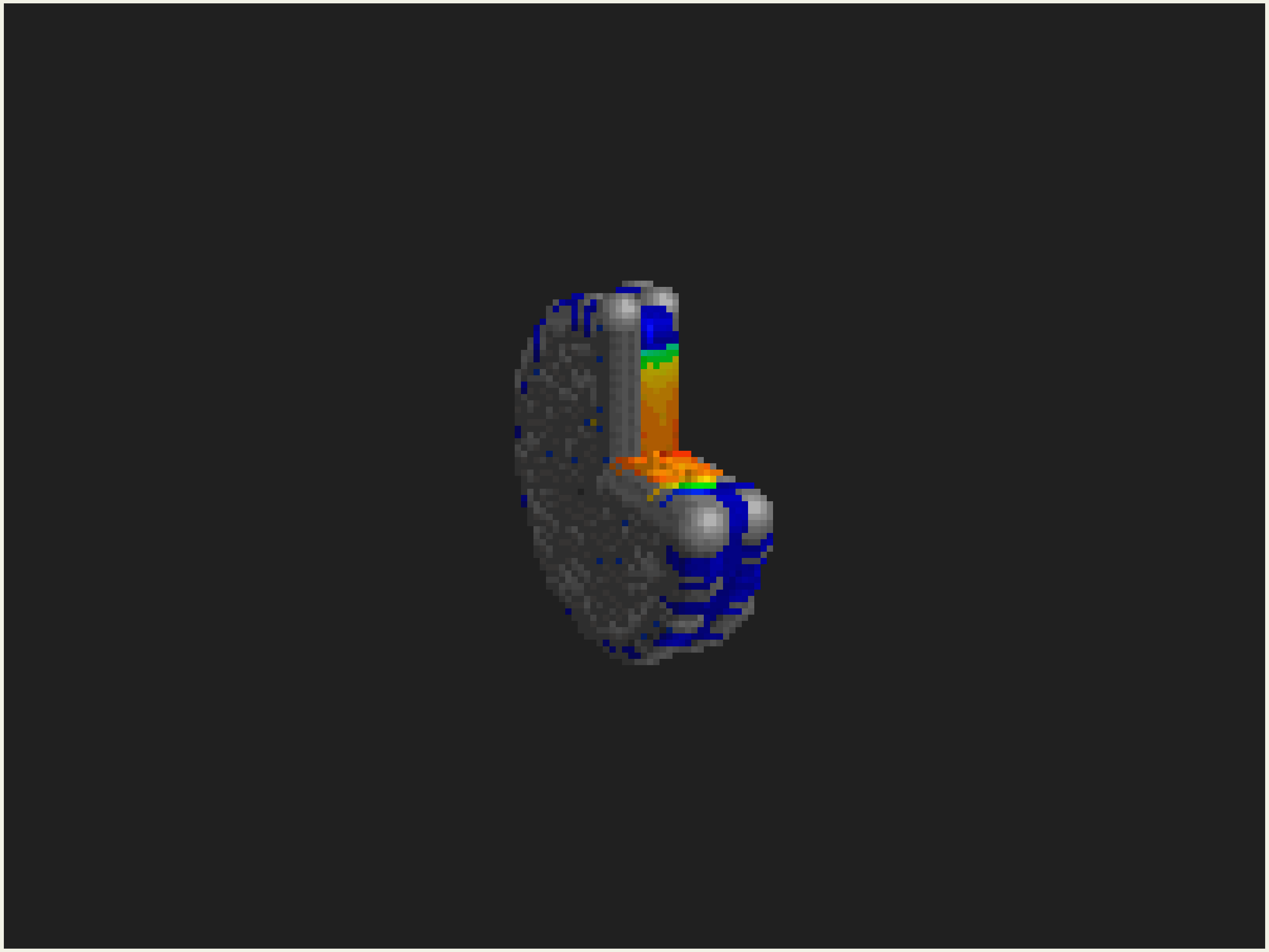

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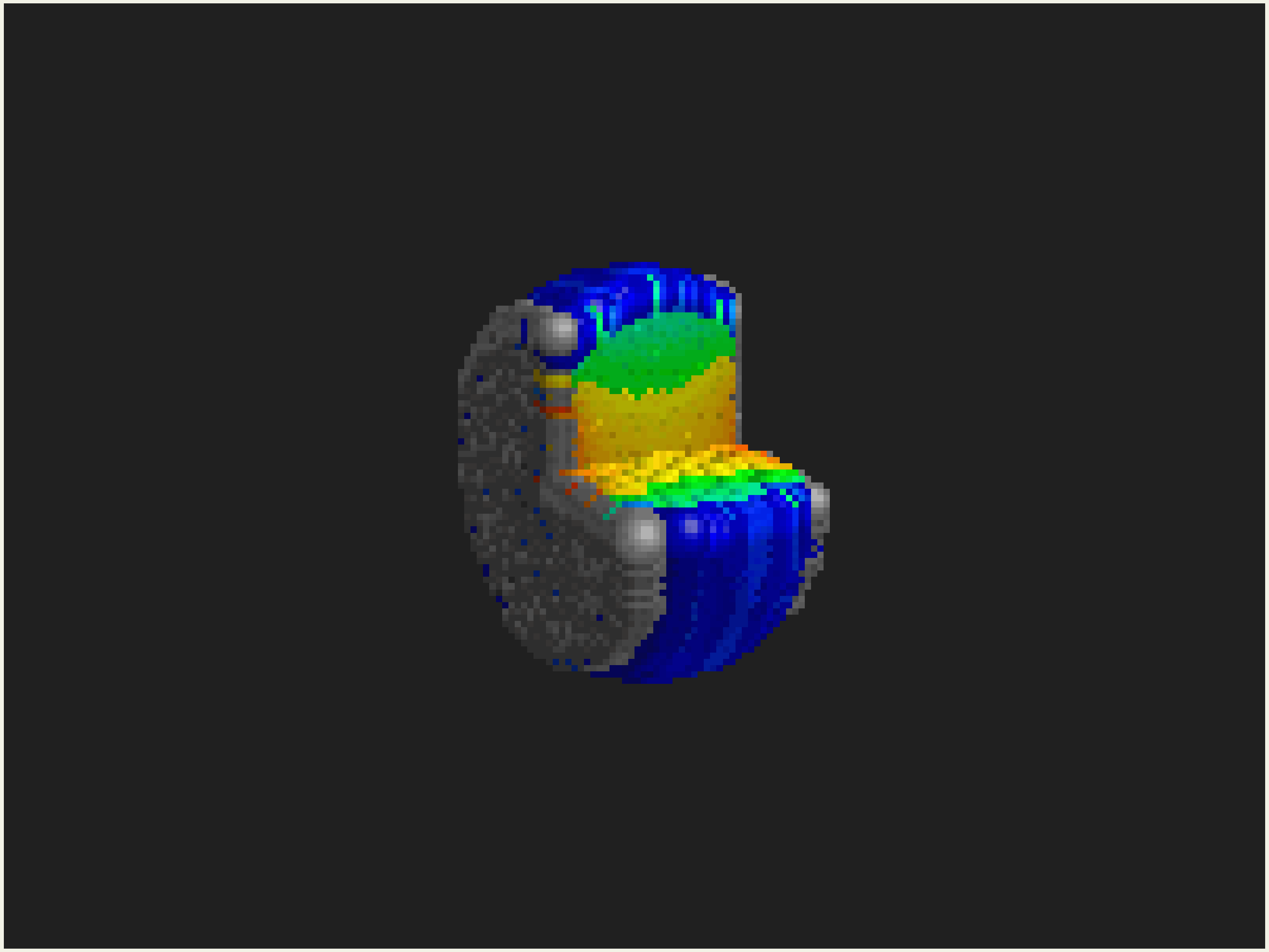

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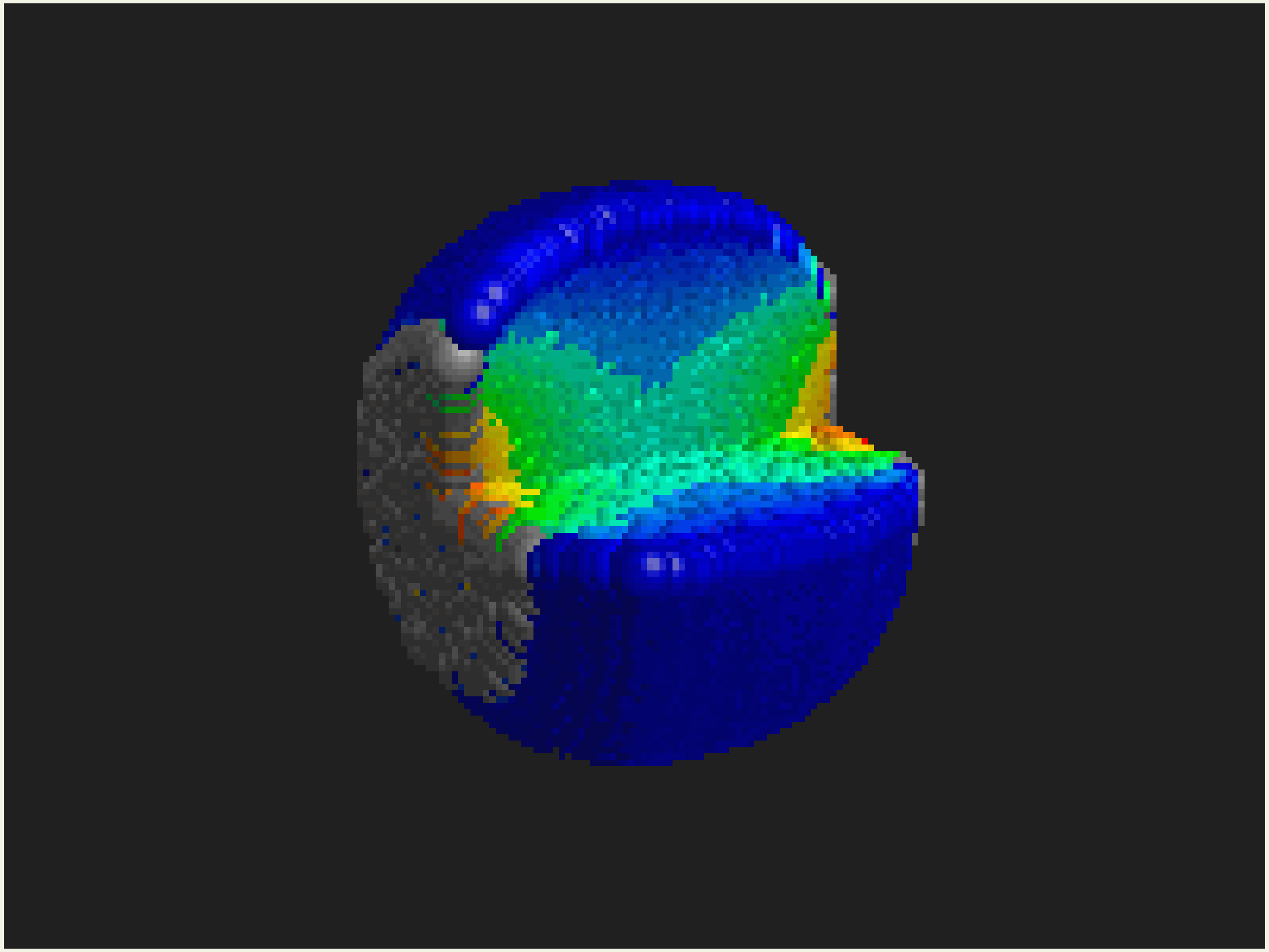

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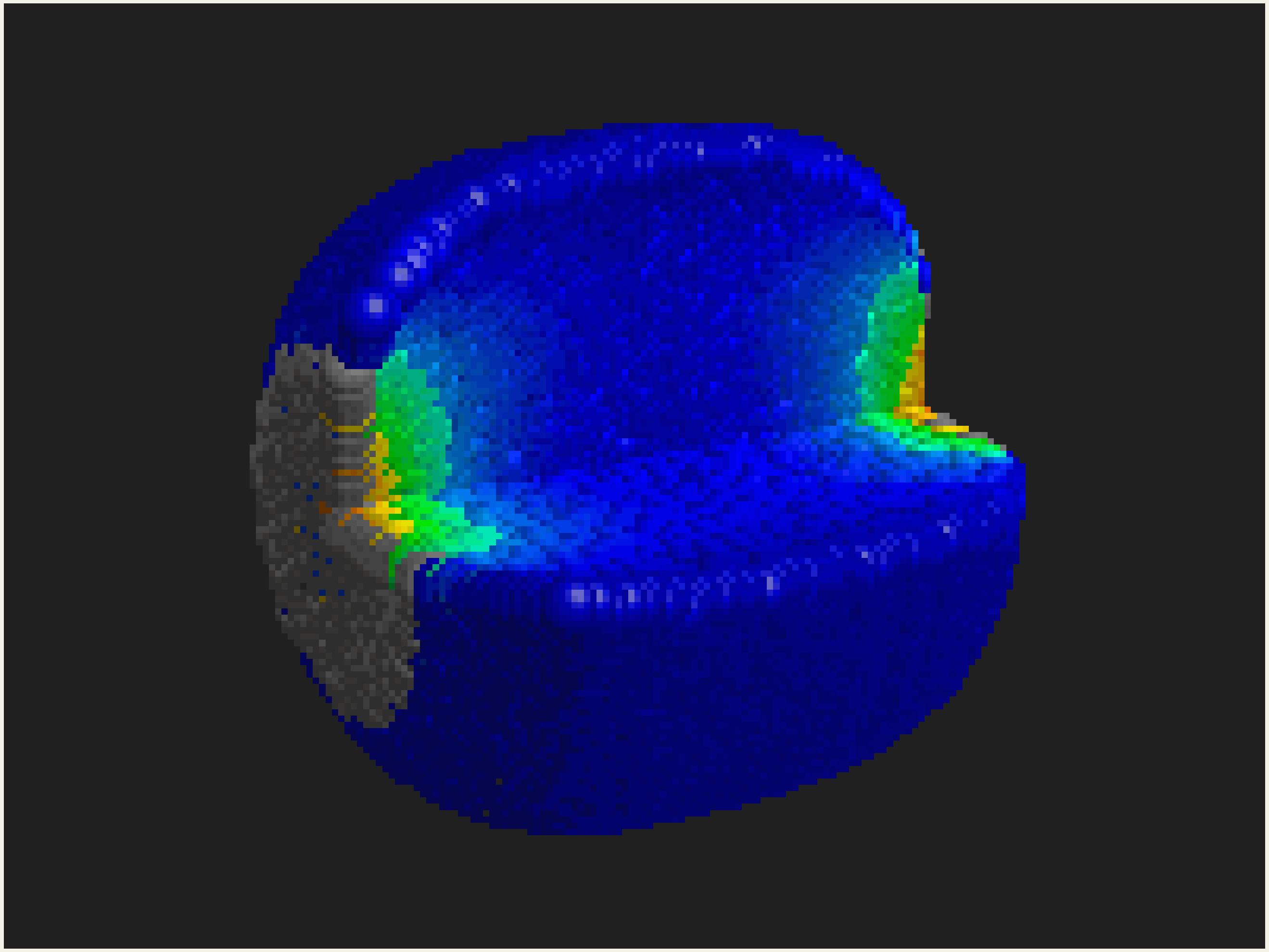
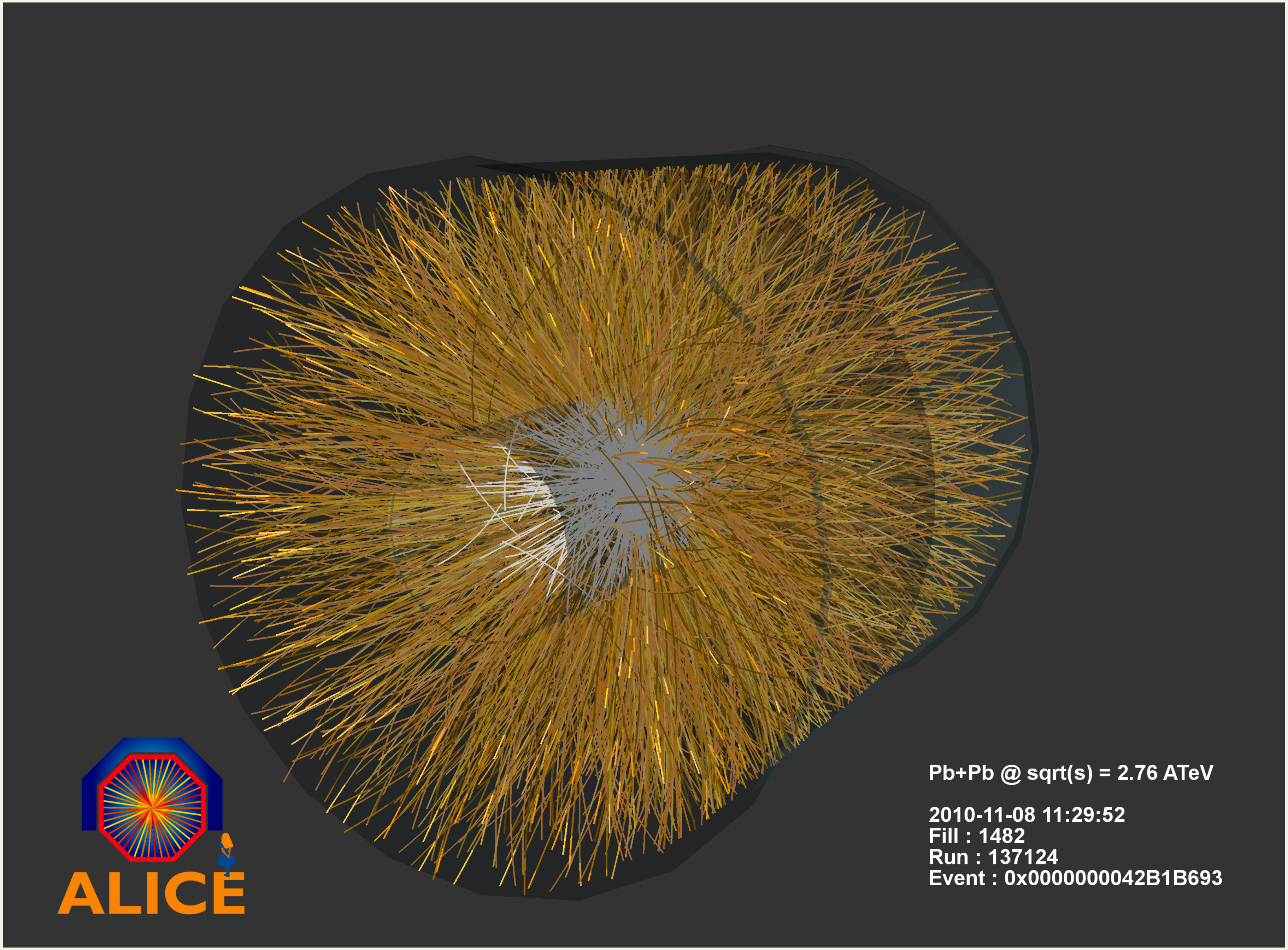
Heavy-ion collision

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Conservation laws

Conservation of energy and momentum:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

Conservation of charge:

$$\partial_\mu N^\mu(x) = 0$$

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Local conservation of particle number and energy-momentum.

$\Longleftrightarrow$ **Hydrodynamics!**

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Conservation of energy and momentum:

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Conservation of charge:

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Local conservation of particle number and energy-momentum.

$$\Longleftrightarrow \quad \textbf{Hydrodynamics!}$$

This can be generalized to multicomponent systems and systems with several conserved charges:

$$\partial_\mu N_i^\mu = 0,$$

$i =$ **baryon number, strangeness, charge. . .**

Consider only baryon number conservation, $i = B$.

$\Rightarrow$ **5 equations contain 14 unknowns!**

$\Rightarrow$ **The system of equations does not close.**

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$\Rightarrow$ **Provide 9 additional equations or**
Eliminate 9 unknowns.

