Entanglement, partial set of measurements, and diagonality of the density matrix in the parton model

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arXiv.2001.01726, with Alex Kovner and Vladimir V. Skokov

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Motivation

- **Proton:**
  - $\tau \sim \infty \Rightarrow H \left| \text{Proton} \right\rangle = M \left| \text{Proton} \right\rangle$, energy eigenstate. ($S = 0$)
  - Parton model treats proton as collection of nearly free particles ($S_f \neq 0$)
  - Suggested resolution of this apparent paradox: quantum entanglement (arXiv.1702.03489, Kharzeev & Levin)
  - Postulation: reduced density matrix for observed parton is diagonal in particle number basis

- **Color Glass Condensate:**
  - Hamiltonian is non-perturbative and unknown, so is the wavefunction
  - A model for proton wavefunction
    \[
    \left| \text{proton} \right\rangle = \sum_{\rho_a} \left| v; \rho_a \right\rangle \otimes \left| s; \rho_a, A_b \right\rangle
    \]
  - Is CGC reduced density matrix diagonal in gluon number basis?
Reduced density matrix in DIS context

Density matrix $\hat{\rho}(A, B) \rightarrow$ reduced density matrix

$$\hat{\rho}_A = \text{Tr}_B \hat{\rho}(A, B)$$

Here $A$ can be probed in DIS partons of the parton model; $B$ is the unobserved part of the parton wavefunction.

The property of interest: if $\hat{\rho}(A, B)$ is pure, non-pure reduced density matrix $\hat{\rho}_A \rightarrow$ entanglement

from arXiv.1904.11974
Quantum entropies

Common entropies in quantum information theory:

- Renyi entropy $S^N_R = \frac{1}{1-N} \ln \text{Tr}\{\hat{\rho}^N\}$

- von Neumann entropy $S_V = \lim_{N \to 1} S^N_R = - \text{Tr}\{\hat{\rho} \ln \hat{\rho}\}$

**Entropy of entanglement:** entropy of the reduced density matrix
Any experimental measurement is limited: one can study only part of the full $\hat{\rho}$.

Parton model: most (if not all) observables probe diagonal components of the density matrix in the number of parton representation.

Ignorance density matrix: replace the off-diagonal elements of the density matrix with zeros. The Ignorance density matrix is positive-definite and is definitely not pure.

Entropy of ignorance: entropy of the ignorance density matrix
Example

Given a pure state $|\phi_{AB}\rangle = \frac{\sqrt{2}}{2} |0_A\rangle \otimes |0_B\rangle + \frac{1}{2} |0_A\rangle \otimes (|0_B\rangle + |1_B\rangle)$

$$\hat{\rho}_A = \frac{1}{4} \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \quad \hat{\rho}_I = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The ignorance density matrix $\rho^I_A$ is defined in particle number basis.

$$S_V(\hat{\rho}_A) \sim 0.426 \quad S_V(\hat{\rho}_I) \sim 0.693$$
Proton wave function

CGC model for Proton wavefunction,

$$|\text{proton}\rangle = \sum_{\rho_a} |v; \rho_a\rangle \otimes |s; \rho_a, A_b\rangle$$

where

- $|v\rangle$ describes the valance dof
- $|s\rangle$ stands for soft gluons
- $\rho_a(x)$ is the color charge density of the valance modes
- $A_b$ is the gluon field generated from the source $\rho_a$
Reduced density matrix for soft gluons in MV model

Our goal is the reduced density matrix for soft gluons

\[ \hat{\rho}_s = \text{Tr}_v (|v\rangle \langle v| \otimes |s\rangle \langle s|) \]

Use MV model

\[
\text{Tr}_v \Rightarrow \int D[\rho_a]
\]

\[
|v\rangle = \exp \left\{ - \int_k \frac{\rho_a(k)\rho_a^*(k)}{2\mu^2} \right\}
\]

\[
|s\rangle = C |0\rangle ; \quad C = \exp \left\{ i \int_k b_i^a(k)\phi_i^a \right\}
\]

\[
b_i^a(k) = \frac{igk^i}{k^2} \rho_a(k) + \mathcal{O}(\rho_a^2)
\]

\[
\phi_i^a(k) = a_{i\dagger}^a(k) + a_i^a(-k)
\]
Entropy of entanglement

\[ \hat{\rho}(\phi, \Phi) = N \int D[\rho_a] e^{-\int_k \frac{\rho_a(k)\rho_a^*(k)}{2\mu^2}} \langle \phi | C | 0 \rangle \langle 0 | C^\dagger | \Phi \rangle \]

