

Entanglement,  
partial set of measurements, and diagonality  
of the density matrix in the parton model

Haowu Duan

North Carolina State University

[arXiv.2001.01726](https://arxiv.org/abs/2001.01726), with Alex Kovner and Vladimir V. Skokov

Initial Stages 2021

# Motivation

- Proton:

$\tau = 1$ )  $H|j\text{Proton}i\rangle = M|j\text{Proton}i\rangle$ , energy eigenstate. ( $S = 0$ )

Parton model treats proton as collection of **nearly** free particles ( $S_f \neq 0$ )

Suggested resolution of this apparent paradox: quantum entanglement  
(arXiv.1702.03489, Kharzeev & Levin)

Postulation: reduced density matrix for observed parton is diagonal in particle number basis

- Color Glass Condensate:

Hamiltonian is non-perturbative and unknown, so is the wavefunction  
A model for proton wavefunction

$$|j\text{proton}i\rangle = \sum_{\rho_a} |jv; \rho_a\rangle |js; \rho_a, A_b\rangle$$

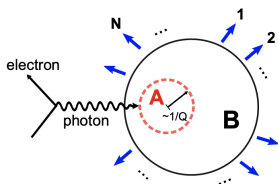
- Is CGC reduced density matrix diagonal in gluon number basis?

## Reduced density matrix in DIS context

Density matrix  $\hat{\rho}(A, B)$  ! reduced density matrix

$$\hat{\rho}_A = \text{Tr}_B \hat{\rho}(A, B)$$

- Here  $A$  can be probed in DIS partons of the parton model;  $B$  is the unobserved part of the parton wavefunction.
- The property of interest: if  $\hat{\rho}(A, B)$  is pure, non-pure reduced density matrix  $\hat{\rho}_A$  ! entanglement



from arXiv.1904.11974

# Quantum entropies

Common entropies in quantum information theory:

- Renyi entropy  $S_R^N = \frac{1}{1-N} \ln \text{Tr}\{\hat{\rho}^N\}$
- von Neumann entropy  $S_V = \lim_{N \rightarrow 1} S_R^N = -\text{Tr}\{\hat{\rho} \ln \hat{\rho}\}$

**Entropy of entanglement:** entropy of the reduced density matrix

## Entropy of ignorance

- Any experimental measurement is limited: one can study only part of the full  $\hat{\rho}$ .
- Parton model: most (if not all) observables probe diagonal components of the density matrix in the number of parton representation.
- Ignorance density matrix: replace the off-diagonal elements of the density matrix with zeros. The Ignorance density matrix is positive-definite and is **definitely** not pure.
- **Entropy of ignorance**: entropy of the ignorance density matrix

## Example

Given a pure state  $|\phi_{AB}\rangle = \frac{\rho_{-}}{2} |0_A\rangle |0_B\rangle + \frac{1}{2} |0_A\rangle |1_B\rangle$

$$\hat{\rho}_A = \frac{1}{4} \begin{bmatrix} \rho_{-} & 0 \\ 0 & 2 \end{bmatrix} \quad \hat{\rho}_A^I = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The ignorance density matrix  $\rho_A^I$  is defined in particle number basis.

$$S_V(\hat{\rho}_A) = 0.426 \quad S_V(\hat{\rho}_A^I) = 0.693$$

# Proton wave function

CGC model for Proton wavefunction,

$$j_{\text{proton}}^i = \sum_{\rho_a} j_{v; \rho_a}^i \quad j_{s; \rho_a, A_b}^i$$

where

- $j_{v}^i$  describes the valance dof
- $j_{s}^i$  stands for soft gluons
- $\rho_a(x)$  is the color charge density of the valance modes
- $A_b$  is the gluon field generated from the source  $\rho_a$

