Emergence of prescaling in far-from-equilibrium quark-gluon plasma

Initial Stages
January 11, 2021

Bruno Scheihing-Hitschfeld
Massachusetts Institute of Technology (MIT)
in collaboration with Jasmine Brewer and Yi Yin
Introduction

• The early stages of a weakly-coupled QGP after a heavy-ion collision constitutes a non-hydrodynamic system, far from thermal equilibrium.

• Yet, there is mounting evidence that even at this early stage the evolution of the plasma is governed by only a handful of degrees of freedom.

  —> “Hydrodynamization”

• However, these “slow” degrees of freedom elicit some questions:
  - Is there systematic way to identify them at all times?
  - In what sense can we extend hydrodynamics to earlier times?
The path to hydrodynamics in the “Bottom-up” thermalization scenario


In the BMSS scenario,

1. Over-occupied hard gluons $f_g \gg 1$ at very early times $1 \ll Q_s \tau \ll \alpha_s^{-3/2}$

2. Hard gluons become under-occupied $f_g \ll 1$, when $\alpha_s^{-3/2} \ll Q_s \tau \ll \alpha_s^{-5/2}$

3. Thermalization of the soft sector after $\alpha_s^{-5/2} \ll Q_s \tau$

We will be working inside this regime

$\rightarrow$ Well before hydrodynamics & thermalization
Motivation

Observation of prescaling in far-from-equilibrium QCD kinetic theory

- Prescaling: time-dependent scaling

\[ f(p_\perp, p_z, \tau) = \tau^{\alpha(\tau)} f_S(\tau^{\beta(\tau)} p_\perp, \tau^{\gamma(\tau)} p_z) \]

([2] A. Mazeliauskas, J. Berges (2019))
Motivation

Observation of prescaling in far-from-equilibrium QCD kinetic theory

• Prescaling: time-dependent scaling

\[ f(p_\perp, p_z, \tau) = \tau^{\alpha(\tau)} f_S(p_\perp, \tau^{\beta(\tau)} p_z, \tau^{\gamma(\tau)} p_z) \]

In [2], the setup of the simulation featured consistent with the first stage of the bottom-up scenario [1].

\[
-\frac{\partial}{\partial \tau} f(p, \tau) + \frac{p_z}{\tau} \frac{\partial}{\partial p_z} f(p, \tau) = \mathcal{C}[f(p)]
\]

Universal scaling exponents, BMSS scenario [1]
**Motivation**

Observation of prescaling in far-from-equilibrium QCD kinetic theory

- **Prescaling:** time-dependent scaling

\[ f(p_\perp, p_z, \tau) = \tau^{\alpha(\tau)} f_S(\tau^{\beta(\tau)} p_\perp, \tau^{\gamma(\tau)} p_z) \]

- In this phase, three “slow” apparent degrees of freedom, \( \alpha, \beta, \gamma \) govern the evolution.

- How do these exponents emerge? Are they actually degrees of freedom?
Adiabatic Hydrodynamization

—> Plenary talk by Jasmine Brewer [4], Friday 17:45

• Look at the kinetic equation as a Hamiltonian system:

\[ \partial_y \vec{\psi} = -\mathcal{H}[y; \{F_i[\vec{\psi}]\}]\vec{\psi}. \]

—> Each eigenstate \( \vec{\psi} \) of \( \mathcal{H} \) constitutes an effective degree of freedom. \( (\mathcal{H} \vec{\psi} = E\vec{\psi}) \)

—> If the rate of change of \( \mathcal{H} \) is smaller than the gap between its eigenvalues, then the lowest energy eigenstates should dominate the “slow” evolution of the system.

—> These modes should describe the “slow” properties of the plasma.

• Since \( \alpha, \beta, \gamma \) are “slow” quantities, is it possible to understand prescaling from this point of view?
Summary of results
(what we will discuss today)

• We show that at early times the time-dependent scaling exponents are determined by the evolution of only one slowly varying parameter.

• We establish the nature of the slow modes in kinetic theory at early times in the small-angle scattering approximation from the adiabatic hydrodynamization perspective.
Early-time prescaling
Solution scheme

• At early times, we can use the small-angle scattering approximation, and that the typical gluon momenta satisfy \( p_z \ll p_{\perp} \approx p \). It follows that

\[
\hat{q} = 4\pi N_c^2 \alpha_s^2 l_{C_b} \int_p (1 + f) f \approx 4\pi N_c^2 \alpha_s^2 l_{C_b} \int_p f^2.
\]

\[
\frac{\partial f}{\partial \tau} - \frac{p_z}{\tau} \frac{\partial f}{\partial p_z} \approx \hat{q}(y) \frac{\partial^2 f}{\partial p_z^2} \approx \hat{q}(y) \nabla_p^2 f,
\]

where \( \hat{q} \) is varying slowly.