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$\Rightarrow$ **5 equations** contain **14 unknowns**!

$\Rightarrow$ **The system of equations does not close.**

$\Rightarrow$ Provide **9 additional equations** or
Eliminate **9 unknowns**.

What are the components of $T^{\mu\nu}$ and N^μ?

- N^μ and $T^{\mu\nu}$ can be decomposed with respect to **arbitrary, normalized, time-like 4-vector** u^μ,

$$u_\mu u^\mu = 1$$

- Define a **projection operator**

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \quad \Delta^{\mu\nu} u_\nu = 0,$$

which **projects on the 3-space** orthogonal to u^μ.

- **Then**

$$\boxed{N^\mu = nu^\mu + \nu^\mu}$$

where

$n = N^\mu u_\mu$	**is (baryon) charge density in the frame where** $u = (1, \mathbf{0})$**, local rest frame, LRF**
$\nu^\mu = \Delta^{\mu\nu} N_\nu$	**is charge flow in LRF,**

and

$$\boxed{T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}}$$

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$ **energy density in LRF**

$P \equiv -\frac{1}{3}\Delta^{\mu\nu} T_{\mu\nu}$ **isotropic pressure in LRF**

$W^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$ **energy flow in LRF**

$\pi^{\mu\nu} \equiv [\frac{1}{2}(\Delta^\mu{}_\alpha \Delta^\nu{}_\beta + \Delta^\nu{}_\beta \Delta^\mu{}_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}] T^{\alpha\beta}$

(trace-free) stress tensor in LRF

- **The 14 unknowns in 5 equations:**

$$\left.\begin{matrix} N^\mu & 4 \\ T^{\mu\nu} & 10 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} n,\ \epsilon,\ P & 3 \\ W^\mu & 3 \\ \nu^\mu & 3 \\ \pi^{\mu\nu} & 5 \end{matrix}\right.$$

- So far u^μ is arbitrary. It attains a physical meaning by relating it to N^μ or $T^{\mu\nu}$:

1. Eckart frame:

$$u^\mu_E \equiv \frac{N^\mu}{\sqrt{N_\nu N^\nu}}$$

   u^μ is 4-velocity of charge flow, $\nu^\mu = 0$.
   The 14 unknowns are n, ϵ, P, W^μ, $\pi^{\mu\nu}, u^\mu$.

2. Landau frame:

$$u^\mu_L \equiv \frac{T^{\mu\nu} u_\nu}{\sqrt{u_\alpha T^{\alpha\beta} T_{\beta\gamma} u^\gamma}}$$

   u^μ is 4-velocity of energy flow, $W^\mu = 0$.
   The 14 unknowns are n, ϵ, P, ν^μ, $\pi^{\mu\nu}, u^\mu$.

- In general, the hydrodynamical equations are not closed and cannot be solved uniquely.

Ideal hydrodynamics

Suppose particles are in **local thermodynamical equilibrium**, i.e., single particle phase space distribution function is given by:

$$f_i(x,k) = \frac{g}{(2\pi)^3}\left[\exp\left(\frac{k_\mu u^\mu(x) - \mu(x)}{T(x)}\right) \pm 1\right]^{-1}$$

where

$T(x)$ and $\mu(x)$:	local temperature and chemical potential
$u(x)^\mu$:	local 4-velocity of fluid flow.

Then kinetic theory definitions give

$$N^\mu(x) \equiv \sum_i q_i \int \frac{\mathrm{d}^3\mathbf{k}}{E}\, k^\mu f_i(x,k) = n(T,\mu)\, u^\mu$$

$$T^{\mu\nu}(x) \equiv \sum_i \int \frac{\mathrm{d}^3\mathbf{k}}{E}\, k^\mu k^\nu f_i(x,k)$$

$$= (\epsilon(T,\mu) + P(T,\mu))\, u^\mu u^\nu - P(T,\mu)\, g^{\mu\nu}$$

where

$$n(T,\mu) = \sum_i q_i \int \mathrm{d}^3\mathbf{k}\, f_i(x,E)$$ **is local charge density,**

$$\epsilon(T,\mu) = \sum_i \int \mathrm{d}^3\mathbf{k}\, E f_i(x,E)$$ **is local energy density and**

$$P(T,\mu) = \sum_i \int \mathrm{d}^3\mathbf{k}\, \frac{\mathbf{k}^2}{3E} f_i(x,E)$$ **is local pressure.**

Note! $f(x,E)$ **is distribution in local rest frame:** $u^\mu = (1,\mathbf{0})$.

$\rightarrow$ **Local thermodynamical equilibrium implies there is no viscosity:**

$$\boxed{\nu^\mu = W^\mu = \pi^{\mu\nu} = 0.}$$

Ideal fluid approximation:

$$
\begin{aligned}
N^\mu &= n u^\mu \\
T^{\mu\nu} &= (\epsilon + P) u^\mu u^\nu - P g^{\mu\mu}
\end{aligned}
$$

- **Now N^μ and $T^{\mu\nu}$ contain 6 unknowns, ϵ, P, n and u^μ, but there are still only 5 equations!**
- **In thermodynamical equilibrium ϵ, P and n are not independent! They are specified by two variables, T and μ.**
- **The equation of state (EoS), $P(T,\mu)$ eliminates one unknown!**
- **Any equation of state of the form**

$$P = P(\epsilon, n)$$

closes the system of hydrodynamic equations and makes it uniquely solvable (given initial conditions).

Entropy in ideal fluid

is conserved!

$$\partial_\mu S^\mu = 0$$

where $S^\mu = s u^\mu$**.**

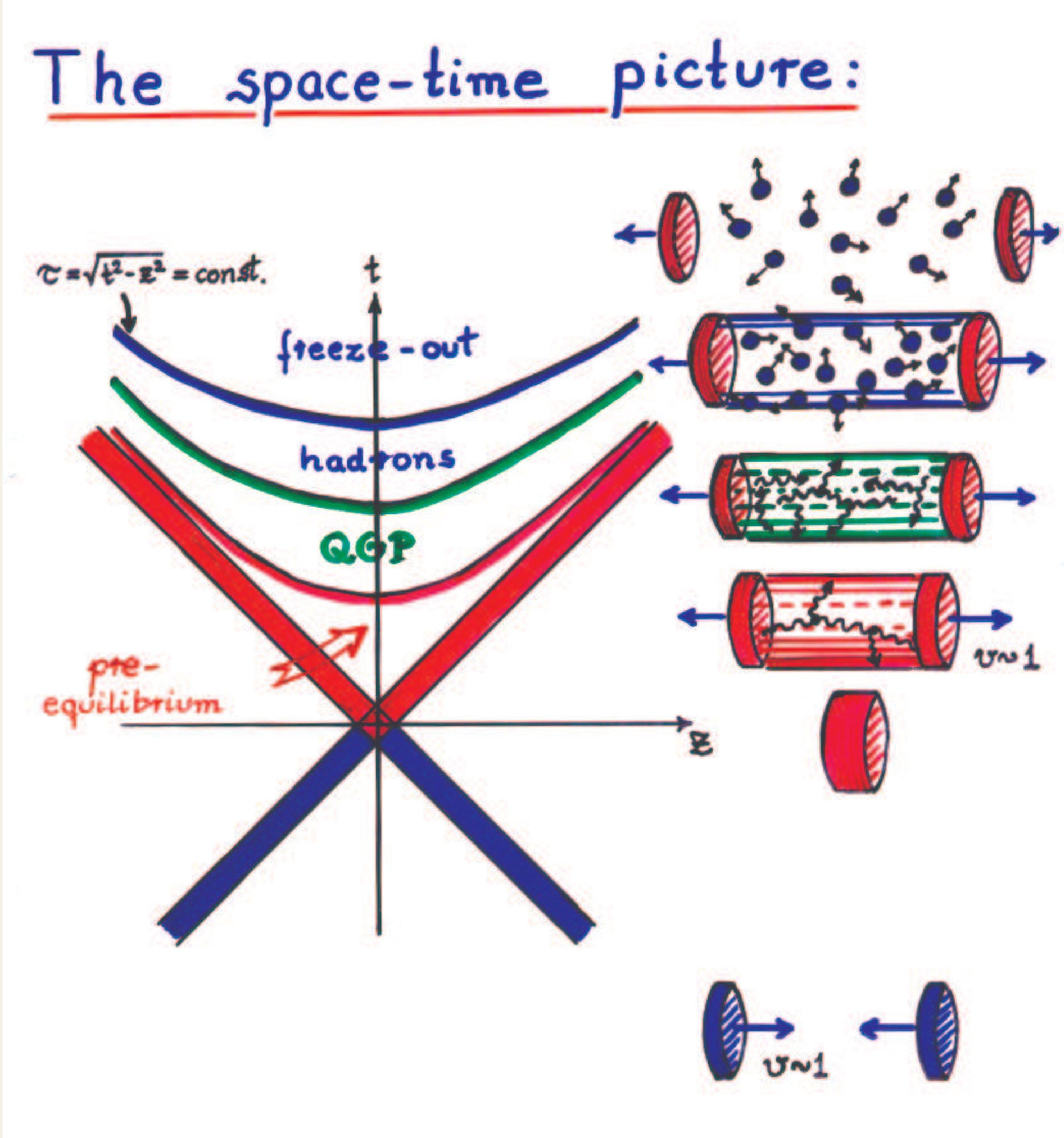
Equations of motion

Conservation laws lead to the equations of motion for relativistic fluid:

$$
\begin{aligned}
Dn &= -n\partial_\mu u^\mu \\
D\epsilon &= -(\epsilon + P)\partial_\mu u^\nu \\
(\epsilon + P)Du^\mu &= \nabla^\mu P,
\end{aligned}
$$

where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu.$$


The space-time picture:
τ = √(t²-z²) = const.
t
freeze-out
hadrons
QGP
pre-equilibrium
z
v~1
v~1

Usefulness of hydro?

- **Initial state:** unknown
- **Equation of state:** unknown
- **Transport coefficients:** unknown
- **Freeze-out:** unknown

$\Rightarrow$ **Predictive power?**

– ***"Hydro doesn't know where to start nor where to end"*** **(M. Prakash)**

Usefulness of hydro?

- **Initial state:** unknown
- **Equation of state:** want to study
- **Transport coefficients:** want to study
- **Freeze-out:** unknown

$\Big\}\Rightarrow$ **Predictive power?**

$\Longrightarrow$ **Need More Constraints!**

"Hydrodynamical method"

1. Use another model to fix unknowns (and add new assumptions. . .)
   - initial: color glass condensate or pQCD+saturation
   - initial and/or final: hadronic cascade
   - etc.

2. Use data to fix parameters:

Principle		Example @ RHIC
• use one set of data	$\Longleftrightarrow$	$\left.\frac{\mathrm{d}N}{\mathrm{d}y\, p_T\, \mathrm{d}p_T}\right\|_{b=0}$ and $\frac{\mathrm{d}N}{\mathrm{d}y}(b)$
• fix parameters to fit it	$\Longleftrightarrow$	$\begin{cases} \epsilon_{0,\mathrm{max}} = 29.6\,\mathbf{GeV/fm}^3 \\ \tau_0 = 0.6\,\mathbf{fm}/c \\ T_{\mathrm{fo}} = 130\,\mathbf{MeV} \end{cases}$
• predict another set of data	$\Longleftrightarrow$	HBT, photons & dileptons, elliptic flow. . .

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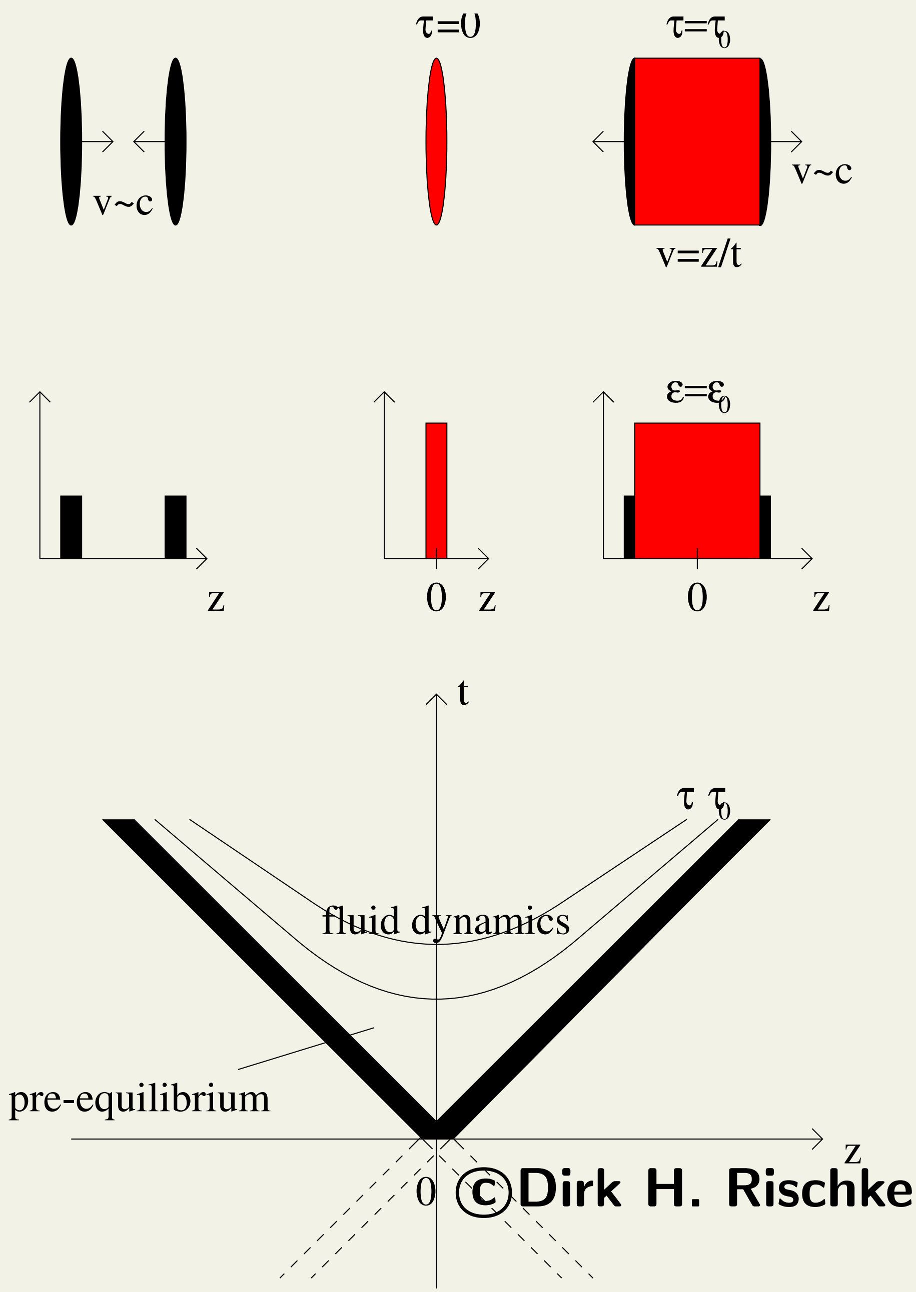
\end{aligned}
$$

where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu.$$

Bjorken hydrodynamics

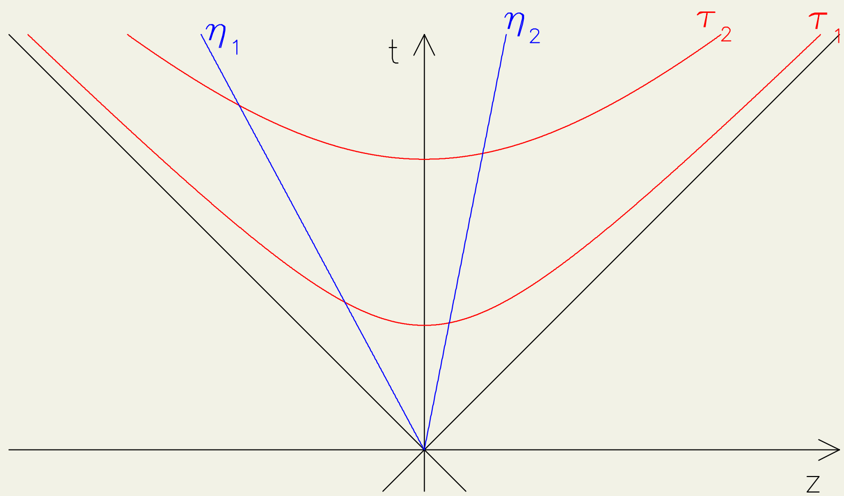


- **At very large energies, $\gamma \to \infty$ and "Landau thickness" $\to 0$**

- **Lack of longitudinal scale**
  $\Rightarrow$ **scaling flow**

$$v = \frac{z}{t}$$

- **Practical coordinates to describe scaling flow expansion are**



  - **Longitudinal proper time τ:**

$$\tau \equiv \sqrt{t^2 - z^2} \quad \Leftrightarrow \quad t = \tau \cosh \eta$$

  - **Space-time rapidity η:**

$$\eta = \frac{1}{2} \ln \frac{t+z}{t-z} \quad \Leftrightarrow \quad z = \tau \sinh \eta$$

- **Scaling flow** $v = z/t \Rightarrow$ **fluid flow rapidity** $y = \eta$**:**

$$y = \frac{1}{2}\ln\frac{1+v}{1-v} = \frac{1}{2}\ln\frac{1+z/t}{1-z/t} = \eta$$

- **Ignore transverse expansion:**
  Hydrodynamic equations turn out to be particularly simple:

$$\left.\frac{\partial\epsilon}{\partial\tau}\right|_{\eta} = -\frac{\epsilon+P}{\tau} \quad (1)$$

$$\left.\frac{\partial P}{\partial\eta}\right|_{\tau} = 0 \quad (2)$$

$$\left.\frac{\partial n}{\partial\tau}\right|_{\eta} = -\frac{n}{\tau} \quad (3)$$

- **Eq. (2)** $\Rightarrow$
  - **No force between fluid elements with different** η**!**
  - $P = P(\tau)$**, no** η**-dependence!**

- **Eq. (2) + thermodynamics:**

$$0 = \left.\frac{\partial P}{\partial \eta}\right|_\tau = s\left.\frac{\partial T}{\partial \eta}\right|_\tau + n\left.\frac{\partial \mu}{\partial \eta}\right|_\tau$$

If $n = 0$, $T = T(\tau) \Rightarrow T =$ const. on $\tau =$ const. surface.

- **In general T and ϵ not constant on $\tau =$ const. surface, but usually they are assumed to be**
  $\Rightarrow$ boost invariance: the system looks the same in all reference frames!