# Reduced density matrix for soft gluons in MV model

Our goal is the reduced density matrix for soft gluons

$$\hat{\rho}_s = \text{Tr}_v(jvi \ hvj \ jsi \ hsj)$$

Use MV model

$$\begin{aligned} \text{Tr}_v ) \int D[\rho_a] \\ hvjvi &= \exp \left\{ \int_k \frac{\rho_a(k)\rho_a(k)}{2\mu^2} \right\} \\ jsi &= C j0i; \quad C = \exp \left\{ i \int_k b_a^i(k)\phi_a^i \right\} \\ b_a^i(k) &= \frac{igk^i}{k^2} \rho_a(k) + O(\rho_a^2) \\ \phi_a^i(k) &= a_a^{iy}(k) + a_a^i(k) \end{aligned}$$



## Entropy of entanglement

$$\hat{\rho}(\phi, \Phi) = N \int D[\rho_a] e^{-\int_k \frac{\rho_a(k) \rho_a(k)}{2\mu^2} h\phi_j C_j 0 i h 0 j C^y j \Phi i}$$

To compute the entanglement entropy, one recall

$$\text{Tr} f \hat{\rho} \ln(\hat{\rho}) g = \lim_{N \rightarrow \infty} \frac{1}{N} \ln(\text{Tr} \{ \hat{\rho}^N \})$$

and in terms of functional integrals

$$\text{Tr} \{ \hat{\rho}^N \} = \int D[\phi_1, \phi_2, \dots, \phi_N] \rho(\phi_1, \phi_2) \rho(\phi_2, \phi_3) \dots \rho(\phi_N, \phi_1)$$

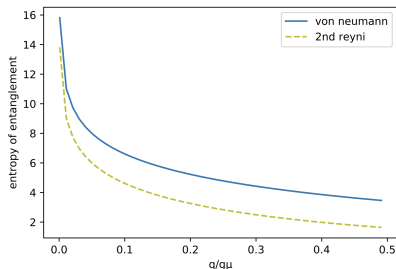
# Analytic results for entropy of entanglement

$$S_R^2 = \frac{1}{2}(N_c^2 - 1)S_\gamma \int \frac{d^2 q}{(2\pi)^2} \ln\left(1 + 4 \frac{g^2 \mu^2}{q^2}\right).$$

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\gamma \int \frac{d^2 q}{(2\pi)^2} \left[ \ln\left(\frac{g^2 \mu^2}{q^2}\right) + \sqrt{1 + 4 \frac{g^2 \mu^2}{q^2}} \ln\left(1 + \frac{q^2}{2g^2 \mu^2} + \frac{q^2}{2g^2 \mu^2} \sqrt{1 + 4 \frac{g^2 \mu^2}{q^2}}\right) \right].$$

- Extensive in terms of transverse area  $S_\gamma$
- $\lim_{q \rightarrow \infty} S(q) = 0$

*ArXiv.1506.05394 by Alex Kovner,  
Michael Lublinsky*



# Soft gluon in particle number basis

Recall the definition of soft gluon state

$$|S\rangle = e^{i \int_k b_a(k) (a_a^\dagger(k) + a_a(k))} |0\rangle$$

- Discretize the momentum  $\int \frac{dk^2}{(2\pi)^2} \rightarrow \sum \frac{\Delta^2}{(2\pi)^2}$
- The coherent operator can be rewritten as

$$C|0\rangle = e^{i \int_k b_a(k) a_a^\dagger(k) + b_a(k) a_a(k)} |0\rangle = e^{i \int_k b_a(k) a_a^\dagger(k)} e^{-\frac{1}{2} \int_k \frac{g^2}{k^2} j \rho_a j^2} |0\rangle$$

- Expanding  $e^{i \frac{\Delta^2}{(2\pi)^2} b_a(k) a_a^\dagger(k)}$  will allow us to do calculation in particle number basis.

