Finite Nc Corrections in the NLO BK Equation

Lappi, Mäntysaari, A. R. [Phys. Rev. D 102 074027] (2020)

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Outline

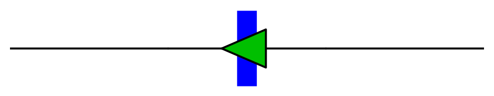
- Preliminaries
  - Colour Glass Condensate
  - Leading order (LO) BK (Balitsky–Kovchegov) equation
  - Gaussian approximation for 2-point correlator
- Next-to-leading (NLO) BK equation
  - Finite $N_c \implies$ 6-point correlators
  - Gaussian approximation parametrisation
- Numerical results
  - 6-point correlators in line configuration
  - NLO BK evolution finite vs large N_c

Colour Glass Condensate

- Effective field theory at small virtuality Q^2, small x_{Bj}
- Wilson lines are basic building blocks

Marquet, Weigert [Nucl.Phys. A843 (2010) 68-97], Blaizot, Gelis, Venugopalan [Nucl.Phys. A743 (2004) 57-91]

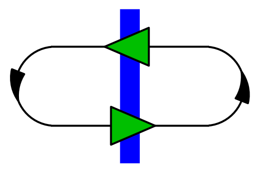
$$U^\dagger_{\boldsymbol{x}} := P\exp\left\{ ig\int_{x^+} \alpha^a_{\boldsymbol{x}}(x^+)t^a \right\} \sim$$



- Enter cross sections explicitly, e.g.

$$\sigma_{\mathrm{DIS}}(Y,Q^2) = 2\int_{\boldsymbol{r}}\int_0^1 d\alpha \left|\Psi(\alpha,r^2,Q^2)\right|^2 \int_{\boldsymbol{b}} \left\langle 1 - S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle,$$

$$S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} := \frac{1}{N_c}\mathrm{tr}\left\{ U_{\boldsymbol{x}} U^\dagger_{\boldsymbol{y}} \right\} \sim$$



$\Longrightarrow$ need to calculate and understand behaviour of correlators

LO BK Equation

- Evolution in rapidity of correlators governed by Balitsky Hierarchy ($\Longleftrightarrow$ JIMWLK equation)
- Infinite set of equations – $\mathcal{O}(n)$ equation needs input from $\mathcal{O}(n+1)$
- First equation is Balitsky equation for $S^{(2)}_{\boldsymbol{x},\boldsymbol{y}}$

Balitsky [Phys. Rev. D60 (1999) 014020], Kovchegov [Phys. Rev. D60 (1999) 034008], Kovchegov [Phys. Rev. D61 (2000)]

$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle = \frac{\alpha_s}{\pi^2} \int_{\boldsymbol{z}} \tilde{\mathcal{K}}_{\boldsymbol{xzy}} \left(\left\langle \frac{1}{N_c} U^{ab}_{\boldsymbol{z}} \mathrm{tr}\left\{ t^a U_{\boldsymbol{x}} t^b U^\dagger_{\boldsymbol{y}} \right\} \right\rangle - C_f \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle \right)$$

- Use Fierz identity for 3-point correlator $\implies$ 4 point correlators
- Take large-N_c limit: expectation values factorise $\left\langle \mathrm{tr}\{\}\,\mathrm{tr}\{\} \right\rangle \to \left\langle \mathrm{tr}\{\} \right\rangle \left\langle \mathrm{tr}\{\} \right\rangle$

  $\implies$ BK equation at LO

$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle = \frac{\alpha_s}{\pi^2} \frac{N_c}{2} \int_{\boldsymbol{z}} \tilde{\mathcal{K}}_{\boldsymbol{xzy}} \left(\left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} \right\rangle \left\langle S^{(2)}_{\boldsymbol{z},\boldsymbol{y}} \right\rangle - \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle \right)$$

Gaussian Approximation

Iancu, Leonidov, McLerran [Nucl.Phys. A692 (2001) 583-645]
Fujii [Nucl.Phys. A709 (2002) 236-250]
Dumitru, Dusling, Gelis, Jalilian-Marian, Lappi, Venugopalan [Phys.Lett. B697 (2011) 21-25]
Dusling, Mace, Venugopalan [Phys.Rev. D97 (2018) no.1, 016014]

- Truncate infinite hierarchy of evolution equations
- Parametrise correlators according to

$$\partial_Y \left\langle \hat{\mathcal{O}} \right\rangle := -\frac{1}{2}\int_{\boldsymbol{uv}} G_{\boldsymbol{uv}}(Y) L^a_{\boldsymbol{u}} L^a_{\boldsymbol{v}} \hat{\mathcal{O}}$$