• To solve this, we will treat \( \hat{q} \) as a time-dependent parameter and look for time-dependent scaling solutions \( \langle p_{\perp}^m p_z^n \rangle = D_{n,m} A(\tau) B(\tau)^m C(\tau)^n \).

\[
\implies \alpha = \alpha[\tau; \hat{q}, \partial_\tau \hat{q}; f_0], \quad \beta = \beta[\tau; \hat{q}, \partial_\tau \hat{q}; f_0], \quad \gamma = \gamma[\tau; \hat{q}, \partial_\tau \hat{q}; f_0].
\]

• The only approximation we make is that \( \partial \log \hat{q} / \partial \log \tau \) is varying slowly.

momentum diffusion constant [5]

derivative of \( \hat{q} \) with respect to \( \log \tau \)
Results

Note that (pre)scaling implies that \( \frac{\partial \log \hat{q}}{\partial \log \tau} = 2\alpha - 2\beta - \gamma. \)

\[ \hat{q} = \alpha[\tau; \hat{q}, \partial \hat{q}] \]
\[ \beta = \beta[\tau; \hat{q}, \partial \hat{q}] \]
\[ \gamma = \gamma[\tau; \hat{q}, \partial \hat{q}] \]

one gets a 1st order ODE for \( \hat{q}. \)

Solving it, \( \hat{q} \) fully determines \( \alpha, \beta, \gamma. \)
Results

Note that (pre)scaling implies that \( \frac{\partial \log \hat{q}}{\partial \log \tau} = 2\alpha - 2\beta - \gamma. \)

\[ \longrightarrow \text{Then, replacing the expressions} \]

\[ \alpha = \alpha[\tau; \hat{q}, \partial \hat{q}] \]
\[ \beta = \beta[\tau; \hat{q}, \partial \hat{q}] \]
\[ \gamma = \gamma[\tau; \hat{q}, \partial \hat{q}] \]

one gets a 1st order ODE for \( \hat{q}. \)

Solving it, \( \hat{q} \) fully determines \( \alpha, \beta, \gamma. \)
Results

Note that (pre)scaling implies that
\[
\frac{\partial \log \hat{q}}{\partial \log \tau} = 2\alpha - 2\beta - \gamma.
\]

\[\rightarrow\] Then, replacing the expressions

\[
\begin{align*}
\alpha &= \alpha[\tau; \hat{q}, \partial_t \hat{q}] \\
\beta &= \beta[\tau; \hat{q}, \partial_t \hat{q}] \\
\gamma &= \gamma[\tau; \hat{q}, \partial_t \hat{q}]
\end{align*}
\]

one gets a 1st order ODE for \( \hat{q} \).

Solving it, \( \hat{q} \) fully determines \( \alpha, \beta, \gamma \).
Results

Note that (pre)scaling implies that \( \frac{\partial \log \hat{q}}{\partial \log \tau} = 2\alpha - 2\beta - \gamma. \)

\[ \rightarrow \] Then, replacing the expressions

\[
\begin{align*}
\alpha &= \alpha[\tau; \hat{q}, \partial_q q] \\
\beta &= \beta[\tau; \hat{q}, \partial_q q] \\
\gamma &= \gamma[\tau; \hat{q}, \partial_q q]
\end{align*}
\]

one gets a 1st order ODE for \( \dot{\hat{q}}. \)

Solving it, \( \dot{\hat{q}} \) fully determines \( \alpha, \beta, \gamma. \)
The adiabatic perspective: scaling

Why is this time-dependent scaling solution preferred?

\[ \partial_y \overline{\psi} = -\mathcal{H}[y; \{ F_i(\overline{\psi}) \}] \overline{\psi}. \]

\[ \text{—> Lowest energy eigenstate (for simplicity } \mathcal{C}[f] \propto \partial_{p_z}^2 f \text{) is given by} \]

\[ |\psi_0\rangle \leftrightarrow \langle p_z^{2n} p_\perp^m \rangle \propto \frac{(2n)!}{n!} \left( \frac{\tau \hat{q}}{2} \right)^n, \]

\[ E_0 = 1. \]

\[ \text{—> This state exhibits time-dependent scaling. It follows that} \]

\[ \alpha = -\frac{1}{2} \frac{\partial \log \hat{q}}{\partial \log \tau} - \frac{3}{2}, \quad \gamma = -\frac{1}{2} - \frac{1}{2} \frac{\partial \log \hat{q}}{\partial \log \tau}, \quad \beta = 0, \quad \text{and} \quad \frac{\partial \log \hat{q}}{\partial \log \tau} = 2\alpha - 2\beta - \gamma. \]

\[ \text{—> Putting these together, } \alpha = -2/3, \quad \gamma = 1/3, \quad \beta = 0. \]
The adiabatic perspective: prescaling

Eigenvalues of $\mathcal{H}$: $E_n = 2n + 1 \implies$ Energy gap.