To compute the entanglement entropy, one recall

\[ -\text{Tr}\{\hat{\rho} \ln(\hat{\rho})\} = \lim_{N \to 1} \frac{1}{1 - N} \ln(\text{Tr}\{\hat{\rho}^N\}) \]

and in terms of functional integrals

\[ \text{Tr}\{\hat{\rho}^N\} = \int D[\phi_1, \phi_2, ..\phi_N] \rho(\phi_1, \phi_2)\rho(\phi_2, \phi_3)\ldots\rho(\phi_N, \phi_1) \]
Analytic results for entropy of entanglement

\[ S_{R}^{2} = \frac{1}{2} (N_{c}^{2} - 1) S_{\perp} \int \frac{d^{2}q}{(2\pi)^{2}} \ln \left( 1 + 4 \frac{g^{2} \mu^{2}}{q^{2}} \right). \]

\[ S_{V} = \frac{1}{2} (N_{c}^{2} - 1) S_{\perp} \int \frac{d^{2}q}{(2\pi)^{2}} \left[ \ln \left( \frac{g^{2} \mu^{2}}{q^{2}} \right) + \sqrt{1 + 4 \frac{g^{2} \mu^{2}}{q^{2}}} \ln \left( 1 + \frac{q^{2}}{2g^{2} \mu^{2}} + \frac{q^{2}}{2g^{2} \mu^{2}} \sqrt{1 + 4 \frac{g^{2} \mu^{2}}{q^{2}}} \right) \right]. \]

- Extensive in terms of transverse area \( S_{\perp} \)
- \( \lim_{q \to \infty} S(q) = 0 \)

*ArXiv.1506.05394 by Alex Kovner, Michael Lublinsky*
Soft gluon in particle number basis

Recall the definition of soft gluon state

\[ |s\rangle = e^{i \int k b_a(k)(a_a^\dagger(k)+a_a(-k))} |0\rangle \]

- Discretize the momentum \( \int \frac{dk^2}{(2\pi)^2} \to \sum \frac{\Delta^2}{(2\pi)^2} \)
- The coherent operator can be rewritten as

\[ C |0\rangle = e^{i \int k b_a(k)a_a^\dagger(k)+b_a^\ast(k)a_a(k)} |0\rangle = e^{i \int k b_a(k)a_a^\dagger(k)} e^{-\frac{1}{2} \int k \frac{g^2}{k^2} |\rho_a|^2} |0\rangle \]

- Expanding \( e^{i \frac{\Delta^2}{(2\pi)^2} b_a(k)a_a^\dagger(k)} \) will allow us to do calculation in particle number basis.
Matrix elements in particle number basis

For a single momentum mode $q$, including normalization

$$\langle n_c(q), m_c(-q) | \hat{\rho}_s(q) | \alpha_c(q), \beta_c(-q) \rangle$$

$$= (1 - R) \frac{(n + \beta)!}{\sqrt{n!m!\alpha!\beta!}} \left( \frac{R}{2} \right)^{n+\beta} \delta_{(n+\beta),(m+\alpha)}; \quad R = \left(1 + \frac{q^2}{2g^2\mu^2}\right)^{-1}$$

Off-diagonal elements: eg, $\langle 0, 0 | \hat{\rho}_s(q) | 1, 1 \rangle = \frac{(1-R)R}{2} \neq 0$

Diagonal matrix elements:

$$\langle n_c(q), m_c(-q) | \hat{\rho}_s(q) | n_c(q), m_c(-q) \rangle$$

$$= (1 - R) \frac{(n_c + m_c)!}{n_c!m_c!} \left( \frac{R}{2} \right)^{n_c+m_c}$$
Ratio between entropies of entanglement and ignorance

\[ \rho_{nm} \propto (1 - R) R^{m_c+n_c} \text{ at momentum } q \]

\[ R = (1 + \frac{q^2}{2g^2 \mu^2})^{-1} \]

- At large \( q \), \( S_I \approx S_E \)
  - \( R \approx 0 \)
  - Vacuum contribution dominates
- At small \( q \), \( S_I > S_E \)
  - \( R \sim O(1) \)
  - Higher states and interference terms are also important

\[ \text{Ratio von Neumann } S_I(q)/S_E(q) \]

\[ \text{Renyi } S^I_R(q)/S_R(q) \]
Experimental observation and the entropy of ignorance

- Observation was done in particle number basis in experiment
  \[ S_{\text{entanglement}} \Rightarrow S_{\text{hadron}} = - \sum P(N_h) \ln(P(N_h)) \]

- Kharzeev & Levin
  - The reduced density matrix \( \hat{\rho}_r = \sum_{N_p} P_{N_p} |N_p\rangle\langle N_p| \)
  - \( S = - \sum P_{N_p} \ln(P_{N_p}) \)
  - At small \( x \), entropy of gluon \( \Rightarrow \) entropy of hadron
    \[ S \approx \ln(xG(x, Q^2)) \]

- \( \ln(xG(x, Q^2)) \) corresponds to entropy of ignorance in our calculation
DIS data from HERA