## Matrix elements in particle number basis

For a single momentum mode  $q$ , including normalization

$$\langle m_c(q), m_c(q) | \hat{\rho}_s(q) | n_c(q), \beta_c(q) \rangle$$

$$= (1 - R)^{\beta} \frac{(n + \beta)!}{n! m! \alpha! \beta!} \left( \frac{R}{2} \right)^{n + \beta} \delta_{(n + \beta), (m + \alpha)}; \quad R = \left( 1 + \frac{q^2}{2g^2 \mu^2} \right)^{-1}$$

**O** -diagonal elements: eg,  $\langle 0, 0 | \hat{\rho}_s(q) | 1, 1 \rangle = \frac{(1 - R)R}{2} \neq 0$

**Diagonal matrix elements:**

$$\langle n_c(q), m_c(q) | \hat{\rho}_s(q) | n_c(q), m_c(q) \rangle$$

$$= (1 - R)^{m_c} \frac{(n_c + m_c)!}{n_c! m_c!} \left( \frac{R}{2} \right)^{n_c + m_c}$$

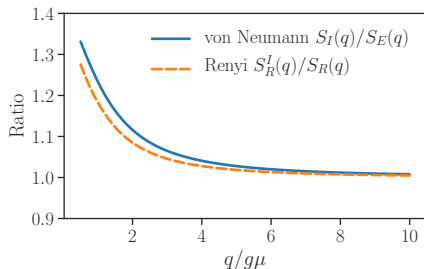
# Ratio between entropies of entanglement and ignorance

- $\rho_{nm} \propto (1 + R)R^{m_c+n_c}$  at momentum  $q$

$$R = \left(1 + \frac{q^2}{2g^2\mu^2}\right)^{-1}$$

- At large  $q$ ,  $S_I \approx S_E$   
 $R \approx 0$   
vacuum contribution dominates

- At small  $q$ ,  $S_I > S_E$   
 $R \approx O(1)$   
higher states and **interference terms** are also important



# Experimental observation and the entropy of ignorance

- Observation was done in particle number basis in experiment

$$S_{\text{entanglement}} \approx S_{\text{hadron}} = - \sum P(N_h) \ln(P(N_h))$$

- Kharzeev & Levin

The reduced density matrix  $\hat{\rho}_r = \sum_{N_p} P_{N_p} |N_p\rangle \langle N_p|$

$$S = - \sum P_{N_p} \ln(P_{N_p})$$

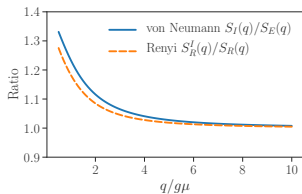
At small  $x$ , entropy of gluon  $\approx$  entropy of hadron

$$S \approx \ln(xG(x, Q^2))$$

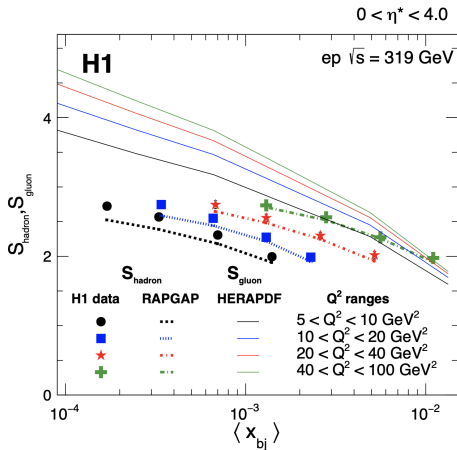
- $\ln(xG(x, Q^2))$  corresponds to entropy of ignorance in our calculation

# DIS data from HERA

- $\ln(xG(x, Q^2))$  overestimates hadron entropy
- Difference becomes small at large  $Q^2$  which is consistent with our analysis



arXiv 2011.01812, H1 Collaboration



In what basis  $S_I = S_E$ ?