$\implies$ After a sufficiently long time the state will be governed by the lowest modes.

$\implies$ initial condition: $|\psi\rangle = A_0 |\psi_0\rangle + A_1 |\psi_1\rangle$.

$\implies$ Solving for the scaling exponents (perturbatively in $A_1/A_0$) gives

$$
\gamma = -\frac{1}{2} \left( 1 + \frac{\partial \log \hat{q}}{\partial \log \tau} \right) + \frac{A_1}{A_0} \frac{(\tau_f/\tau)^2}{4\tau \hat{q}} \left( 3 + \frac{\partial \log \hat{q}}{\partial \log \tau} \right)^2, \quad \beta = 0, \quad \alpha = \gamma - 1.
$$
The adiabatic perspective: prescaling

Eigenvalues of $\mathcal{H}$: $E_n = 2n + 1 \implies$ Energy gap.

$\rightarrow$ After a sufficiently long time the state will be governed by the lowest modes.

$\implies$ initial condition: $|\psi\rangle = A_0 |\psi_0\rangle + A_1 |\psi_1\rangle$.

$\rightarrow$ Solving for the scaling exponents (perturbatively in $A_1/A_0$) gives

$$\gamma = -\frac{1}{2} \left( 1 + \frac{\partial \log \hat{q}}{\partial \log \tau} \right) + \frac{A_1}{A_0} \frac{(\tau_f/\tau)^2}{4\tau\hat{q}} \left( 3 + \frac{\partial \log \hat{q}}{\partial \log \tau} \right)^2,$$

$\beta = 0, \quad \alpha = \gamma - 1.$

"0th order" BMSS exponent

"Perturbative parameter"

Now we solve the ODE

$$\frac{\partial \log \hat{q}}{\partial \log \tau} = 2\alpha - 2\beta - \gamma$$

"1st order" correction

$\rightarrow$ Prescaling
Adiabatic prescaling

- Prescaling emerges as the lowest excited states decay.
- Appears before reaching the time-independent scaling regime, for any initial condition.
- For specific choices of initial conditions (which requires $f_0 \sim f_s$), prescaling can be extended to arbitrarily early times.
Summary

- Scaling and prescaling at early times after a heavy-ion collision can be explained by following the instantaneous eigenstates of lowest energy in the kinetic equation.
  - The use of such states can greatly simplify the analysis of the QGP, even at very early times.

- This analysis extends that of [4] to an earlier stage in the QGP hydrodynamization.

Outlook

- To do: follow the evolution of a “lowest energy” eigenstate from early times until hydrodynamics.
  - Also: study other setups with the adiabatic framework.

- How to probe different scaling regimes: exponent-independent ratios of moments.
  - In particular: cumulants that vanish for specific forms of scaling distributions $f_S$. 
Thanks!
References


Extra slides
How to write the kinetic equation as in the adiabatic theorem of quantum mechanics

Consider a collision integral of the form

\[ \mathcal{C}[f] = - \sum_i \lambda_i(\tau; f)(\mathcal{O}_i f), \]

where \( \lambda_i \) are numbers that may depend non-linearly on \( f \), and \( \mathcal{O}_i \) are linear differential operators acting on \( f \).

Then, by taking moments \( \left( \text{e.g. } n_{n,m} = \int p_{\perp}^{2n} p_{\perp}^{m} f \right) \), one arrives at

\[ \partial_{\log \tau} n_{n,m} = -(2n + 1)n_{n,m} - \sum_i \lambda_i M_{n,m;n',m'} n_{n',m'}, \]

which is of the form \( \partial_y \overline{\psi} = - \mathcal{H}[y; \{ F_i[\overline{\psi}] \}] \overline{\psi} \).
Explicit form of the Hamiltonians