- Operate on $\mathcal{O}$ with Lie derivatives $L^a_{\boldsymbol{u}} L^a_{\boldsymbol{v}} \sim$ add gluon emission/absorption vertices
- E.g. 2-point correlator

$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle = -\frac{1}{2}\int_{\boldsymbol{uv}} G_{\boldsymbol{uv}}(Y) L^a_{\boldsymbol{u}} L^a_{\boldsymbol{v}} \; S^{(2)}_{\boldsymbol{x},\boldsymbol{y}}$$

- Solve differential equation: $\left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle = e^{-C_{\mathrm{F}}\mathcal{G}_{\boldsymbol{xy}}}$, where $\mathcal{G}_{\boldsymbol{xy}} := \int^Y \mathrm{d}Y' \left(G_{\boldsymbol{xy}}(Y') - \frac{1}{2}\left[G_{\boldsymbol{xx}}(Y') + G_{\boldsymbol{yy}}(Y') \right] \right)$

NLO BK Equation

- Repeat this process at NLO
- BK equation at NLO

Balitsky, Chirilli [Nucl. Phys. B822:45-87 (2009)]

$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle = \frac{\alpha_s N_c}{2\pi^2} \int_{\boldsymbol{z}} K_1^{\mathrm{BC}} \langle D_1 \rangle + \frac{\alpha_s^2 {N_c}^2}{16\pi^4} \int_{\boldsymbol{z},\boldsymbol{z}'} \Big(K_{2,1} \langle D_{2,1} \rangle + K_{2,2} \langle D_{2,2} \rangle \Big) + \mathcal{O}(n_f)$$

- Correlators are more complicated than at LO:

$$\langle D_1 \rangle = \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{y}} \right\rangle - \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle$$

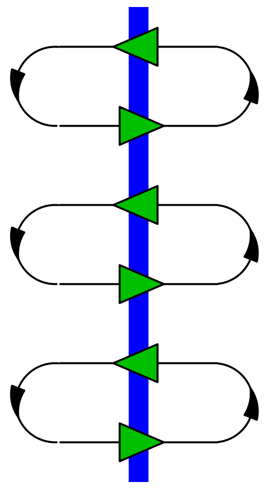
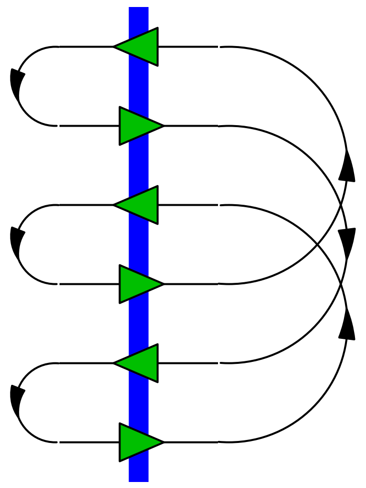
$$\langle D_{2,1} \rangle = \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle - \frac{1}{{N_c}^2} \left\langle S^{(6)}_{\boldsymbol{x},\boldsymbol{z},\boldsymbol{z}',\boldsymbol{y},\boldsymbol{z},\boldsymbol{z}'} \right\rangle - (z' \to z)$$

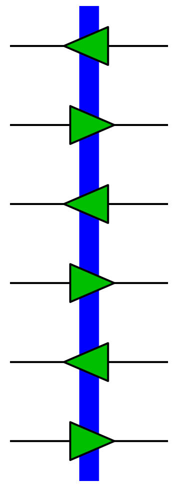
$$\langle D_{2,2} \rangle = \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle - (z' \to z)$$

- Can use large-N_c approximation, e.g. $\left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle \to \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} \right\rangle \left\langle S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} \right\rangle \left\langle S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle$