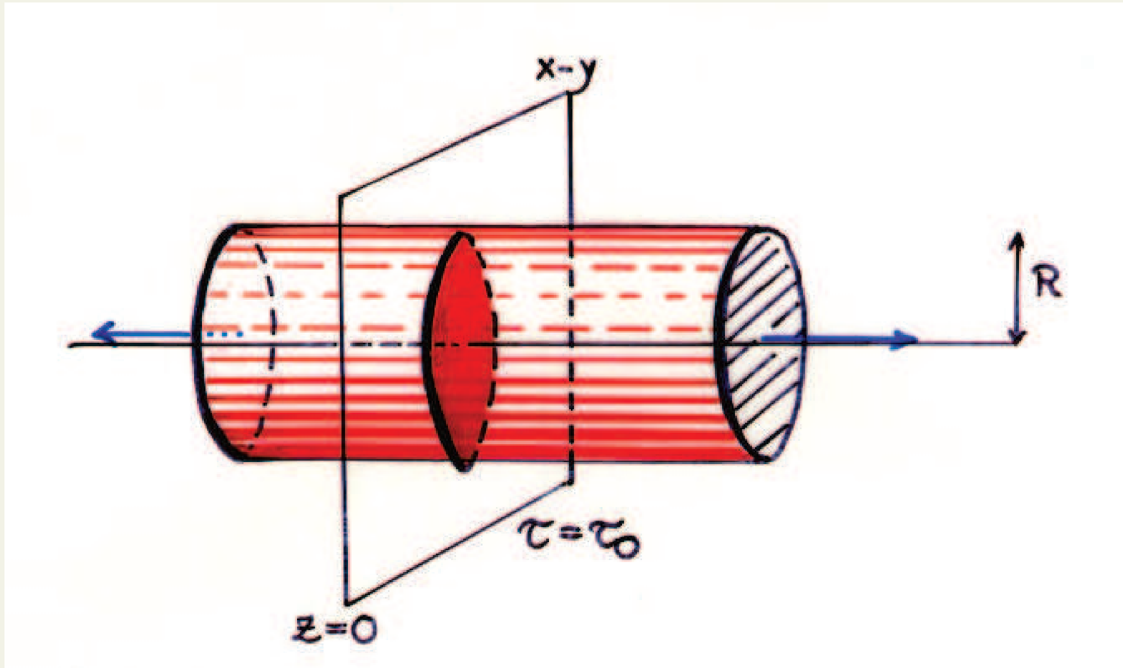
$$\epsilon = \epsilon(\tau), \qquad n = n(\tau)$$

- **Note that still**

$$\frac{\partial}{\partial \eta} T^{\mu\nu} \neq 0 \neq \frac{\partial}{\partial \eta} u^\mu$$

Vector and tensor quantities at finite η Lorentz boosted from values at $\eta = 0$

Transverse expansion and flow



- Transverse expansion will set in latest at $\tau = R/c_s \approx 10$ fm
- Lifetimes in one dimensional expansion ~ 30 fm
- One dimensional expansion an oversimplification

- 2+1D: longitudinal Bjorken, transverse expansion solved numerically
- 3+1D: expansion in all directions solved numerically

Initial conditions

- **Initial time** from early thermalization argument (+finetuning. . .)
- **Total entropy** to fit the multiplicity
- **Density distribution?**
- **Multiplicity is proportional to the number of participants**
- **Glauber model: number of participants/binary collisions**

$$N_{part}(b) = \int \mathrm{d}x\mathrm{d}y\, T_A(x + b/2, y)[\dots$$

where

$$T_A(x, y) = \int_{-\infty}^{\infty} \mathrm{d}z\, \rho(x, y, z) \qquad \text{and} \qquad \rho(x, y, z) = \frac{\rho_0}{1 + e^{\frac{r-R_0}{a}}}$$

are nuclear thickness function and nuclear density distribution

- **"Differential Optical Glauber:"**
  Number of participants per unit area in transverse plane:

$$
\begin{aligned}
n_{\mathrm{WN}}(x,y;b) &= T_A(x+b/2,y)\left[1-\left(1-\frac{\sigma}{B}T_B(x-b/2,y)\right)^B\right] \\
&+ T_B(x-b/2,y)\left[1-\left(1-\frac{\sigma}{A}T_A(x-b/2,y)\right)^A\right]
\end{aligned}
$$

Number of binary collisions per unit area

$$n_{\mathrm{BC}}(x,y;b) = \sigma_{pp}T_A(x+b/2,y)T_B(x-b/2,y)$$

- **MC-Glauber:**
  - **sample $\rho(x,y,z)$ to get the positions of nucleons in 2 nuclei**
  - **count # of nucleons closer than $\sqrt{\sigma_{\mathrm{pp}}/\pi}$ in the collision**
  - **this gives n_{WN} and n_{BC}**
  - **repeat to get enough statistics**

Equation of state

- Final state includes π's, K's, nucleons. . .
  $\Rightarrow$ EoS of interacting hadron gas
  $\Rightarrow$ well approximated by non-interacting gas of hadrons and resonances

$$P(T) = \sum_i \int \mathrm{d}^3p \, \frac{p^2}{3E} \, f(p,T)$$

- Plasma EoS (=massless parton gas) with proper statistics and $\mu_B \neq 0$:

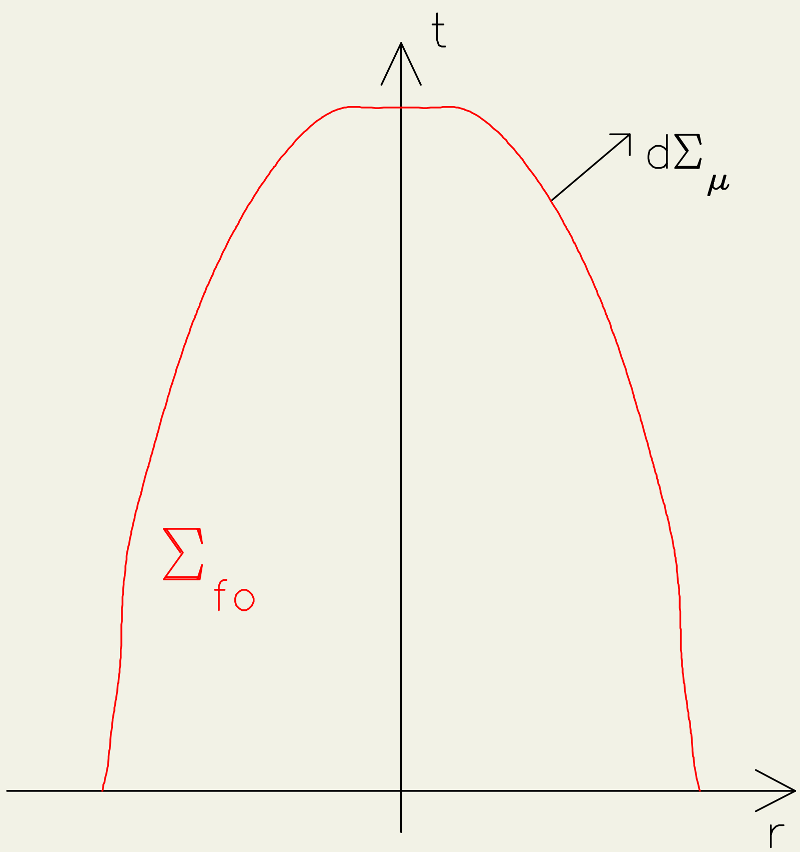
$$P(T,\mu) = \frac{(32+21N_f)\pi^2}{180} T^4 + \frac{1}{9}\mu_B^2 T^2 + \frac{1}{192\pi^2}\mu_B^4 - B$$

  $\Rightarrow$ First order phase transition by Maxwell construction

- OR parametrized lattice result (only at $\mu_B = 0$):
  $\Rightarrow$ match your favourite smoothly to HRG

When to end?

- **Particles are observed, not fluid**
- **How and when to convert fluid to particles?**
- **i.e. how far is hydro valid?**



- **Kinetic equilibrium requires scattering rate** $\gg$ **expansion rate**
- **Scattering rate** $\tau_{\rm sc}^{-1} \sim \sigma n \propto \sigma T^3$
- **Expansion rate** $\theta = \partial_\mu u^\mu$
- **Fluid description breaks down when** $\tau_{\rm sc}^{-1} \approx \theta$

$\rightarrow$ **momentum distributions freeze-out**

- $\tau_{\rm sc}^{-1} \propto T^3 \rightarrow$ **rapid transition to free streaming**
- **Approximation: decoupling takes place on constant temperature hypersurface** $\Sigma_{\rm fo}$, **at** $T = T_{\rm fo}$

Hybrid models

- **End hydro when rescatterings still frequent**
- **Convert fluid to particle ensembles**
- **Describe evolution of particles using hadronic transport**
- **Advantages:**
  - **chemical evolution and dissipation described**
  - **physical decoupling**
- **Disadvantages:**
  - **all the unknowns of hadronic cascade. . .**
  - **where and how to switch?**
- **Note: The switch from fluid to cascade is NOT freeze-out**
  ⇒ particlization

Cooper-Frye

- **Number of particles emitted = Number of particles crossing $\Sigma_{\rm fo}$**

$$\Rightarrow \quad N = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, N^\mu$$

- **Frozen-out particles do not interact anymore: kinetic theory**

$$\Rightarrow \quad N^\mu = \int \frac{\mathrm{d}^3\mathbf{p}}{E} p^\mu f(x, p\cdot u)$$

$$\Rightarrow \quad N = \int \frac{\mathrm{d}^3\mathbf{p}}{E} \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)$$

- **Invariant single inclusive momentum spectrum: (Cooper-Frye formula)**

$$\boxed{E\frac{\mathrm{d}N}{\mathrm{d}\mathbf{p}^3} = \int_{\Sigma_{\rm fo}} \mathrm{d}\Sigma_\mu \, p^\mu f(x, p\cdot u)}$$

Cooper and Frye, PRD 10, 186 (1974)

Blast wave

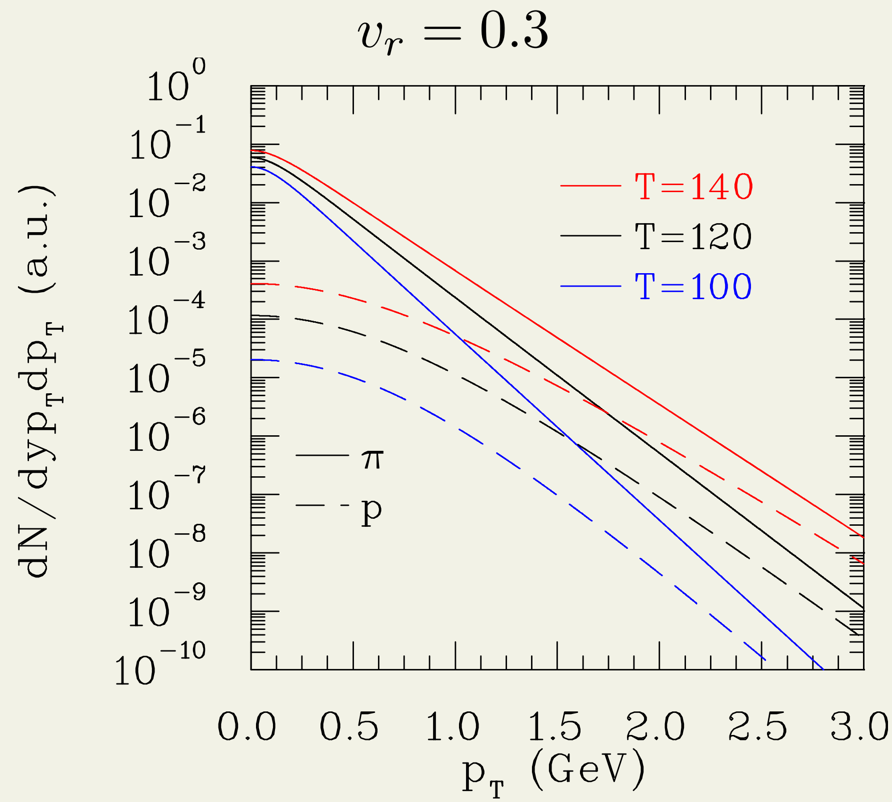
(Siemens and Rasmussen, PRL 42, 880 (1979))

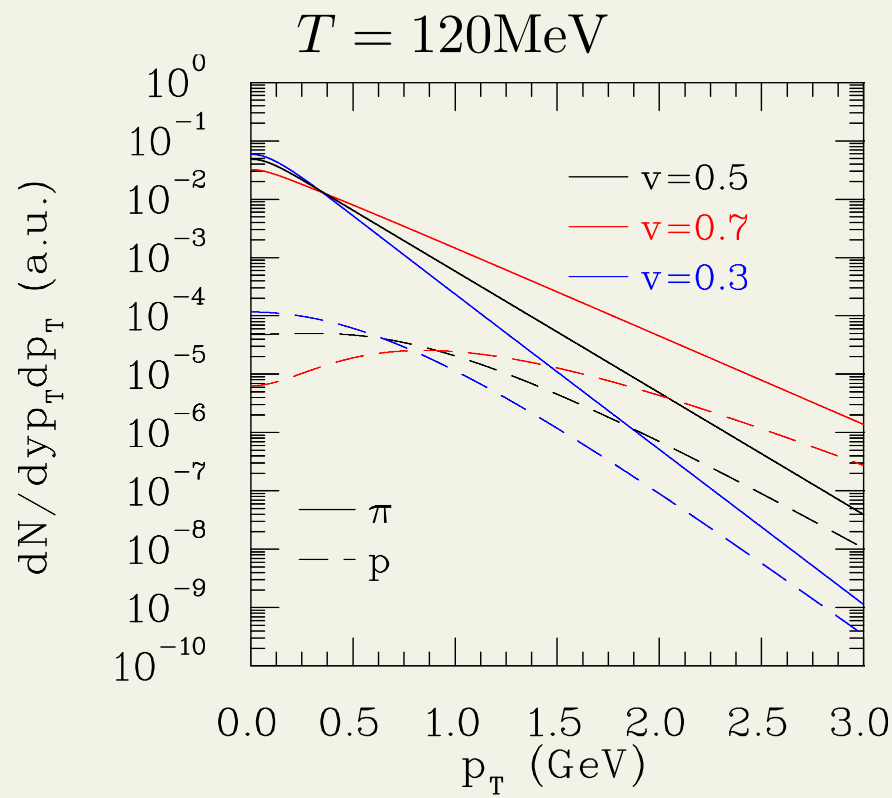
- **Freeze-out surface a thin cylindrical shell radius r, thickness $\mathrm{d}r$, expansion velocity v_r, decoupling time τ_{fo}, boost invariant**

- **Cooper-Frye for Boltzmannions**

$$\frac{\mathrm{d}N}{\mathrm{d}y\, p_T\, \mathrm{d}p_T} = \frac{g}{\pi}\,\tau_{\mathrm{fo}}\, r\, m_T\, \mathrm{I}_0\!\left(\frac{v_r\gamma_r p_T}{T}\right) \mathrm{K}_1\!\left(\frac{\gamma_r m_T}{T}\right)$$

effect of temperature and flow velocity





- **The larger the temperature, the flatter the spectra**
- **The larger the velocity, the flatter the spectra $\Rightarrow$ blueshift**
- **The heavier the particle, the more sensitive it is to flow (shape and slope)**

- **Define speed of sound c_s:**

$$c_s^2 = \left.\frac{\partial P}{\partial \epsilon}\right|_{s/n_b}$$

- **large $c_s \Rightarrow$ "stiff EoS"**

- **small $c_s \Rightarrow$ "soft EoS"**

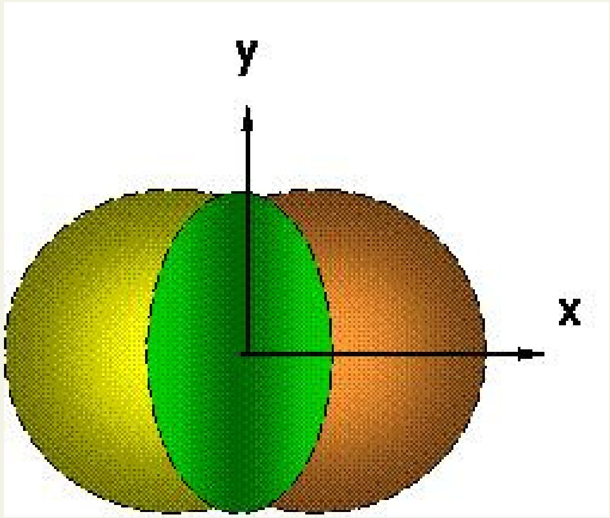
- **For baryon-free matter in rest frame**

$$(\epsilon + P)Du^\mu = \nabla^\mu P \qquad \Longleftrightarrow \qquad \frac{\partial}{\partial \tau} u_\mu = -\frac{c_s^2}{s}\partial_\mu s$$

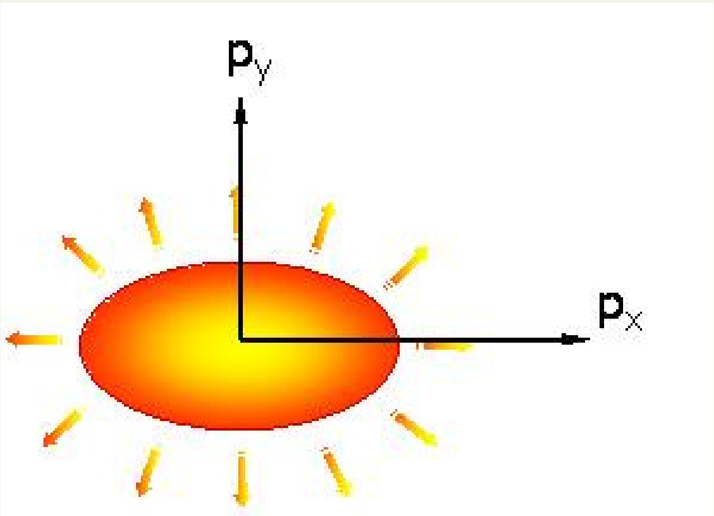
$\Rightarrow$ **The stiffer the EoS, the larger the acceleration**