- If $\mathcal{C}[f] = -q \nabla_p^2 f$, we have (in the $n_{n,m} = \langle p_z^{2n} p_{\perp}^m \rangle$ basis)
  
  $$
  \partial \log \tau n_{n,m} = - (2n + 1)n_{n,m} + \tau \hat{q}(\tau) [2n(2n - 1)n_{n-1,m} + m^2 n_{n,m-2}],
  $$

  which means that

  $$
  \mathcal{H}_{n,m;n',m'} = (2n + 1) \delta_{n,n'} \delta_{m,m'} - \tau \hat{q}(\tau) [2n(2n - 1) \delta_{n-1,n'} \delta_{m,m'} + m^2 \delta_{n,n'} \delta_{m-1,m'}].
  $$

- If we only keep the longitudinal momentum derivatives, $\mathcal{C}[f] = -q \partial_{p_z}^2 f$,

  $$
  \partial \log \tau n_{n,m} = - (2n + 1)n_{n,m} + \tau \hat{q}(\tau) 2n(2n - 1)n_{n-1,m},
  $$

  $$
  \mathcal{H}_{n,m;n',m'} = (2n + 1) \delta_{n,n'} \delta_{m,m'} - \tau \hat{q}(2n)(2n - 1) \delta_{n-1,n'} \delta_{m,m'}. 
  $$
Scaling around the late-time attractor:

- Recently, Almalool, Kurkela, Strickland (2020) showed that an aHydro ansatz

\[ f(p; \tau) = f_{\text{Bose}} \left( \frac{\sqrt{p^2 + \xi^2(\tau)p_z^2}}{\Lambda(\tau)} \right), \]

fixing \( \xi, \Lambda \) such that the energy-momentum tensor matches that of a full kinetic theory simulation.

\(--> \) This is also a time-dependent scaling distribution. How do we understand this?
Relaxation Time Approximation (RTA) (near Hydro)

• As an illustrative example, consider the RTA approximation to the kinetic equation:

$$\frac{\partial}{\partial \tau} f - \frac{p_z}{\tau} \frac{\partial}{\partial p_z} f = -\frac{1}{\tau_R(\tau)} \left( f - f_{eq} \right).$$

• After the transients have died out, most of the moments behave as

$$n_{n,m} \sim T^{(2n+m+3)},$$

where $T$ is determined by the energy density of the system.

$$\Rightarrow \quad \alpha = 0, \quad \beta = \gamma = -\frac{\partial_y T}{T}.$$

• This is also a time-dependent scaling regime, but the shape of the distribution function is different.
A way to distinguish the two regimes: a “phase transition” of the distribution function

• (Time-dependent) Scaling greatly simplifies the dynamics of a system. However,

  \[ \rightarrow \alpha, \beta, \gamma \text{ do not give information on the shape of the distribution function.} \]

  \[ \rightarrow \text{Moreover, for scaling to take place, that shape must remain fixed, and it must be independent of } \alpha, \beta, \gamma. \]

• One can use this fact to find quantities independent of \( \alpha, \beta, \gamma \) that remain constant.

  \[ \rightarrow \text{For instance, under (time-dependent) scaling, the ratio } \frac{\langle p_z^2 p_\perp^2 \rangle^2}{\langle p_z^4 \rangle \langle p_\perp^4 \rangle} \text{ is constant.} \]

  \[ \rightarrow \text{If this ratio changes, then scaling must be broken, signaling a “phase transition” out of that regime. One can use it as an order parameter to distinguish different “phases.”} \]
Explicit solutions to the scaling exponents’ ODE

Motivated by [2], if the shape of the initial distribution is Gaussian, with \( \langle p_z^2 \rangle = \sigma_z^2 \) and \( \langle p_\perp^2 \rangle = \sigma_\perp^2 \), we find (\( y \equiv \log(\tau/\tau_I) \), \( q \equiv \tau \hat{q} \), \( g_q \equiv -\partial_y q/q \))

\[
\gamma = 1 - \frac{qe^{2y}(1 - g_q/2)}{qe^{2y} - q_0 + \sigma_z^2(1 - g_q/2)}
\]

\[
\beta = -\frac{qg_q/2}{q_0 - q + \sigma_\perp^2 g_q(0)/2}
\]

\[
\alpha = -\frac{qe^{2y}(1 - g_q/2)}{qe^{2y} - q_0 + \sigma_z^2(1 - g_q/2)} - \frac{qg_q}{q_0 - q + \sigma_\perp^2 g_q(0)/2} = \gamma - 1 + 2\beta
\]
Scaling exponents for higher initial occupancy

- In [2], it was also considered an initial occupancy 6 times higher.
- Prescaling starts later than in the case considered in the main section.

→ The comparison to our results starts later.

\[ \mathcal{C}[f] \propto \nabla_p^2 f \]