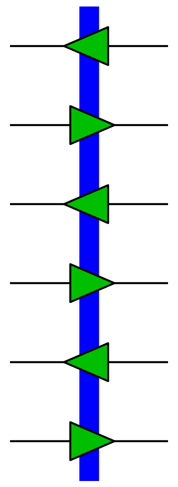
Correlators at Finite N_c

- NLO BK requires 6-point correlators

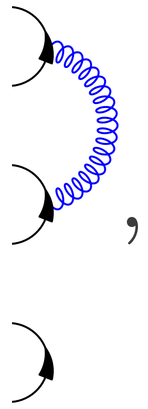
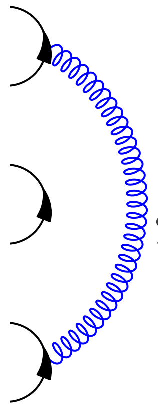
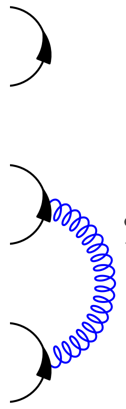
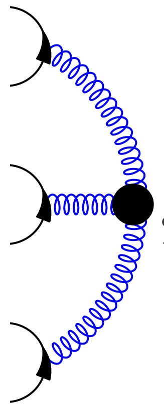
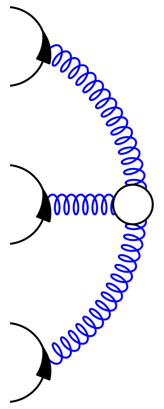
$$\left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle =$$  and $$\left\langle S^{(6)}_{\boldsymbol{x},\boldsymbol{z},\boldsymbol{z}',\boldsymbol{y},\boldsymbol{z},\boldsymbol{z}'} \right\rangle =$$ 

  with 2 repeated coordinates (only 4 unique coordinates)
- Apply Gaussian approximation $\partial_Y \left\langle \hat{\mathcal{O}} \right\rangle := -\frac{1}{2} \int_{\boldsymbol{uv}} G_{\boldsymbol{uv}}(Y) L^a_{\boldsymbol{u}} L^a_{\boldsymbol{v}} \hat{\mathcal{O}}$
- Now $\hat{\mathcal{O}}$ is 6×6 matrix, because 6 ways to form multiplets from 
- Write matrix equation for Gaussian approximation $\partial_Y \langle \mathcal{A}(Y) \rangle = -\mathcal{M}(Y)\mathcal{A}(Y)$
- Exponentiate to solve differential equation (as for 2-point correlator $\left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle = e^{-C_F \mathcal{G}_{\boldsymbol{xy}}}$)

Naive Solution

- **STEP 1**: Choose orthonormal basis for operator 

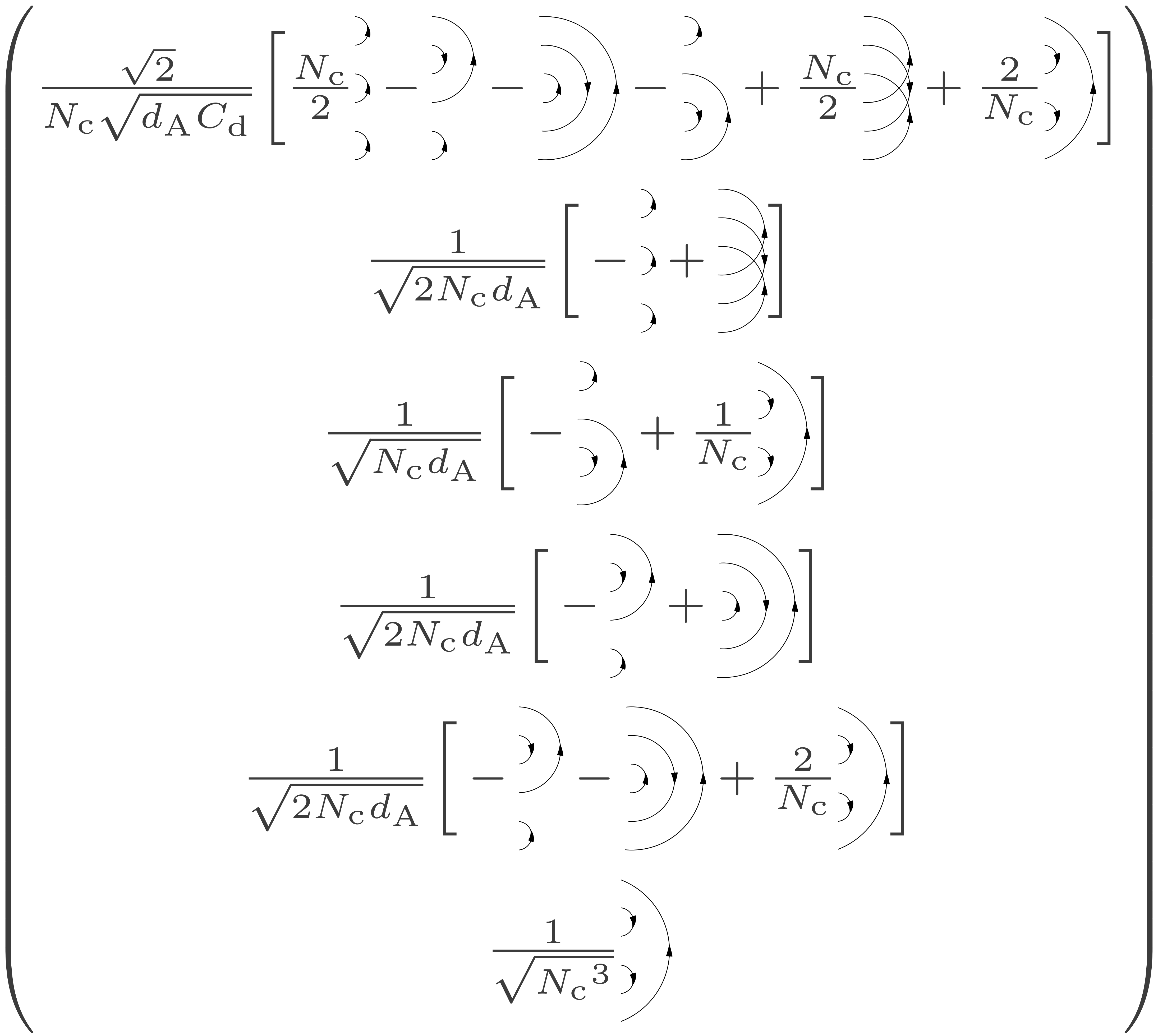
Naive choice is simplest orthonormal set