Elliptic flow v_2

spatial anisotropy $\rightarrow$ final azimuthal momentum anisotropy



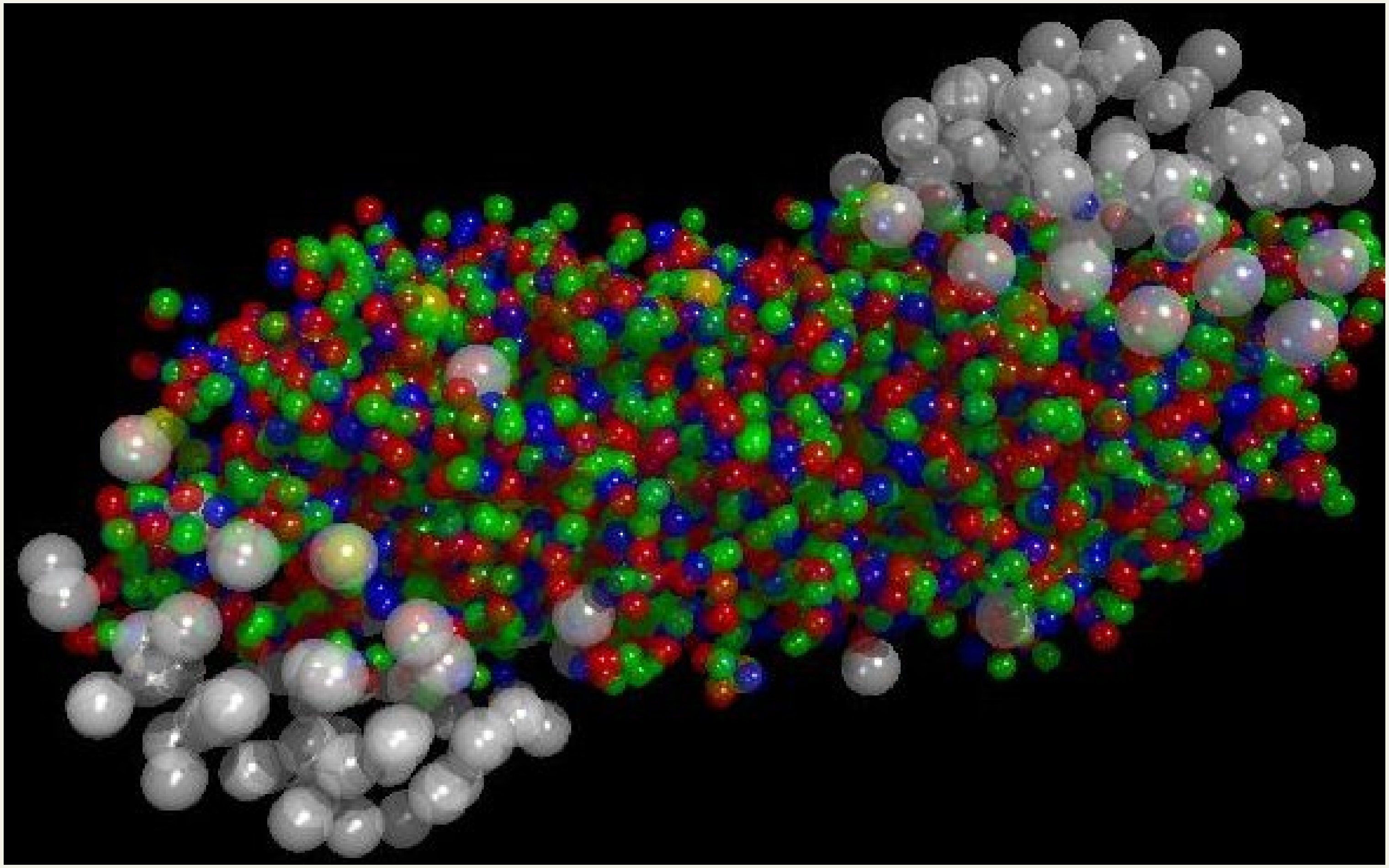
$\Rightarrow$



- Anisotropy in coordinate space + rescattering

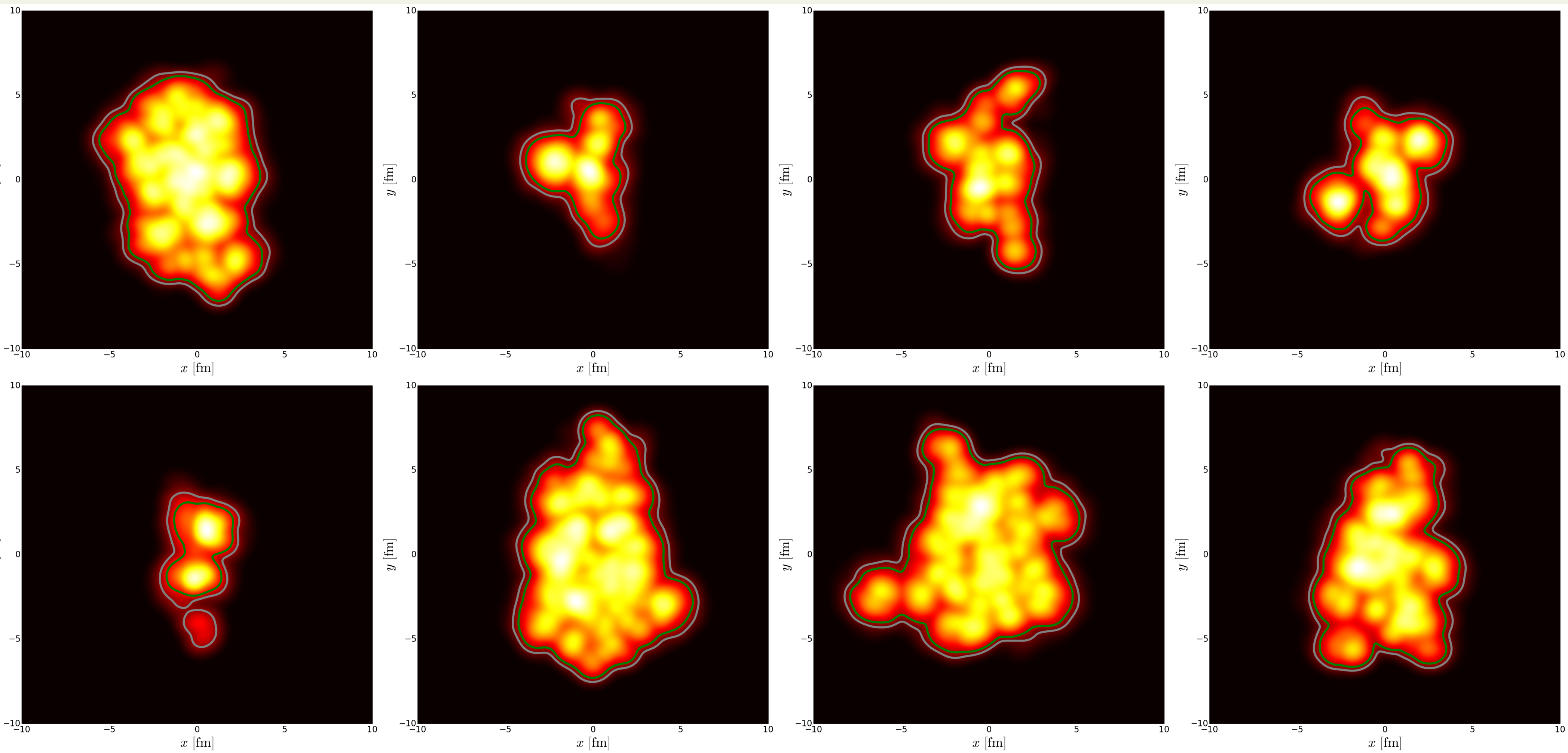
$\Rightarrow$ Anisotropy in momentum space

$$\frac{\partial}{\partial \tau} u_x = -\frac{c_s^2}{s}\frac{\partial}{\partial x} s \qquad \text{and} \qquad \frac{\partial}{\partial \tau} u_y = -\frac{c_s^2}{s}\frac{\partial}{\partial y} s$$



Initial state fluctuates

temperature profiles in transverse plane



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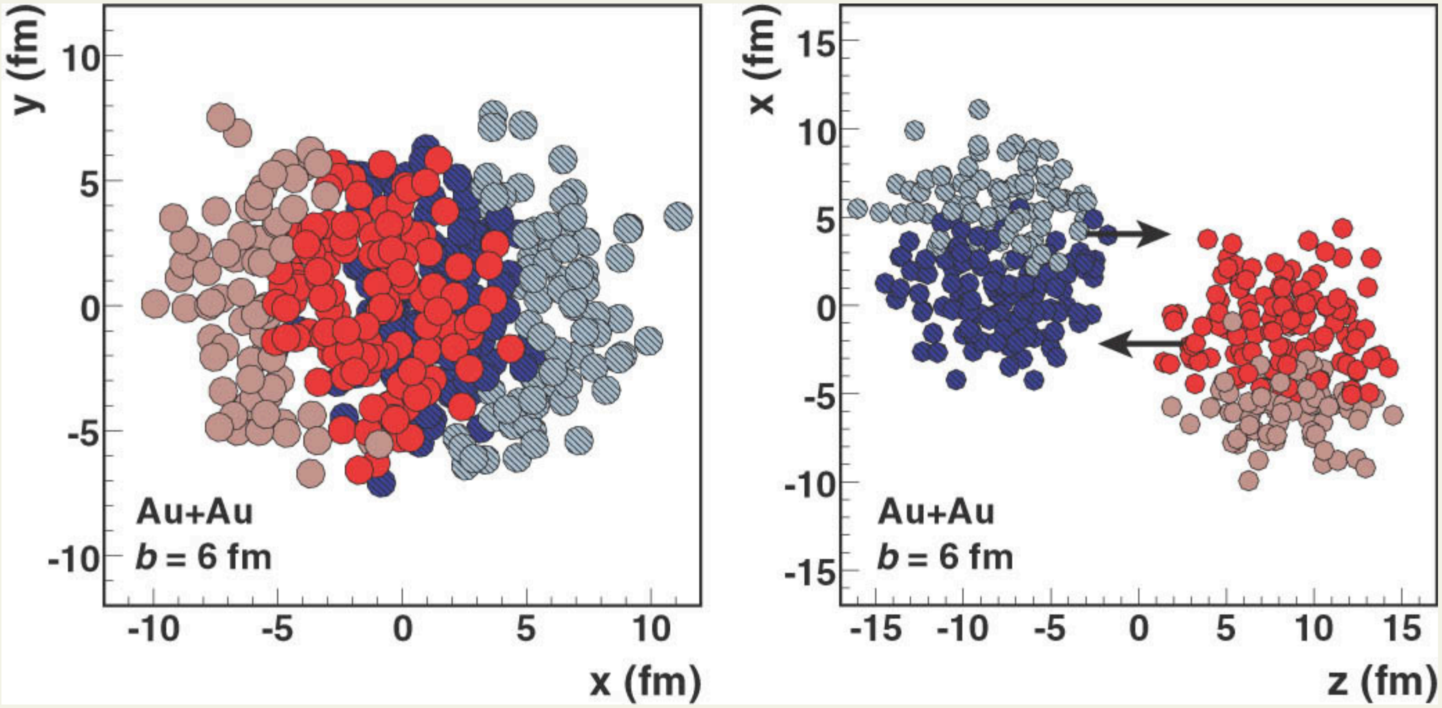
event-by-event



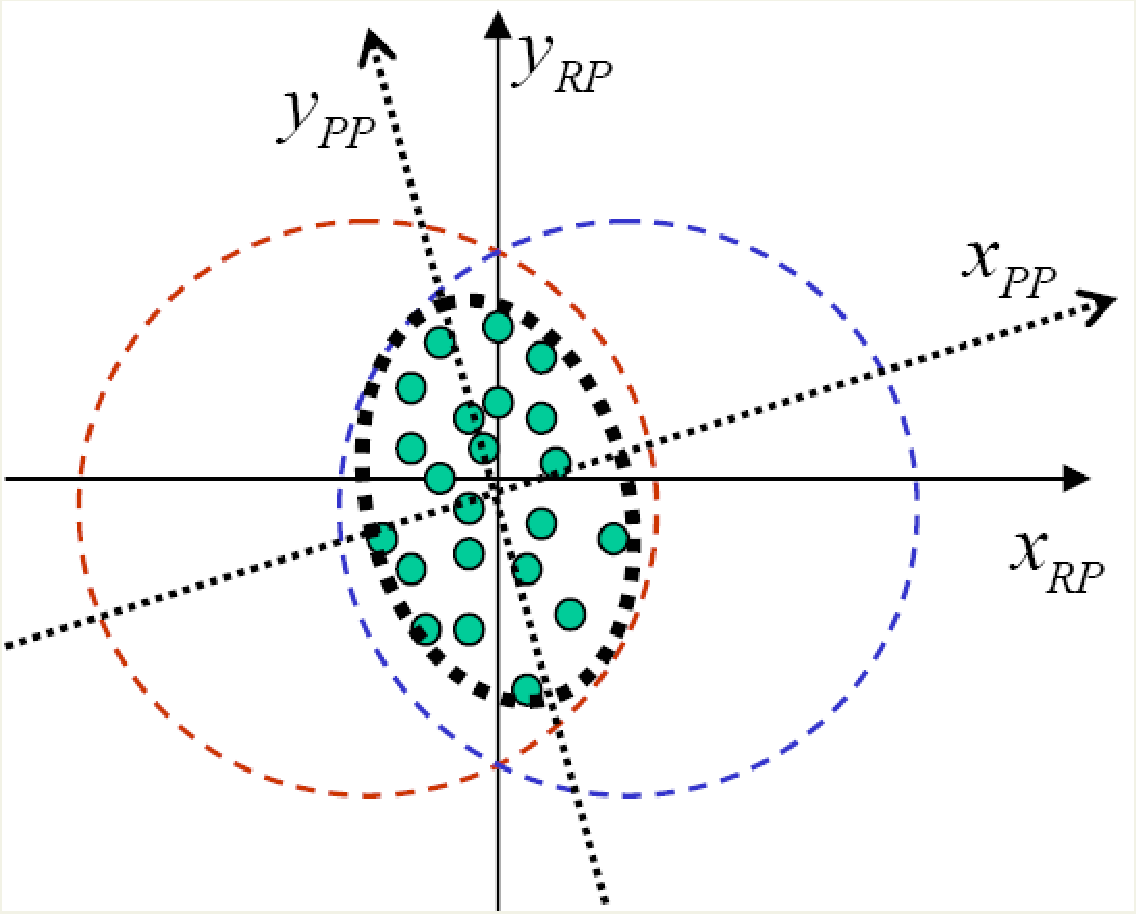
Miller *et al.*, Ann.Rev.Nucl.Part.Sci. 57, 205 (2007)

- shape fluctuates event-by-event
- all coefficients v_n finite

$$\frac{\mathrm{d}N}{\mathrm{d}y\mathrm{d}\phi} = \frac{\mathrm{d}N}{\mathrm{d}y}\left[1 + \sum_n 2v_n \cos(2(\phi - \Psi_n))\right]$$

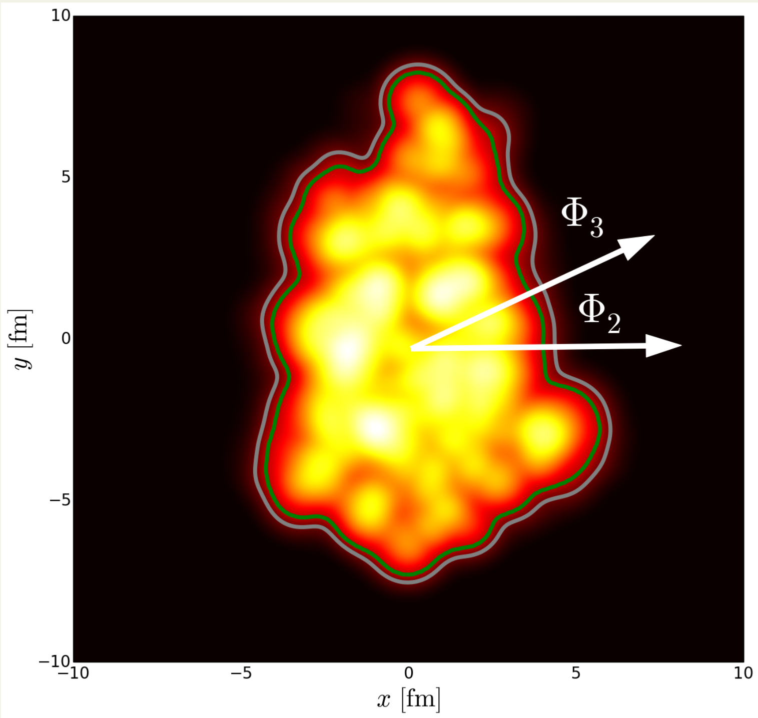
All the planes. . .

Voloshin *et al.* Phys. Lett. B 659, 537 (2008)

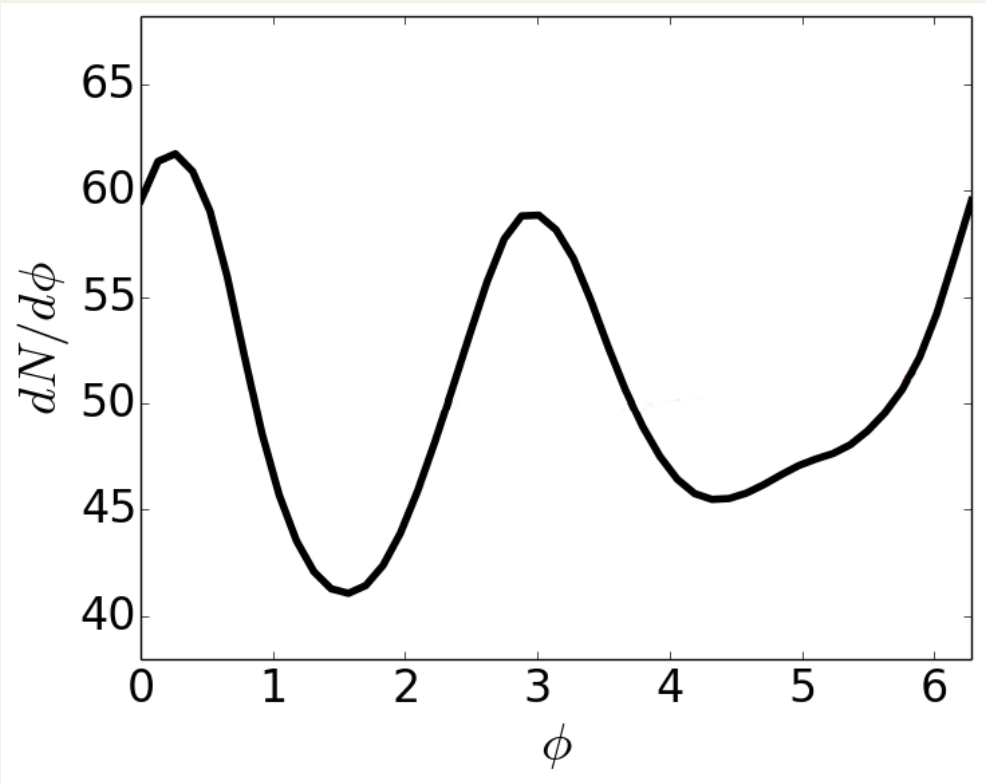


- X_{RP}: **Reaction plane, spanned by beam and impact parameter**
- X_{PP}: **Participant plane, maximises spatial anisotropy** ϵ_n
- Ψ_n: **Event plane, maximises anisotropy** v_n

From fluid to distribution



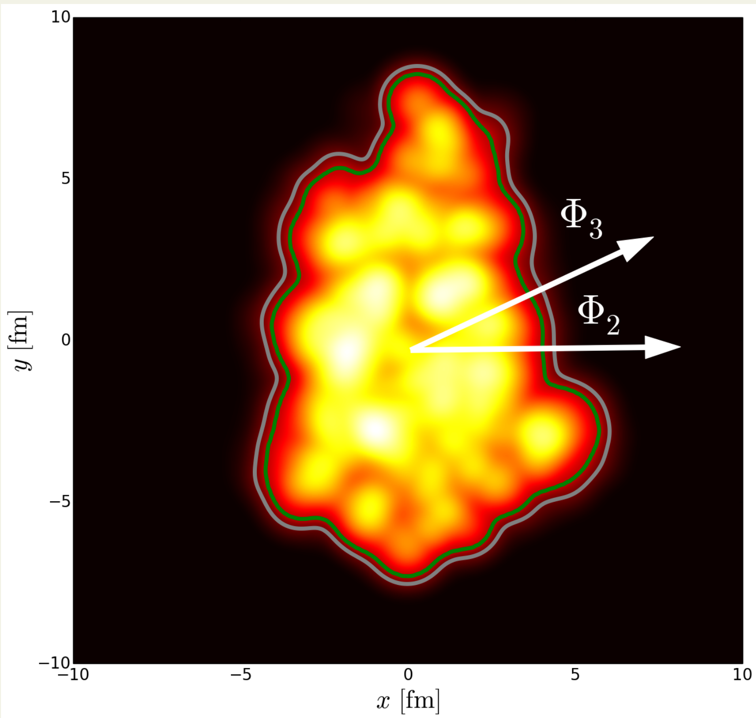
$\Longrightarrow$



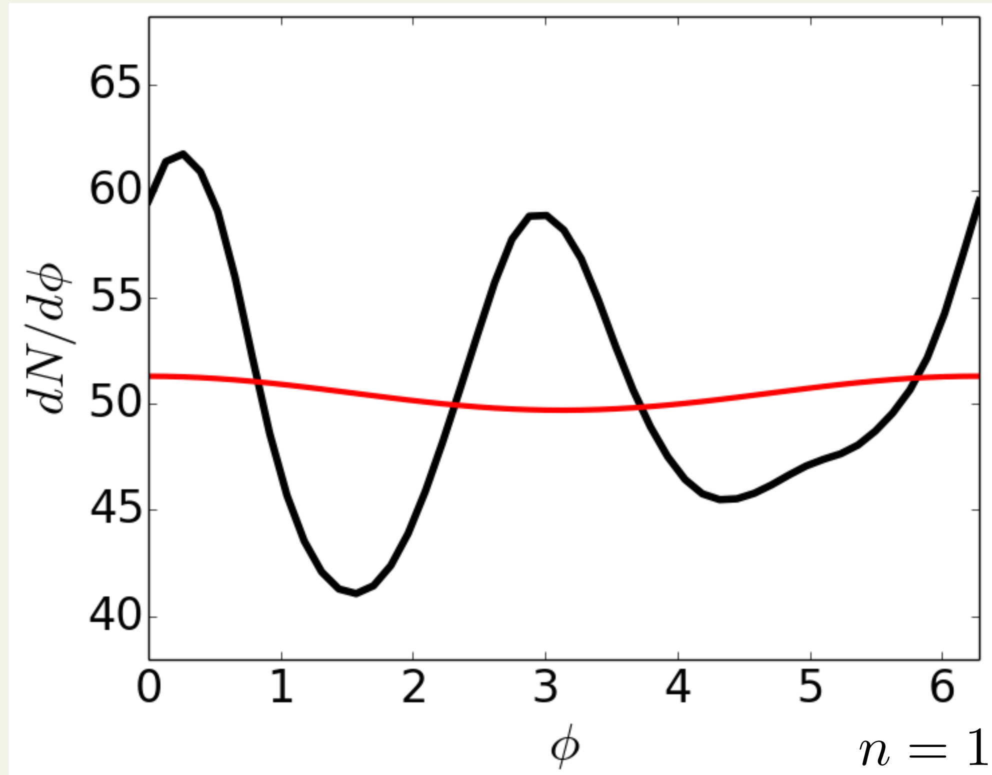
$$\epsilon_n, \Phi_n \Longrightarrow v_n, \Psi_n$$

$$\frac{\mathrm{d}N}{\mathrm{d}y\mathrm{d}\phi} = \frac{\mathrm{d}N}{\mathrm{d}y}\left[1 + \sum_n 2v_n \cos(n(\phi - \Psi_n))\right]$$