- \( \ln(xG(x, Q^2)) \) overestimates hadron entropy
- Difference becomes small at large \( Q^2 \) which is consistent with our analysis

**Graph:**
- von Neumann \( S_I(q)/S_E(q) \)
- Renyi \( S^R(q)/S_R(q) \)

**Legend:**
- H1 data
- RAPGAP
- HERAPDF
- Q^2 ranges
  - 5 < Q^2 < 10 GeV^2
  - 10 < Q^2 < 20 GeV^2
  - 20 < Q^2 < 40 GeV^2
  - 40 < Q^2 < 100 GeV^2

**Graph Details:**
- 0 < \( \eta^* < 4.0 \)
- ep \( \sqrt{s} = 319 \text{ GeV} \)
- H1 Collaboration

**Reference:**
arXiv 2011.01812, H1 Collaboration
In what basis $S_I = S_E$?
A different perspective, consider the following reduced density matrix

\[ \hat{\rho}_r = (1 - e^{-\beta \omega_0}) \sum_{n=0} \left| n \right\rangle \left\langle n \right| \]

where \( n \) is the energy level, and define \( f = \frac{1}{e^{\beta \omega_0} - 1} \). The corresponding von Neumann entropy is

\[ S_V = (1 + f) \ln(1 + f) - f \ln(f) \]
A further examination of CGC $S_V$

$$S_V = \frac{1}{2} (N_c^2 - 1) S_\perp \int \frac{d^2 q}{(2\pi)^2} \left[ \ln \left( \frac{g^2 \mu^2}{q^2} \right) + \sqrt{1 + 4 \frac{g^2 \mu^2}{q^2}} \ln \left( 1 + \frac{q^2}{2g^2 \mu^2} + \frac{q^2}{2g^2 \mu^2} \sqrt{1 + 4 \frac{g^2 \mu^2}{q^2}} \right) \right]$$

If set $\beta \omega_0 = 2 \ln \left( \frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2 \mu^2}} \right)$, we recover the same structure

$$S_V = \frac{1}{2} (N_c^2 - 1) S_\perp \int \frac{d^2 q}{(2\pi)^2} \left[ (1 + f) \ln(1 + f) - f \ln(f) \right]$$

Which indicates the leading order CGC density matrix describe a thermal system of quasi-particles

$$c(q) = \frac{1}{2} \left( \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} \right) a(q) + \frac{1}{2} \left( \sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right) a^\dagger(-q)$$

where $\alpha = \sqrt{1 + \frac{4g^2 \mu^2}{q^2}}$
Conclusion

- In CGC, density matrix for soft gluons is not diagonal in particle number basis; this contradicts to Kharzeev and Levin’s assumption
- Entropy of ignorance overestimates entropy of hadrons
- CGC reduced density matrix can be diagonalized into thermal form (work in progress)
Through diagonalization of $\hat{\rho}_r$

In field basis

$$\hat{\rho}_r = \int D[\phi, \Phi] \rho_r(\phi, \Phi)|\phi\rangle\langle\Phi|$$

To diagonalize it, we construct a wave functional

$$|\Psi\rangle = \int D[\psi]f(\psi)|\psi\rangle$$

and we then have the eigen-equation

$$\hat{\rho}_r |\Psi\rangle = \lambda|\Psi\rangle$$
Through diagonalization of $\hat{\rho}_r$

In terms of field explicitly

$$\int D[\Phi] \rho_r(\phi, \Phi) f(\Phi) = \lambda f(\phi)$$

Our assumption is based on quantum harmonic oscillator such that the ground state is given by

$$f(\phi) = \exp\{-\alpha \phi \phi^*\}$$

One can build higher excited states use ladder operators.
Thermal eigenvalues

It turns out, the reduced density matrix can be exactly diagonalized in the "quantum harmonic oscillator" basis, with Boltzmann weight eigenvalues.

\[ \lambda_n = \exp \left\{ - \left( \frac{1}{2} + n \right) \omega \beta \right\} \]

where \( n = 0, 1, 2, \ldots \).

\[ \beta \omega = 2 \ln \left( \frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2\mu^2}} \right) \]
PP collision data from LHC

ArXiv.1904.11974, by Zhoudunming Tu, Dmitri E. Kharzeev, Thomas Ullrich

\[ S_{\text{Parton}} = \ln(xG(x, Q^2)) \]
\[ S_{\text{hadron}} = -\sum P(N) \ln P(N) \]
Experimental data

**arXiv 2011.01812, H1 Collaboration**

**moving $\eta_{lab}$ window**

$e^+ p \sqrt{s} = 319$ GeV

$\eta_{lab} \sim \ln(x)$

- **H1 data**
- **RAPGAP**
- **HERAPDF**

**$Q^2$ ranges**
- $5 < Q^2 < 10$ GeV$^2$
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**$S_{hadron}$**

**$S_{gluon}$**

**$\langle x_{bj} \rangle$**