$\frac{1}{\sqrt{N_c{}^3}}$ , $\sqrt{\frac{4}{N_c d_A}}$ , $\sqrt{\frac{4}{N_c d_A}}$ , $\sqrt{\frac{4}{N_c d_A}}$ , $\frac{1}{i}\sqrt{\frac{8}{N_c d_A}}$ , $\sqrt{\frac{8}{C_d d_A}}$ 

- **STEP 2**: Form "correlator matrix" $\mathcal{A}(Y)$
- **STEP 3**: Construct "transition matrix" $\mathcal{M}(Y)$ by taking sum of all possible diagrams with 1 gluon added
- **STEP 4**: Solve $\partial_Y \langle \mathcal{A}(Y) \rangle = -\mathcal{M}(Y)\mathcal{A}(Y) \implies$ exponentiate 6×6 matrix!

Better Solution

- Exploit redundancy in coordinates in $\left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle$ and $\left\langle S^{(6)}_{\boldsymbol{x},\boldsymbol{z},\boldsymbol{z}',\boldsymbol{y},\boldsymbol{z},\boldsymbol{z}'} \right\rangle$ $\Longrightarrow$ **STEP 1**: Better basis



Matrix Equation in New Basis

- **STEP 2**: Form correlator matrix $\mathcal{A}(Y)$
- **STEP 3**: Construct transition matrix $\mathcal{M}(Y)$: clever basis choice $\Longrightarrow$ block diagonalisation

$$\lim_{\substack{\boldsymbol{u}\to\boldsymbol{z}\\ \boldsymbol{v}\to\boldsymbol{z}'}} \mathcal{M}(Y) = \begin{pmatrix} \mathcal{M}_1^{(3\times3)}(Y) & 0 & 0 \\ 0 & \mathcal{M}_2^{(2\times2)}(Y) & 0 \\ 0 & 0 & \mathcal{M}_3^{(1\times1)}(Y) \end{pmatrix}$$

- **STEP 4**: Solve $\partial_Y \langle \mathcal{A}(Y) \rangle = -\mathcal{M}(Y)\mathcal{A}(Y) \Longrightarrow$ no long need to exponentiate 6×6 matrix
- 1×1 equation: $\mathcal{M}_3(Y) = C_\mathrm{F}\mathcal{G}_{\boldsymbol{x}_3,\boldsymbol{y}_2}$ gives 2-point correlator (as seen before)
- 2×2 equation: $\mathcal{M}_2(Y)$ gives known 4-point correlator

$$\left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{y}} \right\rangle = \frac{1}{{N_\mathrm{c}}^2} e^{-C_\mathrm{F}\mathcal{G}_{\boldsymbol{x},\boldsymbol{y}}} + \frac{2C_\mathrm{F}}{N_\mathrm{c}} e^{-C_\mathrm{F}\mathcal{G}_{\boldsymbol{x},\boldsymbol{y}}} e^{-\frac{N_\mathrm{c}}{2}(\mathcal{G}_{\boldsymbol{x},\boldsymbol{z}}+\mathcal{G}_{\boldsymbol{y},\boldsymbol{z}}-\mathcal{G}_{\boldsymbol{x},\boldsymbol{y}})}$$