From fluid to distribution



$\Longrightarrow$



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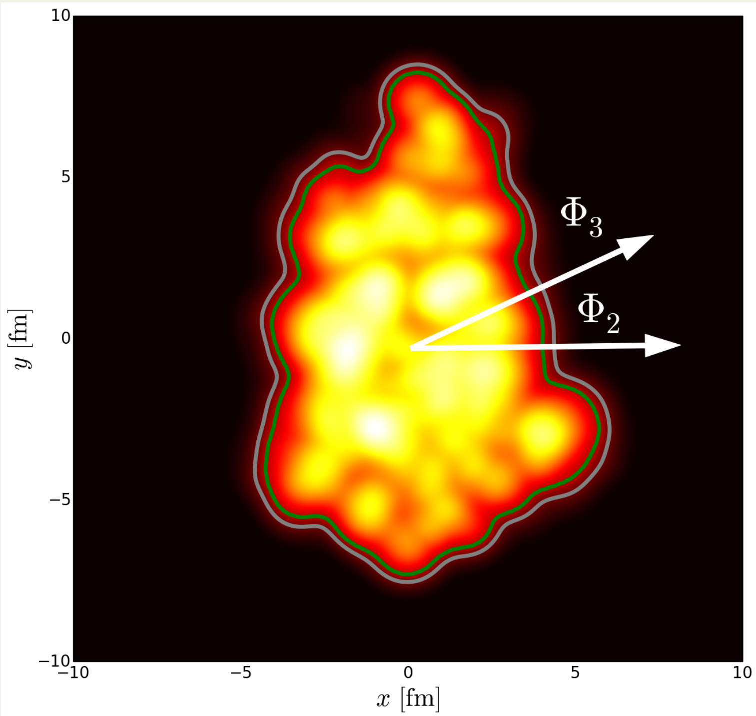
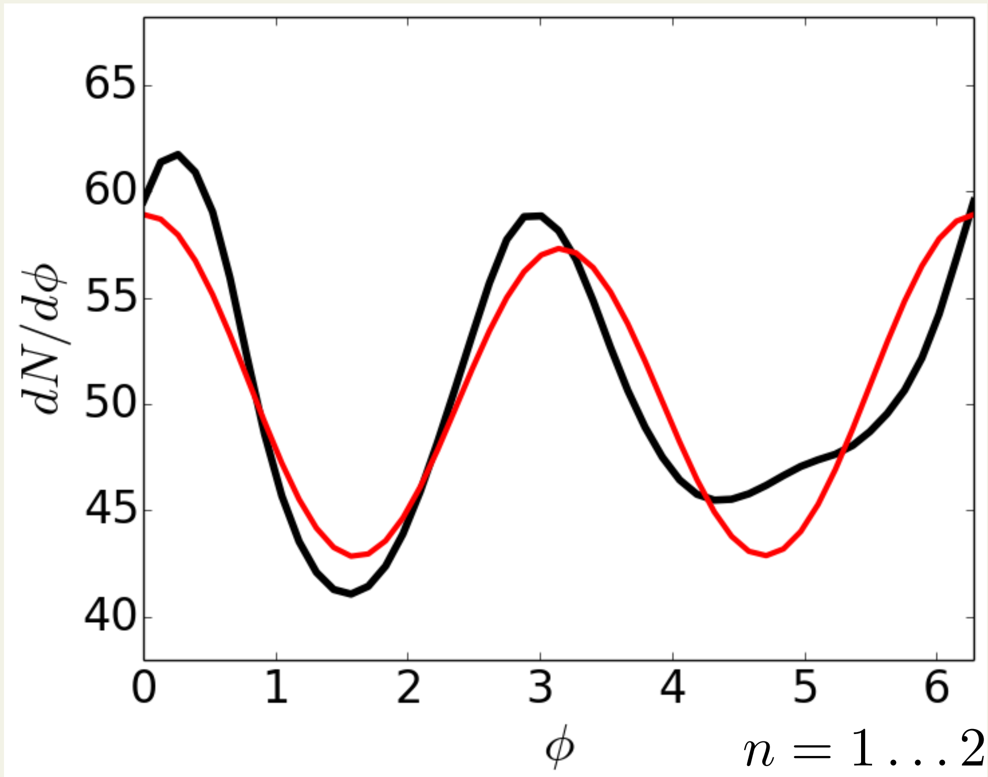


$\Longrightarrow$



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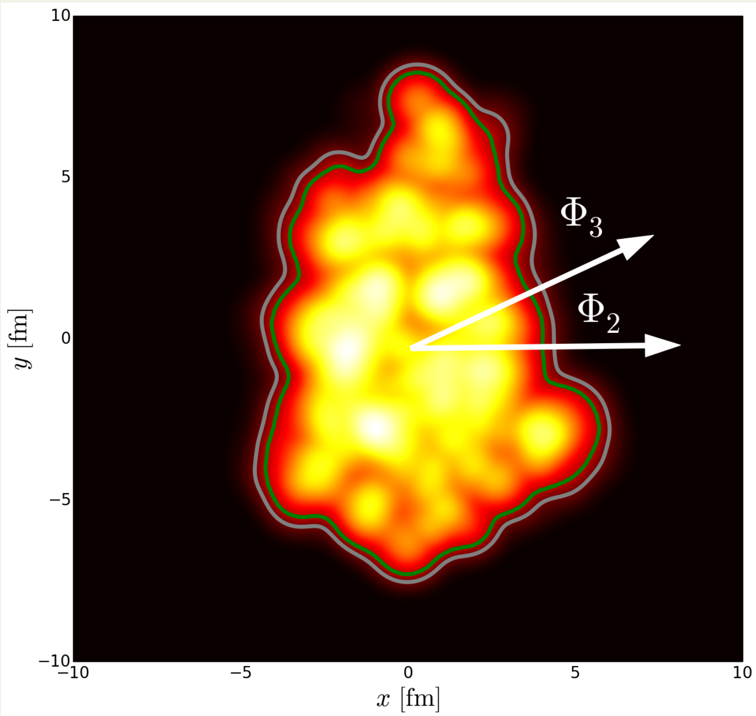
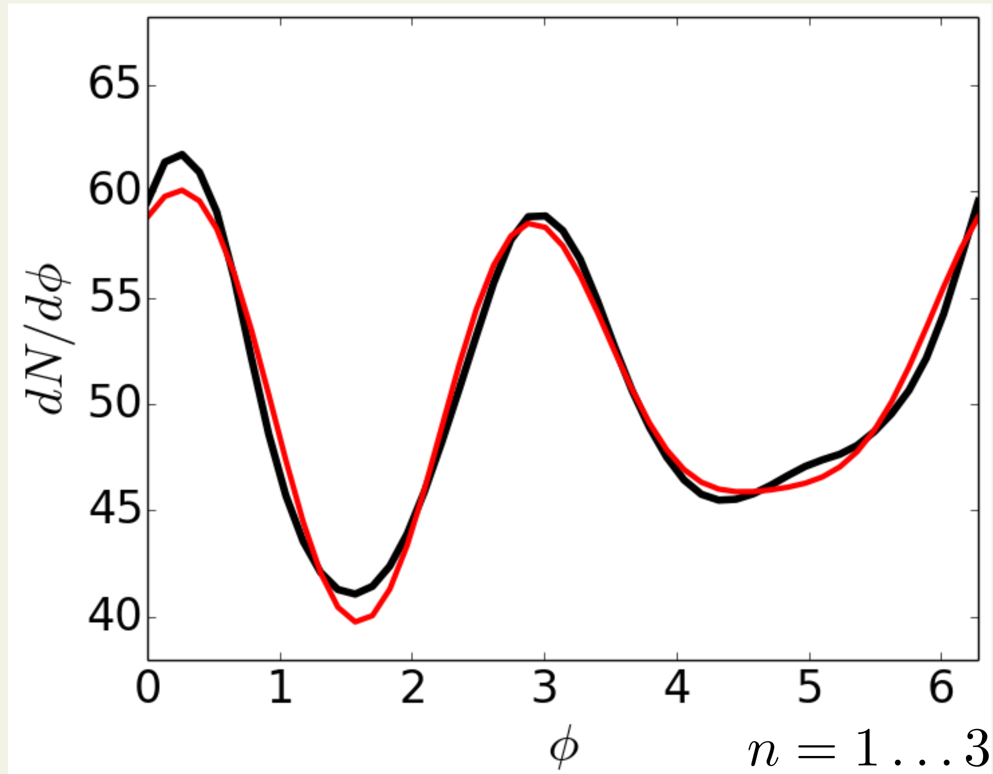
From fluid to distribution



$\Longrightarrow$



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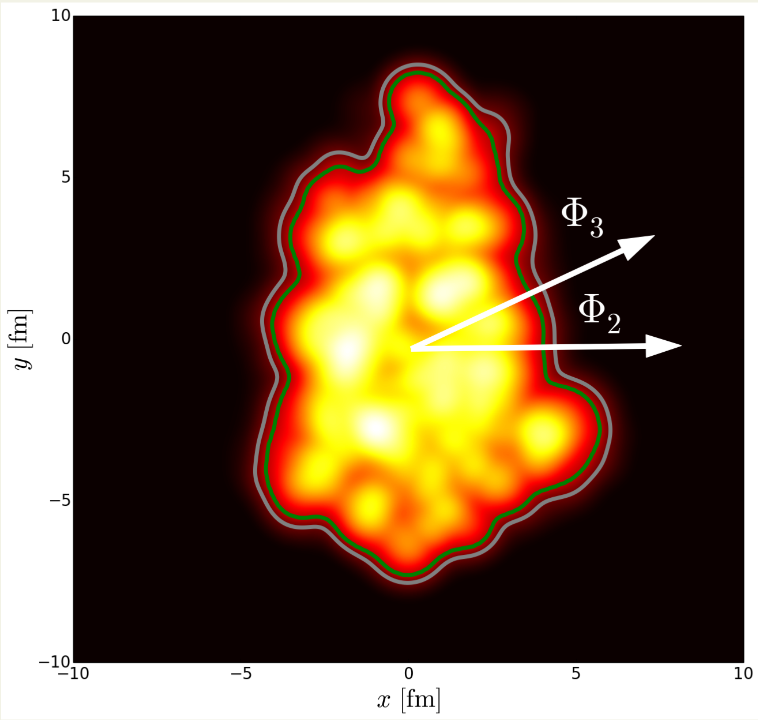
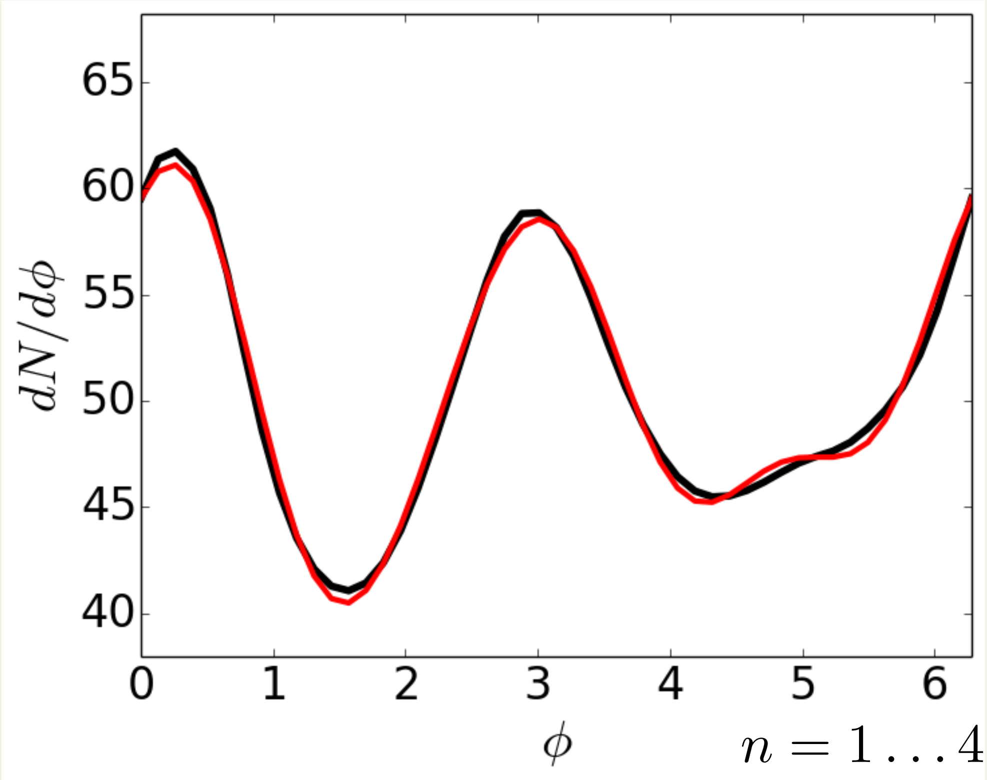
From fluid to distribution



$\Longrightarrow$



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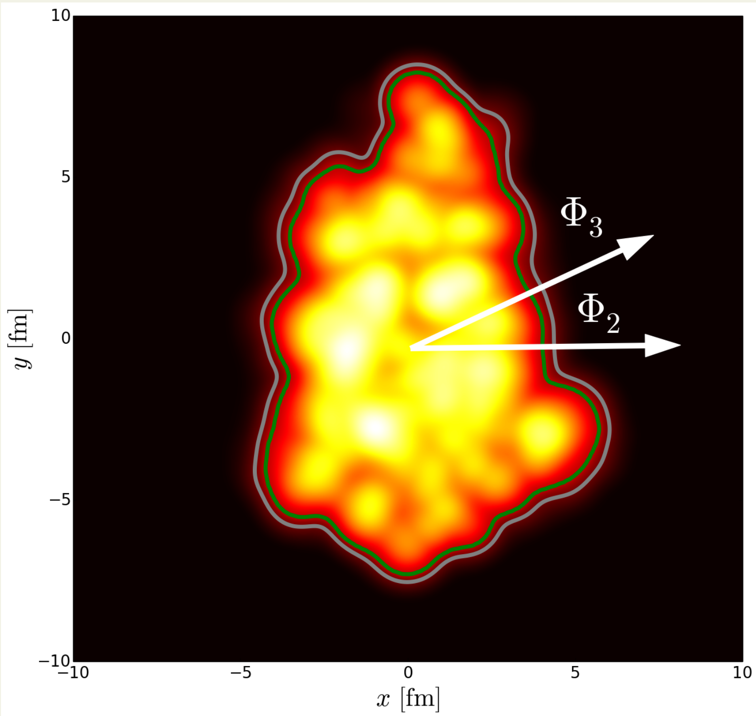

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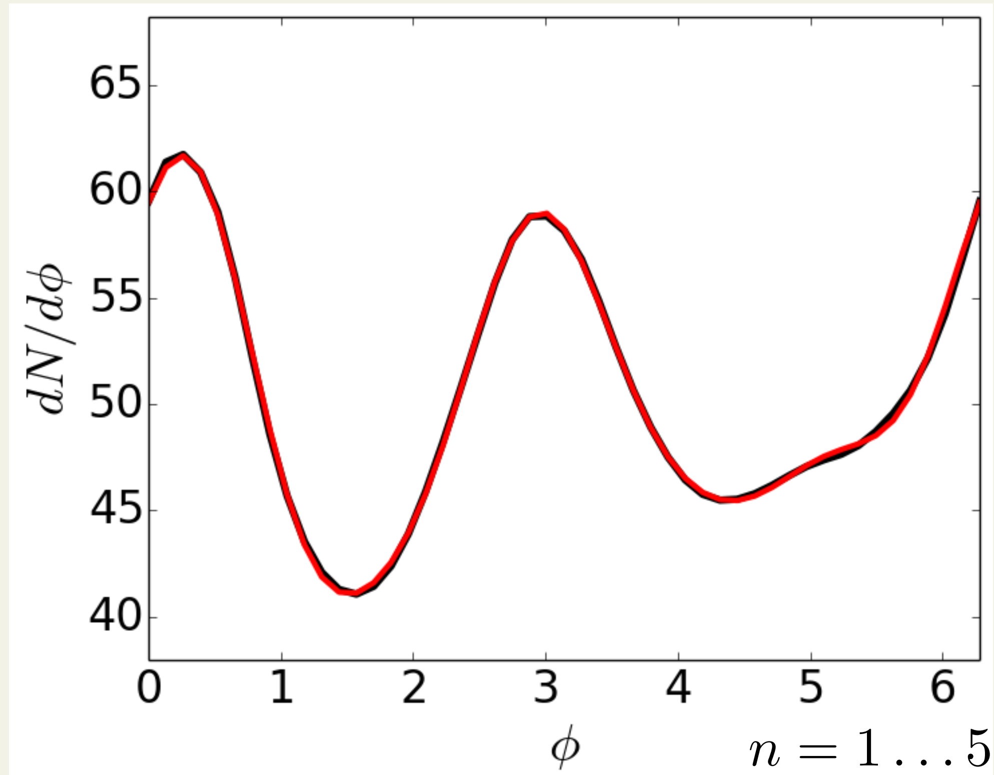

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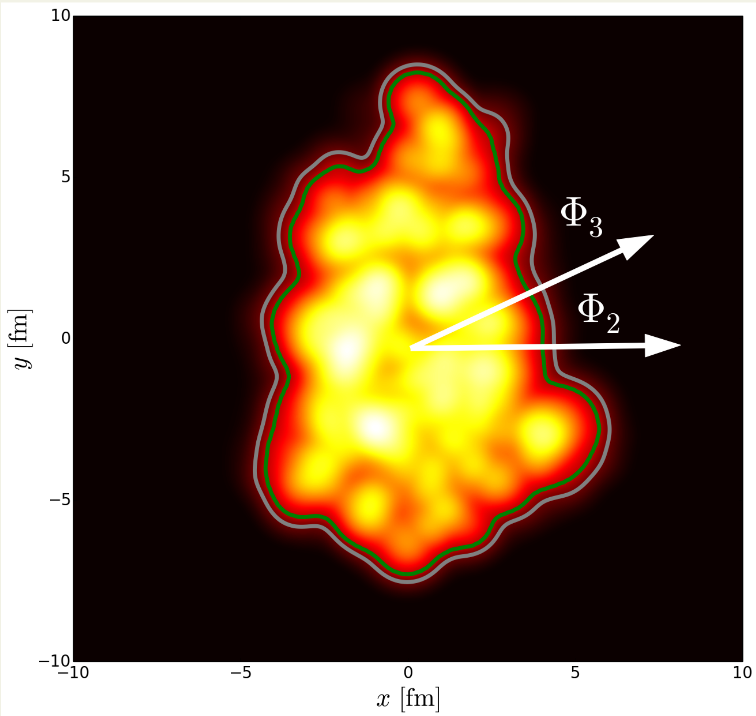
From fluid to distribution