## Thermal density matrix

A different perspective, consider the following **reduced** density matrix

$$\hat{\rho}_r = (1 + e^{-\beta\omega_0})^{-1} \sum_{n=0}^{\infty} e^{-n\beta\omega_0} |n\rangle\langle n|$$

where  $n$  is the energy level, and define  $f = \frac{1}{e^{\beta\omega_0} + 1}$ . The corresponding von Neumann entropy is

$$S_V = -(1+f) \ln(1+f) - f \ln(f)$$

## A further examination of CGC $S_V$

$$S_V = \frac{1}{2}(N_c^2 - 1)S_F \int \frac{d^2q}{(2\pi)^2} \left[ \ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2} \sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right) \right]$$

If set  $\beta\omega_0 = 2 \ln\left(\frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2\mu^2}}\right)$ , we recover the same structure

$$S_V = \frac{1}{2}(N_c^2 - 1)S_F \int \frac{d^2q}{(2\pi)^2} [(1 + f) \ln(1 + f) - f \ln(f)]$$

Which indicates the leading order CGC density matrix describe a thermal system of *quasi particles*

$$c(q) = \frac{1}{2} \left( \rho_{\alpha} + \frac{1}{\alpha} \right) a(q) + \frac{1}{2} \left( \rho_{\alpha} - \frac{1}{\alpha} \right) a^y(q)$$

where  $\alpha = \sqrt{1 + \frac{4g^2\mu^2}{q^2}}$

## Conclusion

- In CGC, density matrix for soft gluons is not diagonal in particle number basis; this contradicts to Kharzeev and Levin's assumption
- Entropy of ignorance overestimates entropy of hadrons
- CGC reduced density matrix can be diagonalized into thermal form (work in progress)

## Through diagonalization of $\hat{\rho}_r$

In field basis

$$\hat{\rho}_r = \int D[\phi, \Phi] \rho_r(\phi, \Phi) j\phi i h\Phi j$$

To diagonalize it, we construct a wave functional

$$j\Psi i = \int D[\psi] f(\psi) j\psi i$$

and we then have the eigen-equation

$$\hat{\rho}_r j\Psi i = \lambda j\Psi i$$

## Through diagonalization of $\hat{\rho}_r$

In terms of field explicitly

$$\int D[\Phi] \rho_r(\phi, \Phi) f(\Phi) = \lambda f(\phi)$$

Our assumption is based on quantum harmonic oscillator such that the ground state is given by

$$f(\phi) = \exp\left[-\frac{\alpha}{2} \phi^2\right]$$

One can build higher excited states use ladder operators.

## Thermal eigenvalues

It turns out, the reduced density matrix can be exactly diagonalized in the "quantum harmonic oscillator" basis, with Boltzmann weight eigenvalues.

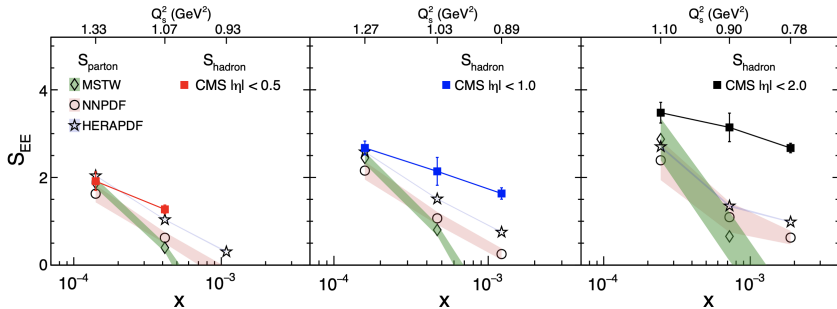
$$\lambda_n = \exp\left\{ \left(\frac{1}{2} + n\right)\omega\beta \right\}$$

where  $n=0, 1, 2, \dots$

$$\beta\omega = 2 \ln\left( \frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2\mu^2}} \right)$$

# PP collision data from LHC

ArXiv.1904.11974, by Zhoudunming Tu, Dmitri E. Kharzeev,  
Thomas Ullrich



$$S_{parton} = \ln(xG(x, Q^2))$$

$$S_{hadron} = \sum P(N) \ln P(N)$$

# Experimental data

arXiv 2011.01812, H1 Collaboration

moving  $\eta_{lab}$  window