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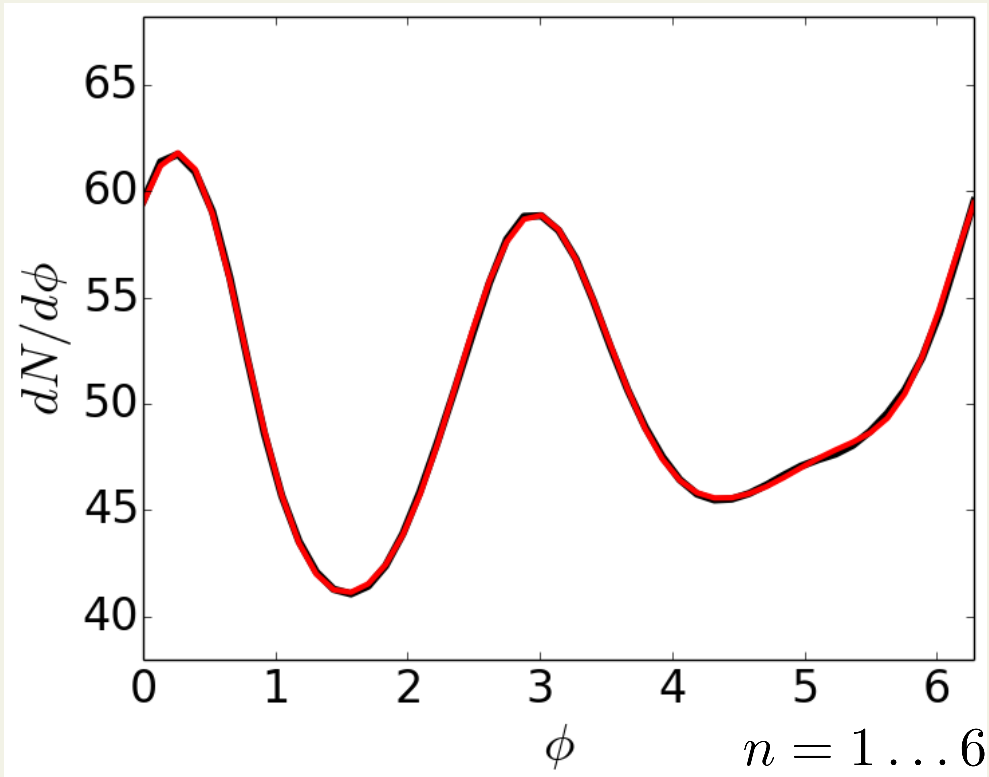

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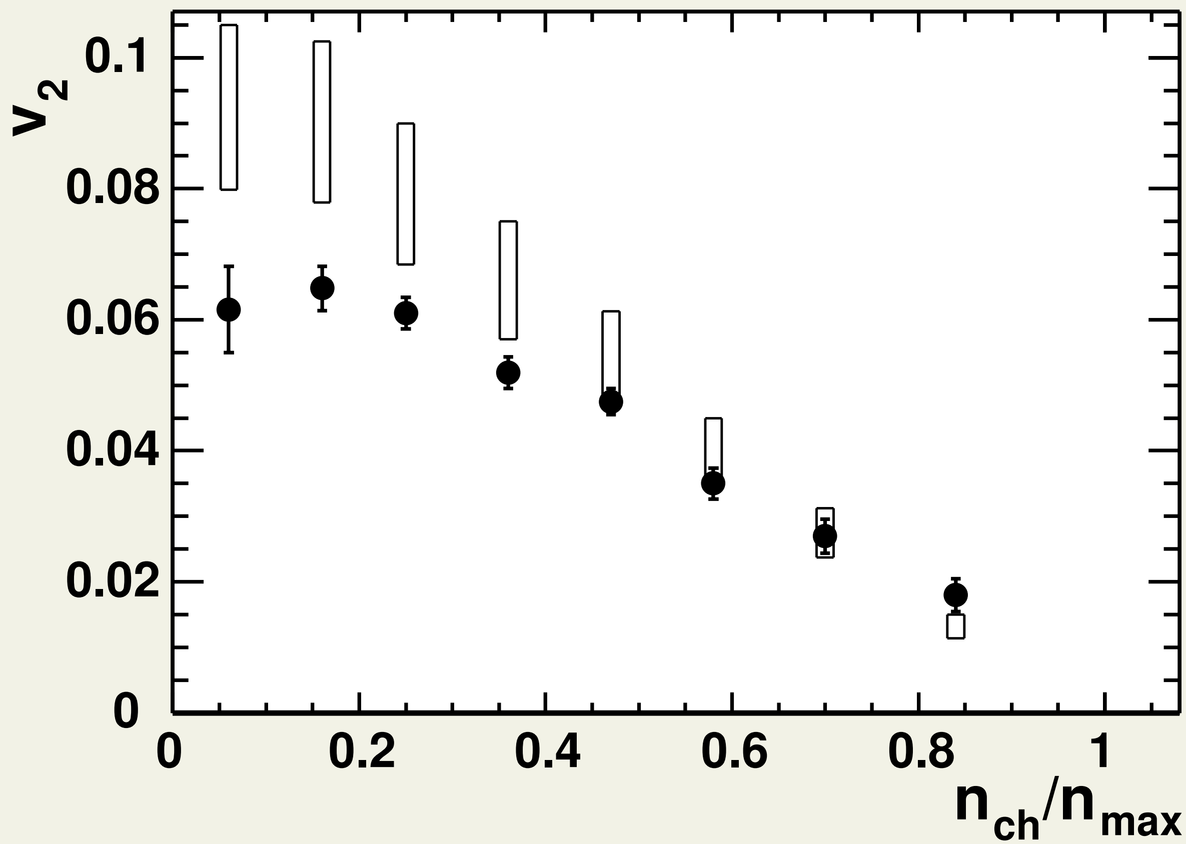
Success of ideal hydrodynamics

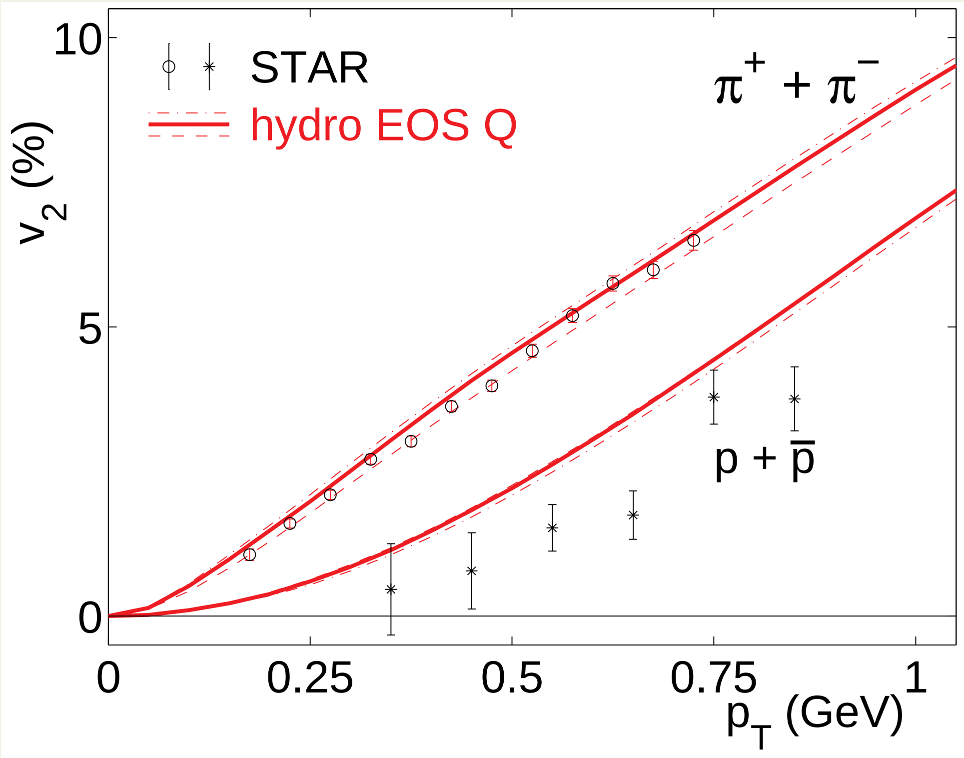
- p_T-averaged v_2 of charged hadrons:

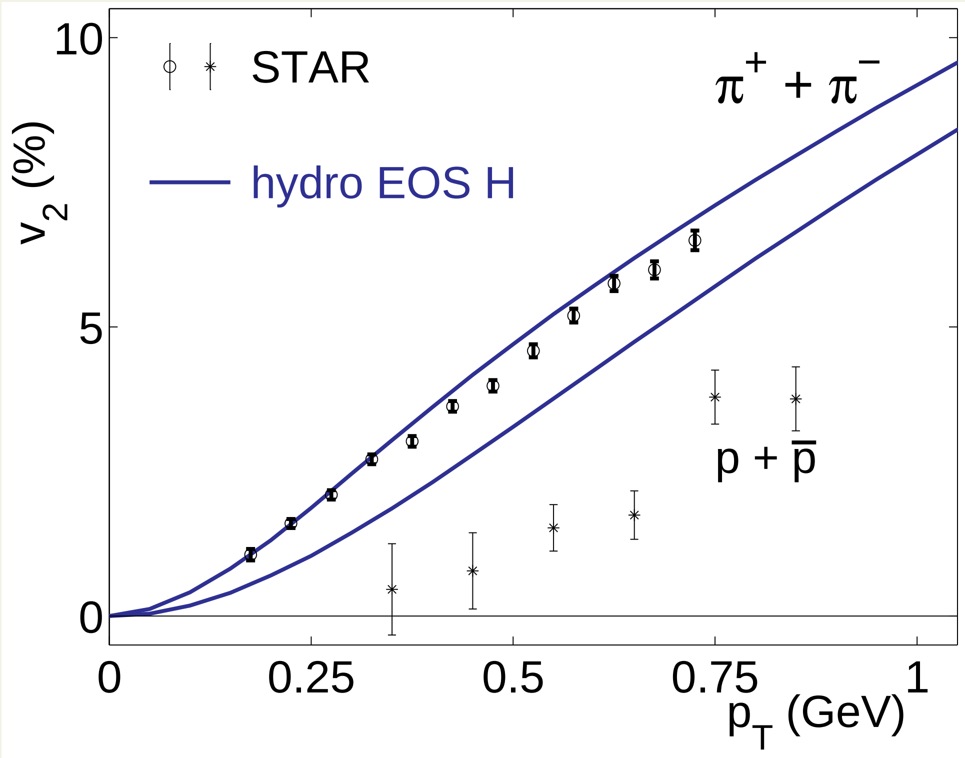


- works beautifully in central and semi-central collisions

Success of ideal hydrodynamics

Kolb, Heinz, Huovinen et al ('01) minbias Au+Au at RHIC





not perfect agreement but plasma EoS favored

Dissipative hydrodynamics

In general

$$N^\mu = nu^\mu + \nu^\mu$$
$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P+\Pi)\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}$$

In Landau frame,

$$W^\mu \equiv 0, \qquad \nu^\mu = -\frac{q^\mu}{h} = -\frac{n}{\epsilon + P} q^\mu \tag{4}$$

and thus

$$N^\mu = nu^\mu + \nu^\mu$$
$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P_{\rm eq}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

Dissipative hydrodynamics

In Landau frame,

$$N^\mu = nu^\mu + \nu^\mu$$
$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P_{\rm eq} + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

Need 9 additional equations to determine

$$\Pi,\ \pi^{\mu\nu},\ \nu^\mu,\ P_{\rm eq}$$

Equation of state

$$P_{\rm eq} = P(T,\mu)$$

Matching conditions

ideal fluid $\Longleftrightarrow$ **exact local kinetic equilibrium**

dissipation $\Longleftrightarrow$ **deviations from thermal distribution**

Non-equilibrium thermodynamics?

- **What are entropy and pressure?**
- **EoS? Temperature?**

Matching conditions

ideal fluid $\Longleftrightarrow$ exact local kinetic equilibrium

dissipation $\Longleftrightarrow$ deviations from thermal distribution

Non-equilibrium thermodynamics?

- What are entropy and pressure?
- EoS? Temperature?

Energy and particle number defined for arbitrary system:

$$\epsilon = u_\mu T^{\mu\nu} u_\nu \qquad \text{and} \qquad n = N^\mu u_\mu$$

apply equilibrium EoS:

$$s = s_0(\epsilon, n) \qquad \text{and} \qquad P = P_0(\epsilon, n)$$

i.e. we match the system to an equilibrium system of *the same* ϵ and n

relativistic Navier-Stokes

Entropy four-current:

$$S^\mu = su^\mu + \frac{\mu}{T}\frac{q^\mu}{h}$$

where

$$h = \frac{\epsilon + P}{n}$$

Require non-decrease of entropy:

$$0 \leq \partial_\mu S^\mu = -\Pi \nabla^\mu u_\mu - q_\mu \frac{T}{e+p} \nabla^\mu \frac{\mu}{T} + \pi_{\mu\nu} \nabla^{\langle\mu} u^{\nu\rangle}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2}(\Delta^\mu_\sigma \Delta^\nu_\tau + \Delta^\nu_\tau \Delta^\mu_\sigma) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right] A^{\sigma\tau}$$

and

$$\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

relativistic Navier-Stokes

$$0 \leq \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

is always valid if we identify

$$\Pi \propto X, \qquad q^\mu \propto X^\mu, \qquad \pi^{\mu\nu} \propto X^{\mu\nu}$$

dissipative currents small corrections linear in gradients

$$\begin{aligned}
\Pi &= -\zeta \nabla^\mu u_\mu \\
q^\mu &= -\kappa \frac{T}{e+p} \nabla^\mu \frac{\mu}{T} \\
\pi^{\mu\nu} &= 2\eta \nabla^{\langle\mu} u^{\nu\rangle}
\end{aligned}$$

η, ζ **shear and bulk viscosities,** κ **heat conductivity**

Navier-Stokes equations of motion

$$
\begin{aligned}
Dn &= -n\partial_\mu u^\mu - \partial_\mu\left(\kappa\frac{Tn}{h^2}\nabla^\mu\frac{\mu}{T}\right) \\
D\epsilon &= -(\epsilon + P - \zeta\nabla^\alpha u_\alpha)\partial_\mu u^\nu + 2\eta\nabla^{\langle\alpha}u^{\beta\rangle}\nabla_{\langle\alpha}u_{\beta\rangle} \\
(\epsilon + P - \zeta\nabla^\alpha u_\alpha)Du^\mu &= \nabla^\mu(P - \zeta\nabla^\alpha u_\alpha) - 2\Delta^\mu_\alpha\partial_\beta(\eta\nabla^{\langle\alpha}u^{\beta\rangle})
\end{aligned}
$$

where

$$D = u^\mu\partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

Navier-Stokes equations of motion

$$
\begin{aligned}
Dn &= -n\partial_\mu u^\mu - \partial_\mu \left(\kappa \frac{Tn}{h^2} \nabla^\mu \frac{\mu}{T}\right) \\
D\epsilon &= -(\epsilon + P - \zeta \nabla^\alpha u_\alpha)\partial_\mu u^\nu + 2\eta \nabla^{\langle\alpha} u^{\beta\rangle} \nabla_{\langle\alpha} u_{\beta\rangle} \\
(\epsilon + P - \zeta\nabla^\alpha u_\alpha) D u^\mu &= \nabla^\mu (P - \zeta \nabla^\alpha u_\alpha) - 2\Delta^\mu_\alpha \partial_\beta (\eta \nabla^{\langle\alpha} u^{\beta\rangle})
\end{aligned}
$$

where

$$D = u^\mu \partial_\mu \qquad \text{and} \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

but these are parabolic. . .

Parabolic partial differential equations

PDE of the form

$$A\frac{\partial^2}{\partial x^2}u + B\frac{\partial^2}{\partial x \partial y}u + C\frac{\partial^2}{\partial y^2}u + D\frac{\partial}{\partial x}u + E\frac{\partial}{\partial y}u + F = 0$$

is parabolic if

$$B^2 - AC = 0$$

Such equations provide infinite speed for signal propagation

Müller ('76), Israel & Stewart ('79) ...

Solutions are unstable

Hiscock & Lindblom, PRD31, 725 (1985) ...

Hyperbolic partial differential equations

PDE of the form

$$A\frac{\partial^2}{\partial x^2}u + B\frac{\partial^2}{\partial x \partial y}u + C\frac{\partial^2}{\partial y^2}u + D\frac{\partial}{\partial x}u + E\frac{\partial}{\partial y}u + F = 0$$

is hyperbolic if

$$B^2 - AC > 0$$

For example one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Solutions stable and with finite propagation speed.

Causal viscous hydro

To obtain causal equations we have to replace

$$\Pi = -\zeta \nabla^\mu u_\mu$$

by

$$\tau_\Pi D\Pi + \Pi = -\zeta \nabla^\mu u_\mu + \cdots$$

or something similar.

Causal viscous hydro

Israel & Stewart:

Entropy four-flow including terms second order in dissipative fluxes:

$$S^\mu = \quad su^\mu + \frac{\mu}{T}\frac{q^\mu}{h} \quad -\left(\beta_0\Pi^2 - \beta_1 q_\nu q^\nu + \beta_2 \pi_{\lambda\nu}\pi^{\lambda\nu}\right)\frac{u^\mu}{2T} - \frac{\alpha_0 q^\mu \Pi}{T} + \frac{\alpha_1 q_\nu \pi^{\nu\mu}}{T}$$

⇒ **"Second order theory"**

or, rather, Transient fluid dynamics

Evolution equation for shear

Require non-decrease of entropy:

$$0 \le \partial_\mu S^\mu = \Pi X + q_\mu X^\mu + \pi_{\mu\nu} X^{\mu\nu}$$

Identify $\pi^{\mu\nu} = 2\eta X^{\langle\mu\nu\rangle}$**:**

$$\begin{aligned}\pi^{\mu\nu} \quad &= \quad 2\eta \left[\nabla^{\langle\mu} u^{\nu\rangle} - \beta_2 \langle u^\lambda \partial_\lambda \pi^{\mu\nu} \rangle - \frac{1}{2} \pi^{\mu\nu} T \partial_\lambda \left(\frac{\tau_\pi u^\lambda}{2\eta T} \right) \right] \\ &+ 2\eta \left[\alpha_1 \nabla^{\langle\mu} q^{\nu\rangle} + a_1' q^{\langle\mu} u^\lambda \partial_\lambda u^{\nu\rangle} \right]\end{aligned}$$

where

$$A^{\langle\mu\nu\rangle} = \left[\frac{1}{2} (\Delta^\mu_\sigma \Delta^\nu_\tau + \Delta^\nu_\tau \Delta^\mu_\sigma) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\sigma\tau} \right] A^{\sigma\tau}$$

and

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu .$$

Israel-Stewart evolution equations

$$
\begin{aligned}
D\Pi &= -\frac{1}{\tau_\Pi}(\Pi + \zeta\nabla_\mu u^\mu) - \frac{1}{2}\Pi\left(\nabla_\mu u^\mu + D\ln\frac{\beta_0}{T}\right) \\
&\quad + \frac{\alpha_0}{\beta_0}\partial_\mu q^\mu - \frac{a_0'}{\beta_0} q^\mu D u_\mu \\
Dq^\mu &= -\frac{1}{\tau_q}\left[q^\mu + \kappa_q \frac{T^2 n}{\varepsilon + p}\nabla^\mu\left(\frac{\mu}{T}\right)\right] - u^\mu q_\nu D u^\nu \\
&\quad - \frac{1}{2} q^\mu \left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) - \omega^{\mu\lambda} q_\lambda \\
&\quad - \frac{\alpha_0}{\beta_1}\nabla^\mu\Pi + \frac{\alpha_1}{\beta_1}(\partial_\lambda \pi^{\lambda\mu} + u^\mu \pi^{\lambda\nu}\partial_\lambda u_\nu) + \frac{a_0}{\beta_1}\Pi D u^\mu - \frac{a_1}{\beta_1}\pi^{\lambda\mu} D u_\lambda \\
D\pi^{\mu\nu} &= -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu}u^{\nu\rangle}\right) - (\pi^{\lambda\mu}u^\nu + \pi^{\lambda\nu}u^\mu) D u_\lambda \\
&\quad - \frac{1}{2}\pi^{\mu\nu}\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_2}{T}\right) - 2\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} \\
&\quad - \frac{\alpha_1}{\beta_2}\nabla^{\langle\mu} q^{\nu\rangle} + \frac{a_1'}{\beta_2} q^{\langle\mu} D u^{\nu\rangle}
\end{aligned}
$$

Israel-Stewart evolution. . .

bulk pressure Π, shear stress $\pi^{\mu\nu}$ charge diffusion μ^{μ} treated as independent dynamical quantities that relax to their Navier-Stokes value on time scales $\tau_{\Pi}(e,n)$, $\tau_{\pi}(e,n)$, $\tau_{q}(e,n)$

Equations of motion	5 equations
evolution of bulk	1 equation
evolution of charge diffusion	3 equations
evolution of shear stress	5 equations
14 equations, 14 unknowns	

These equations are causal and stable

But what are the parameters α_0, α_1, $\beta_0, \beta_1, \beta_2$?

Or how to obtain ζ, κ, η?

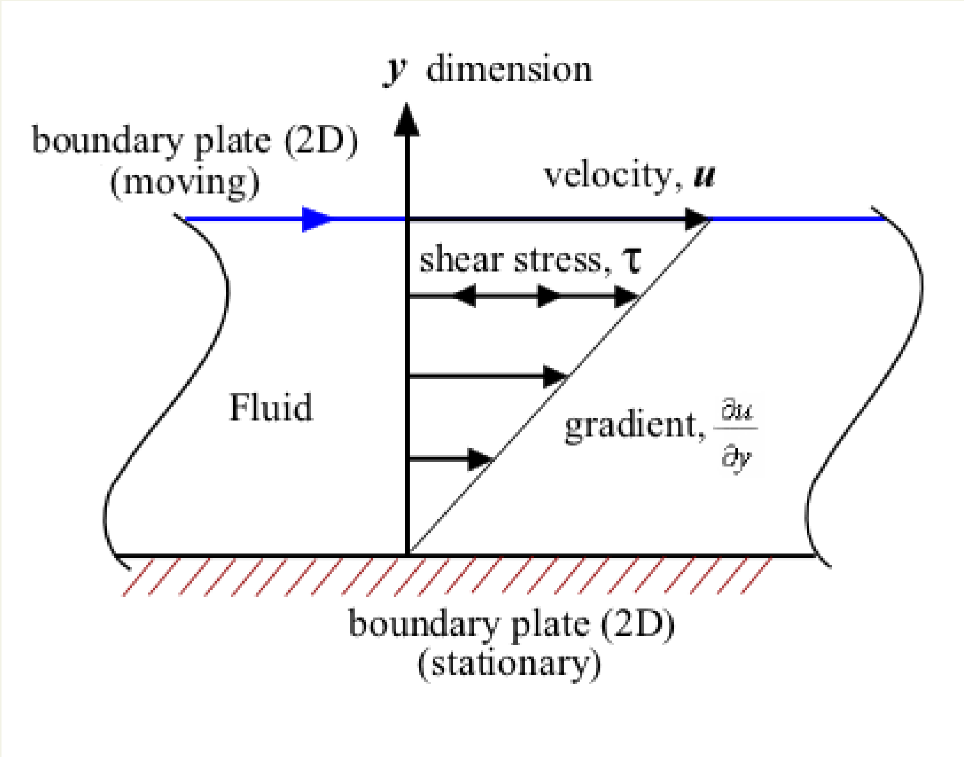
$\Longrightarrow$ use kinetic theory

Or some other microscopic theory

Shear viscosity

Newton:

$$T_{xy} = -\eta \frac{\partial u_x}{\partial y}$$



acts to reduce velocity gradients

in closed system: **energy conserved**

kinetic energy gets converted to internal energy

$\Rightarrow$ **dissipation**

Shear in 1D-bjorken

Navier-Stokes stress

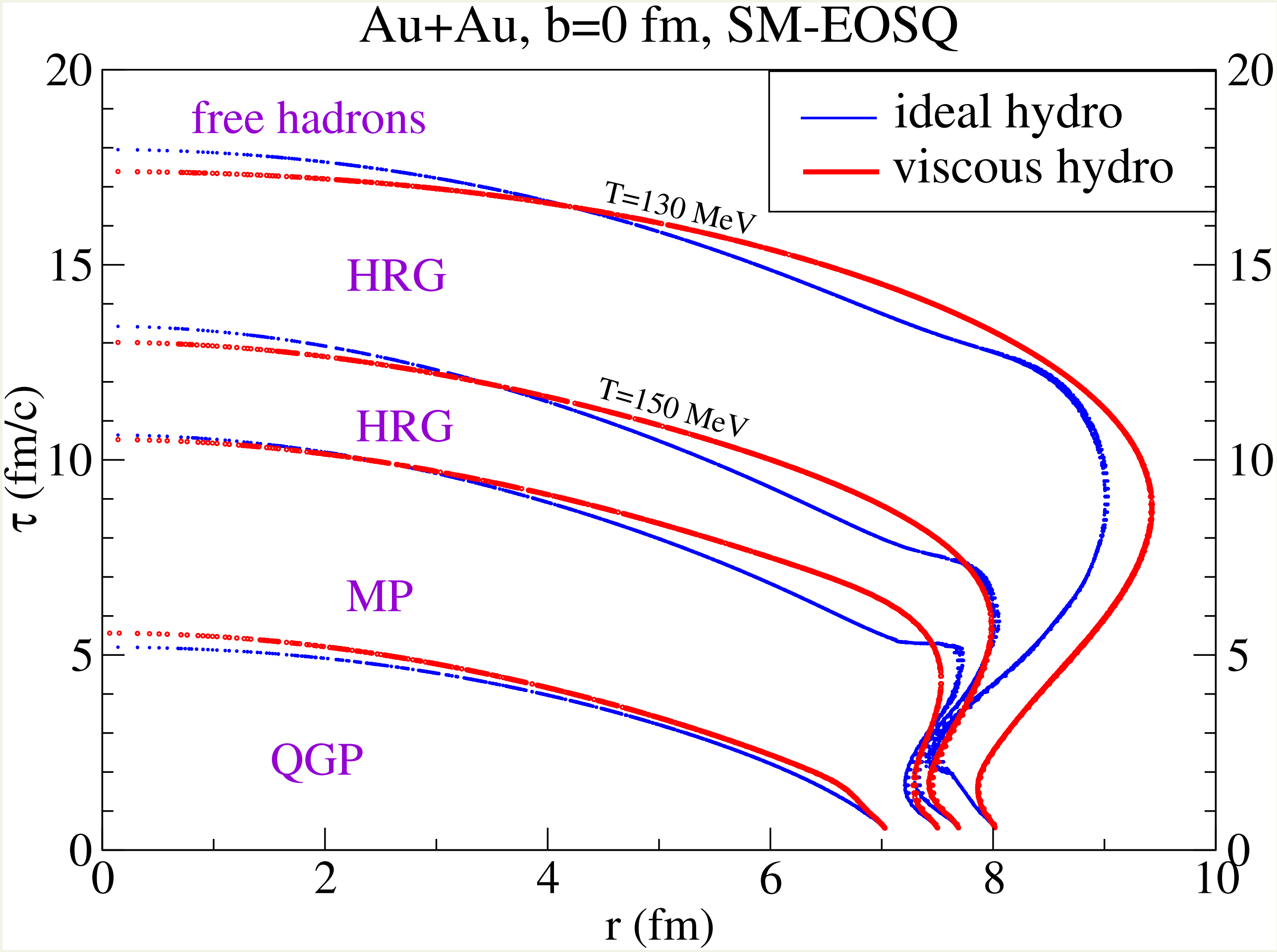
$$\pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} = \mathrm{diag}(0, \frac{2\eta}{3\tau}, \frac{2\eta}{3\tau}, -\frac{4\eta}{3\tau})$$
$$T^{\mu\nu} = \mathrm{diag}(\epsilon, P - \frac{\pi_L}{2}, P - \frac{\pi_L}{2}, P + \pi_L)$$

where $\pi_L = \pi^{\eta\eta} = -\dfrac{4\eta}{3\tau}$

Effective longitudinal pressure $P + \pi_L < P$
Effective transverse pressure $P - \pi_L/2 > P$

Shear slows down longitudinal expansion and accelerates transverse expansion

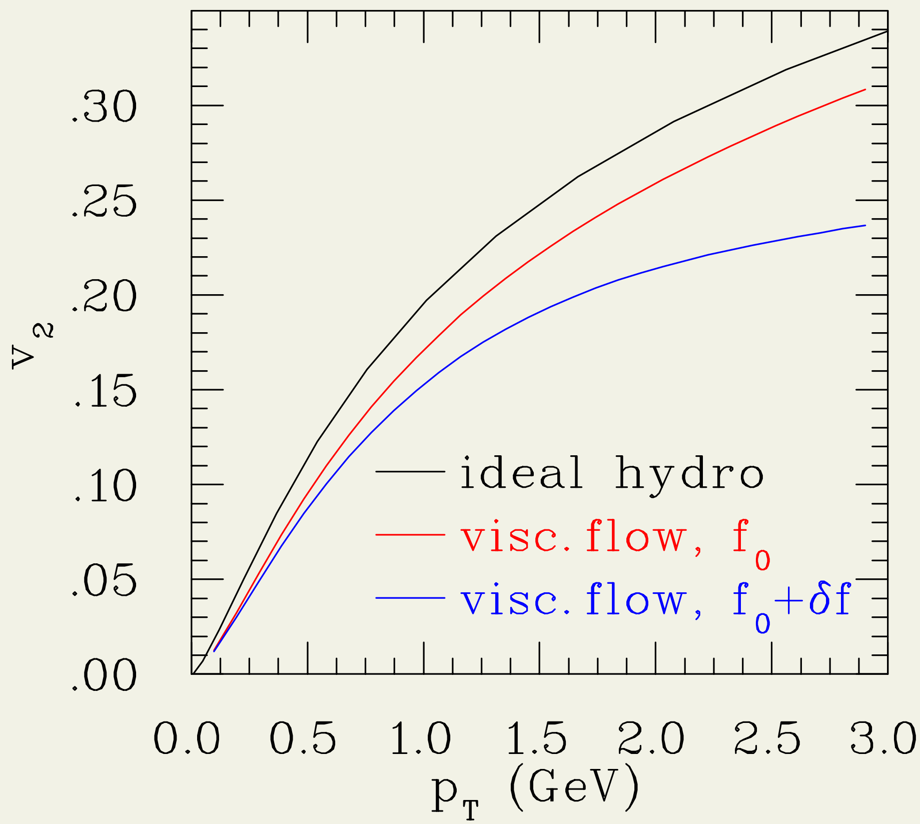
Effect on temperature



©Huichao Song

- **Edges expand further and stay hotter**
- **At first core cools slower, later faster**

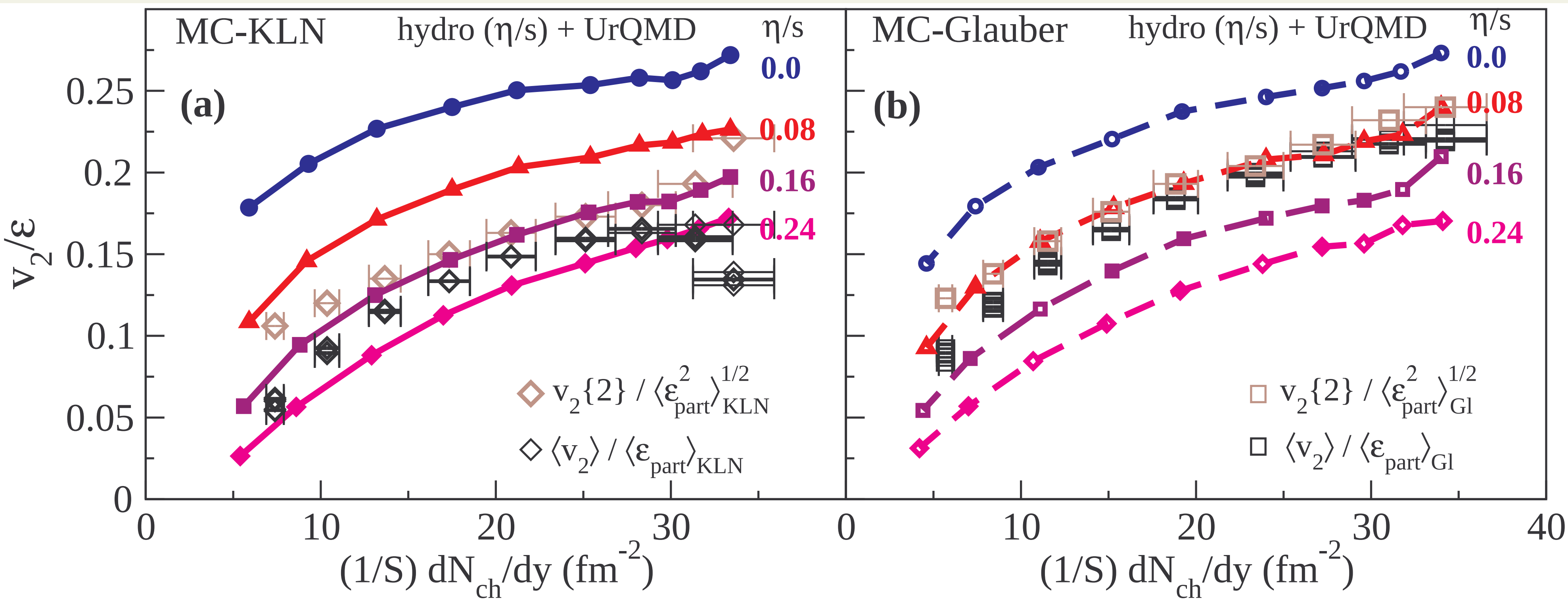
Effect on v_2



- **massless particles**
- **Note: both change in flow and distributions affect v_2**

η/s from v_2

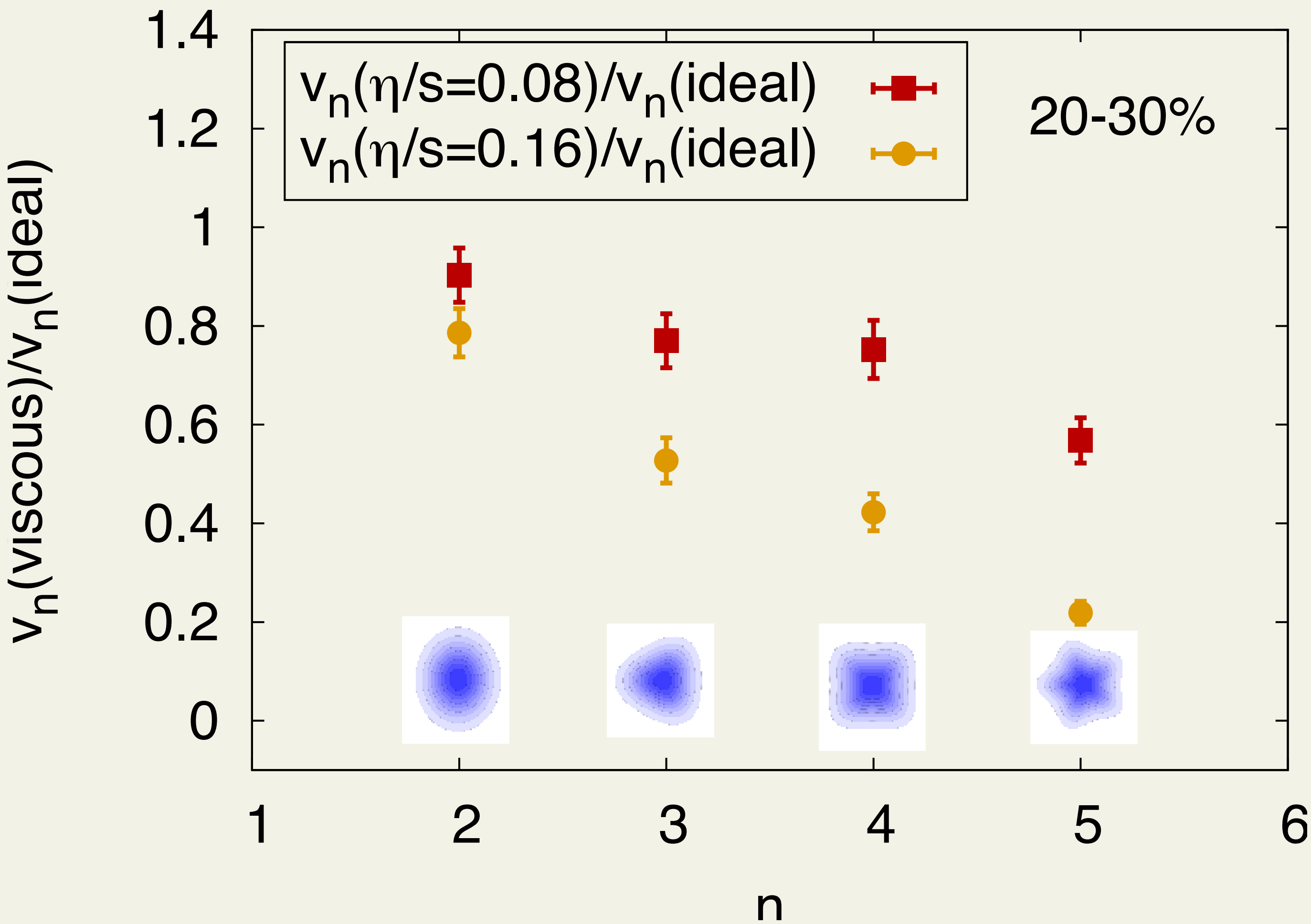
Shen *et al.* J.Phys.G38:124045,2011



- **MC-Glauber initialization:** $\eta/s = 0.08$
- **MC-KLN initialization:** $\eta/s = 0.2$

Sensitivity to η/s

Schenke *et al.* Phys.Rev.C85:024901,2012



- **higher coefficients are suppressed more by dissipation**

Distributions of v_n event-by-event

Scale out the average

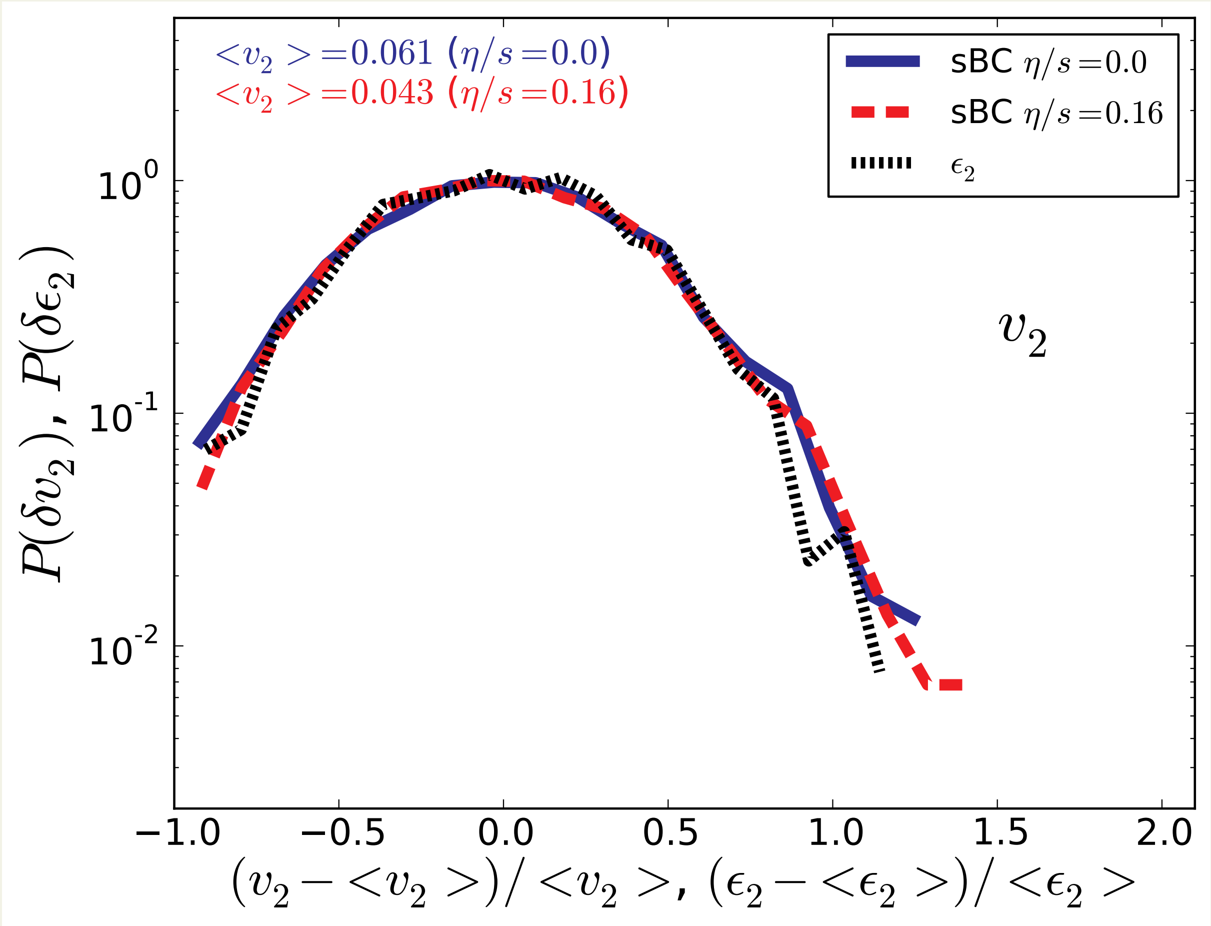
$$\delta v_2 = \frac{v_2 - \langle v_2 \rangle}{\langle v_2 \rangle}$$

$\Downarrow$

$$P(\delta v_2) = P(\delta \epsilon_2)$$

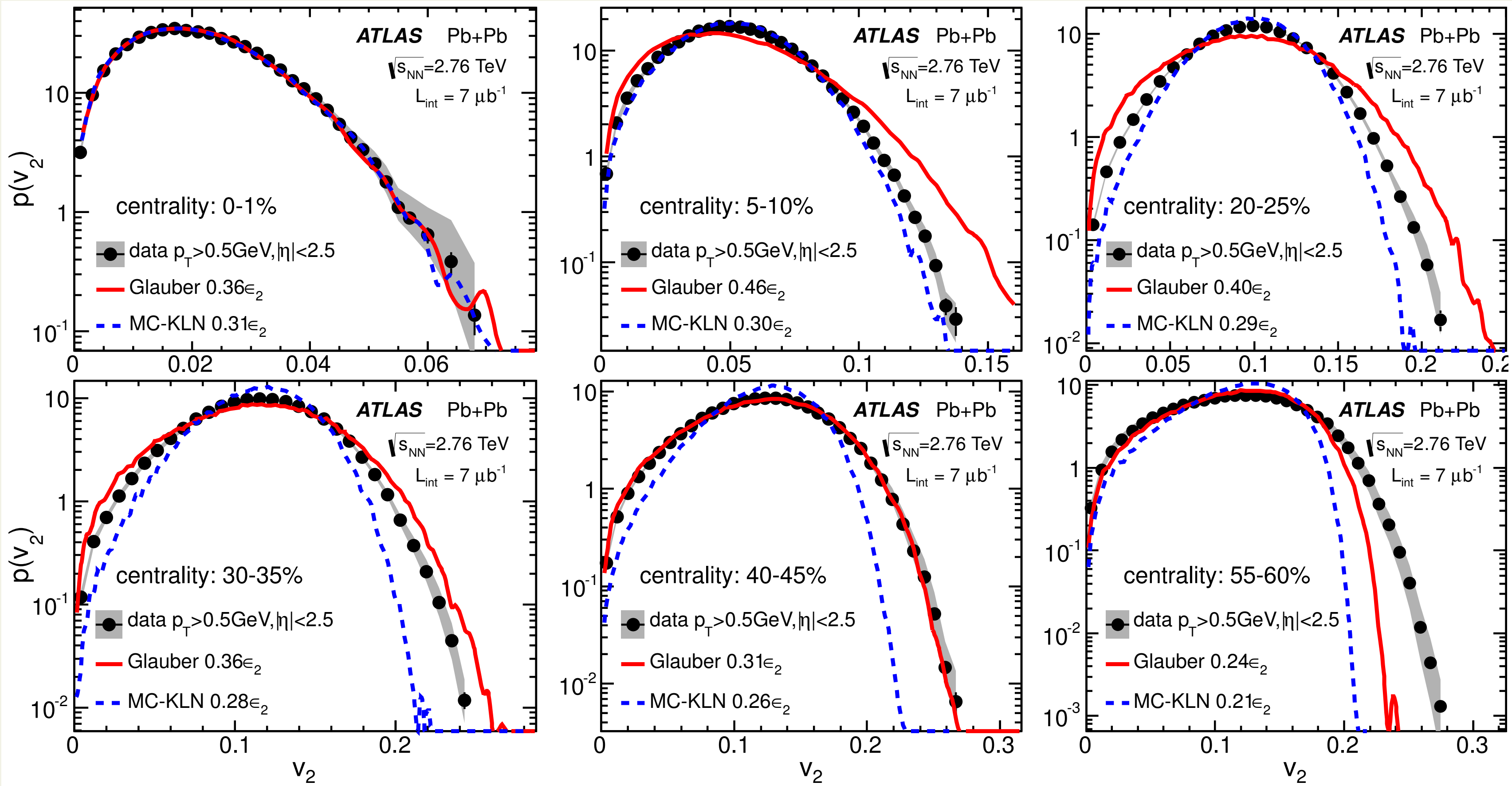
independent of viscosity

Niemi *et al.* Phys.Rev.C87,054901,2013



Flow fluctuations

Aad *et al.* [ATLAS Collaboration] JHEP 1311:183,2013



- $P(v_2)$ **compared to MC-Glauber and MC-KLN** $P(\varepsilon_2)$
- **MC-Glauber initialization: too wide**
- **MC-KLN initialization: too narrow**

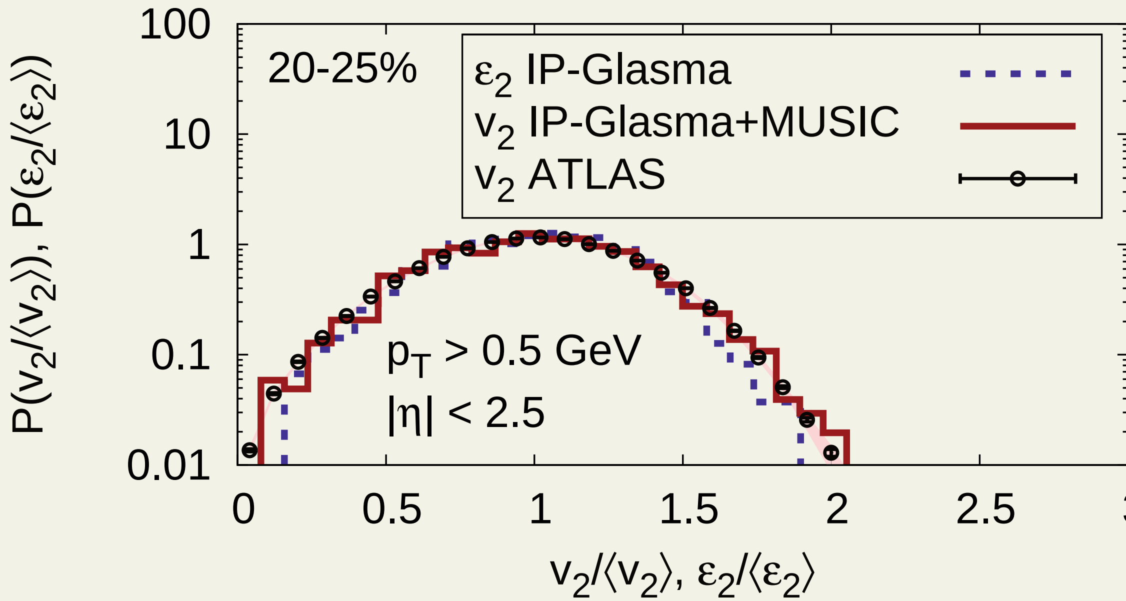
- IP-Glasma
(Color Glass + Yang-Mills)

and

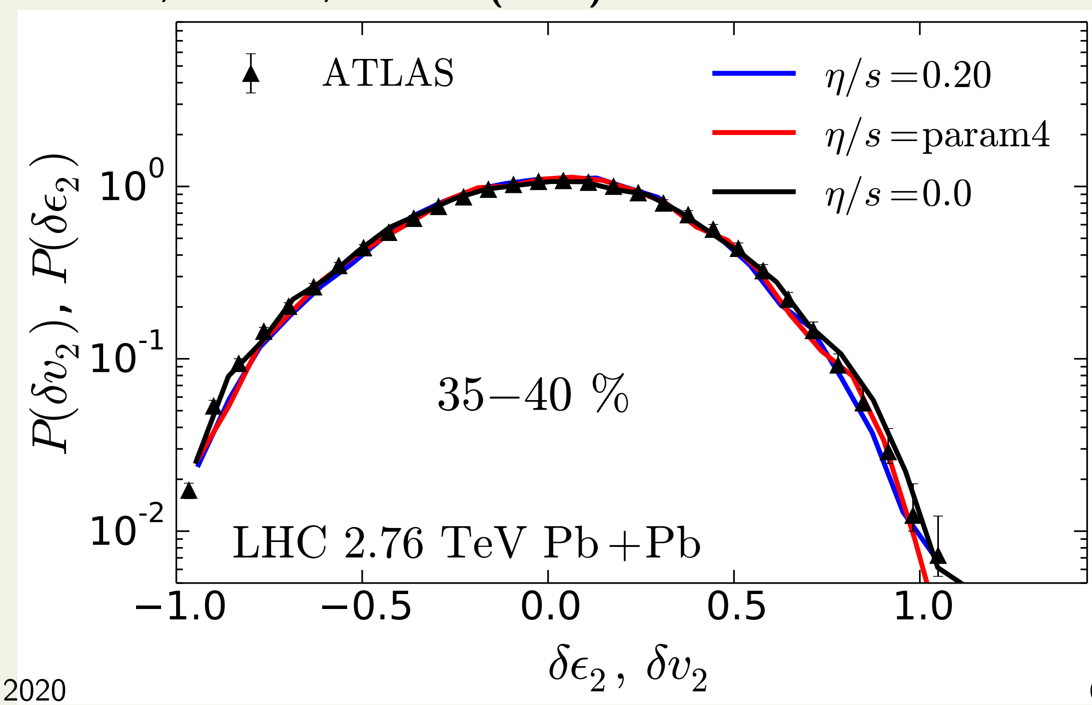
- EKRT
(pQCD + saturation)

initial states work

Gale *et al.*, PRL 110, 012302 (2013)



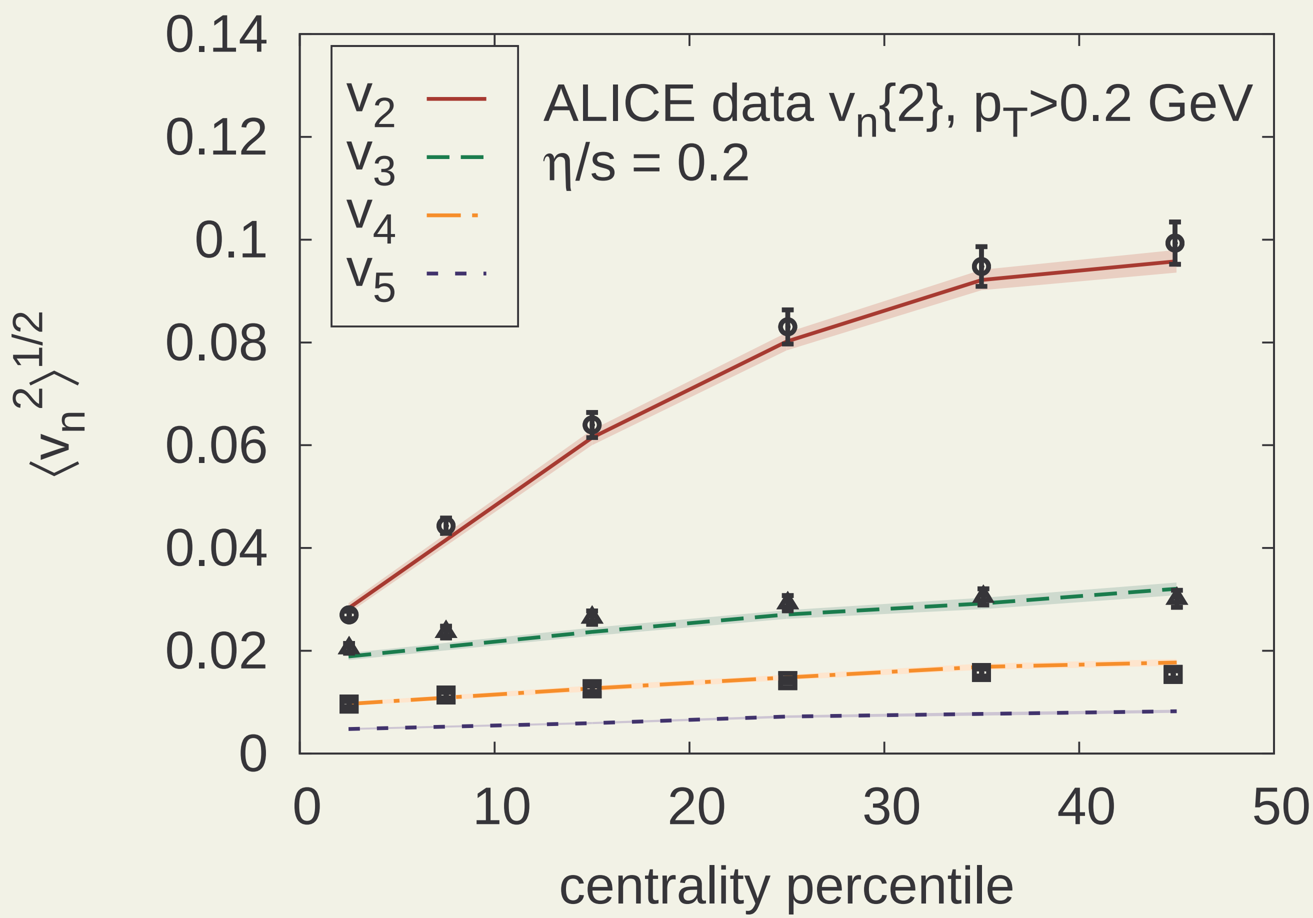
Niemi *et al.*, PRC 93, 024907 (2016)



State of the art

- **IP-Glasma initial state: Color Glass plus Yang-Mills evolution**

Gale *et al.*, PRL 110, 012302 (2013)

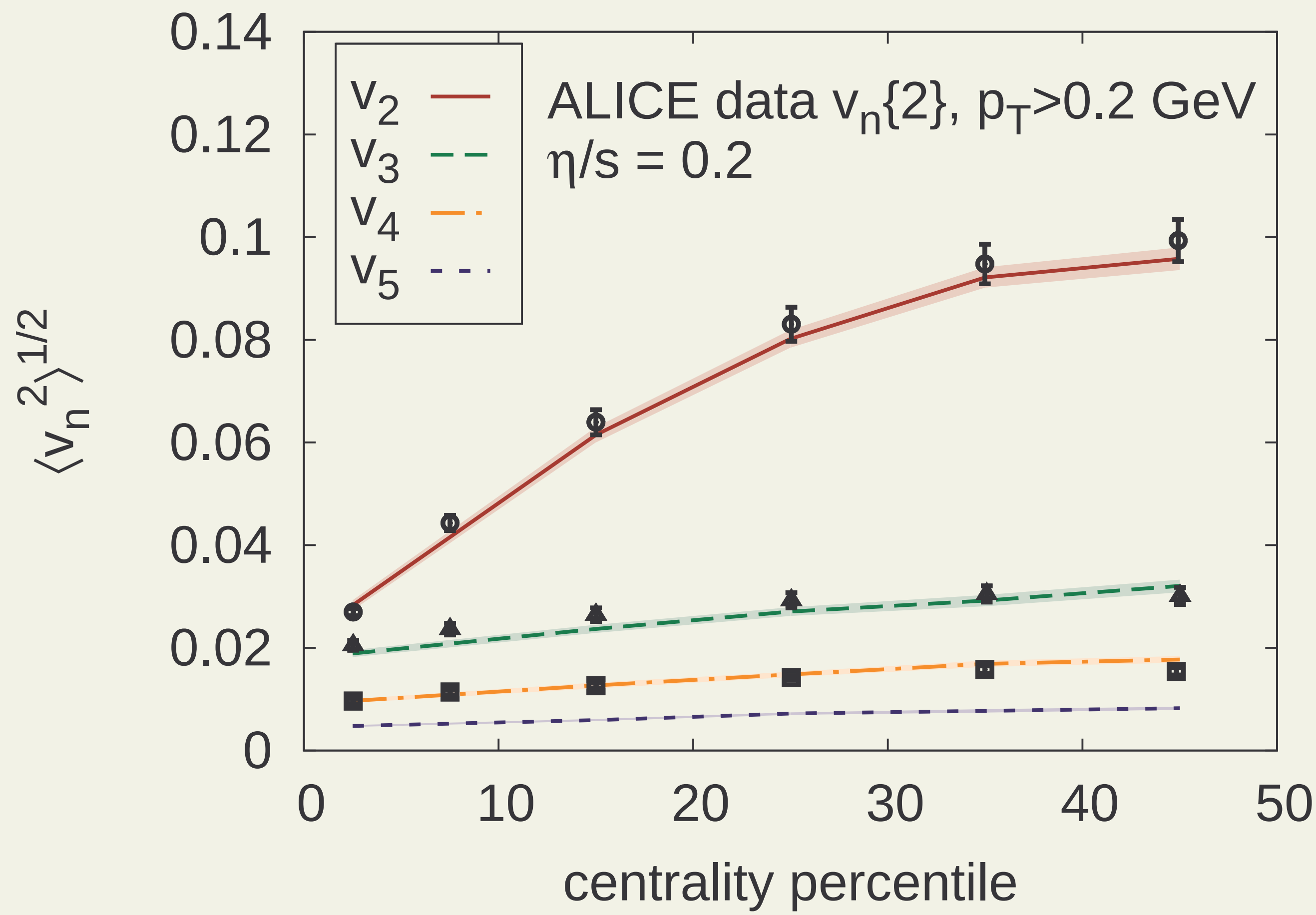


- $\eta/s = 0.2$ **favoured**

State of the art

- **IP-Glasma initial state: Color Glass plus Yang-Mills evolution**

Gale *et al.*, **PRL 110, 012302 (2013)**



- $\eta/s = 0.2$ **favoured at LHC**
- $\eta/s = 0.12$ **favoured at RHIC**

Summary

- **Hydrodynamics is a useful tool to model collision dynamics**
  - **approximation at its best**
  - **but it can reproduce (some of) the data**

- **we have observed hydrodynamical behaviour at RHIC and LHC**

♻ **This lecture consisted of 100% recycled electrons**

Literature:

Textbook:

– W. Florkowski,
Phenomenology of Ultra-relativistic Heavy-ion Collisions, (World-Scientific, 2010)

Reviews:

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– P. Romatschke, Int. J. Mod. Phys. E 19, 1 (2010)
– C. Gale, S. Jeon and B. Schenke, Int. J. Mod. Phys. A 28, 1340011 (2013)
– U. W. Heinz and R. Snellings, arXiv:1301.2826 [nucl-th]

Hydrodynamics lecture notes:

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Dissipative hydro from kinetic theory:

– S. R. de Groot, W. A. van Leeuwen, and Ch. G. van Weert,
Relativistic kinetic theory – Principles and applications, (North-Holland, 1980)
– G. S. Denicol, H. Niemi, E. Molnar and D. H. Rischke, Phys. Rev. D 85, 114047 (2012)