- 3×3 equation: exponentiate matrix $\mathcal{M}_1(Y)$ – doable analytically

3×3 Equation Solution

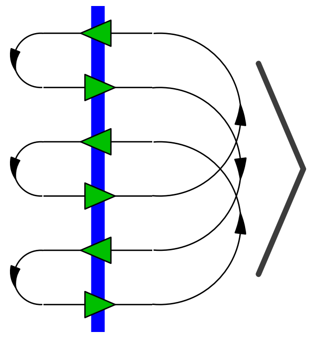
- Exponentiation of $\mathcal{M}_1(Y)$ gives

$$\mathcal{A}_1(Y) = \begin{pmatrix} \sum_{i=1}^{3} \frac{m_{11}}{d}(z_i) & -\sqrt{C_{\mathrm{d}} N_{\mathrm{c}}}\Gamma_2 \sum_{i=1}^{3} \frac{m_{12}}{d}(z_i) & -2\sqrt{2C_{\mathrm{d}} N_{\mathrm{c}}}\Gamma_2^2 \sum_{i=1}^{3} \frac{m_{13}}{d}(z_i) \\ \cdots & \sum_{i=1}^{3} \frac{m_{22}}{d}(z_i) & 2\sqrt{2}\Gamma_2 \sum_{i=1}^{3} \frac{m_{23}}{d}(z_i) \\ \cdots & \cdots & \sum_{i=1}^{3} \frac{m_{33}}{d}(z_i) \end{pmatrix}$$

- Roots z and functions of roots m_{ij} and d are simple polynomials in $\mathcal{G}$

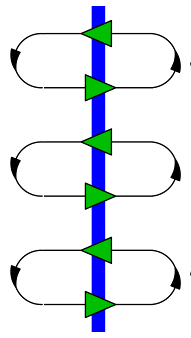
6-point Correlators for NLO BK

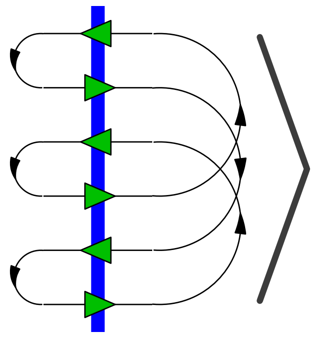
- **Final analytical solution**: two required 6-point correlators are

$$\lim_{\substack{\boldsymbol{u}\to\boldsymbol{z}\\ \boldsymbol{v}\to\boldsymbol{z}'}} \left\langle \frac{1}{N_\mathrm{c}}\right.$$  $$\left.\right\rangle = \left\langle \frac{1}{N_\mathrm{c}} \mathrm{tr}\left\{ U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{z}'} U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}} U^\dagger_{\boldsymbol{z}'} U_{\boldsymbol{y}} \right\} \right\rangle$$

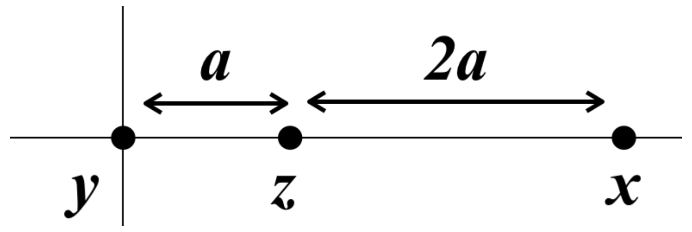
$$= \frac{1}{{N_\mathrm{c}}^2} e^{-C_\mathrm{F}\mathcal{G}_{\boldsymbol{x},\boldsymbol{y}}} + \frac{d_\mathrm{A}}{{N_\mathrm{c}}^2} e^{\frac{1}{2N_\mathrm{c}}\mathcal{G}_{\boldsymbol{x},\boldsymbol{y}}} \left(e^{-\frac{N_\mathrm{c}}{2}(\mathcal{G}_{\boldsymbol{z},\boldsymbol{x}}+\mathcal{G}_{\boldsymbol{z},\boldsymbol{y}})} + e^{-\frac{N_\mathrm{c}}{2}(\mathcal{G}_{\boldsymbol{z}',\boldsymbol{x}}+\mathcal{G}_{\boldsymbol{z}',\boldsymbol{y}})} \right)$$

$$+ C_\mathrm{F} C_\mathrm{d} \mathcal{A}_1^{(11)} - N_\mathrm{c} C_\mathrm{F} \mathcal{A}_1^{(22)} - C_\mathrm{F}\sqrt{8{N_\mathrm{c}}^3 C_\mathrm{d}}\,\mathcal{A}_1^{(13)} + 2N_\mathrm{c} C_\mathrm{F} \mathcal{A}_1^{(33)}$$

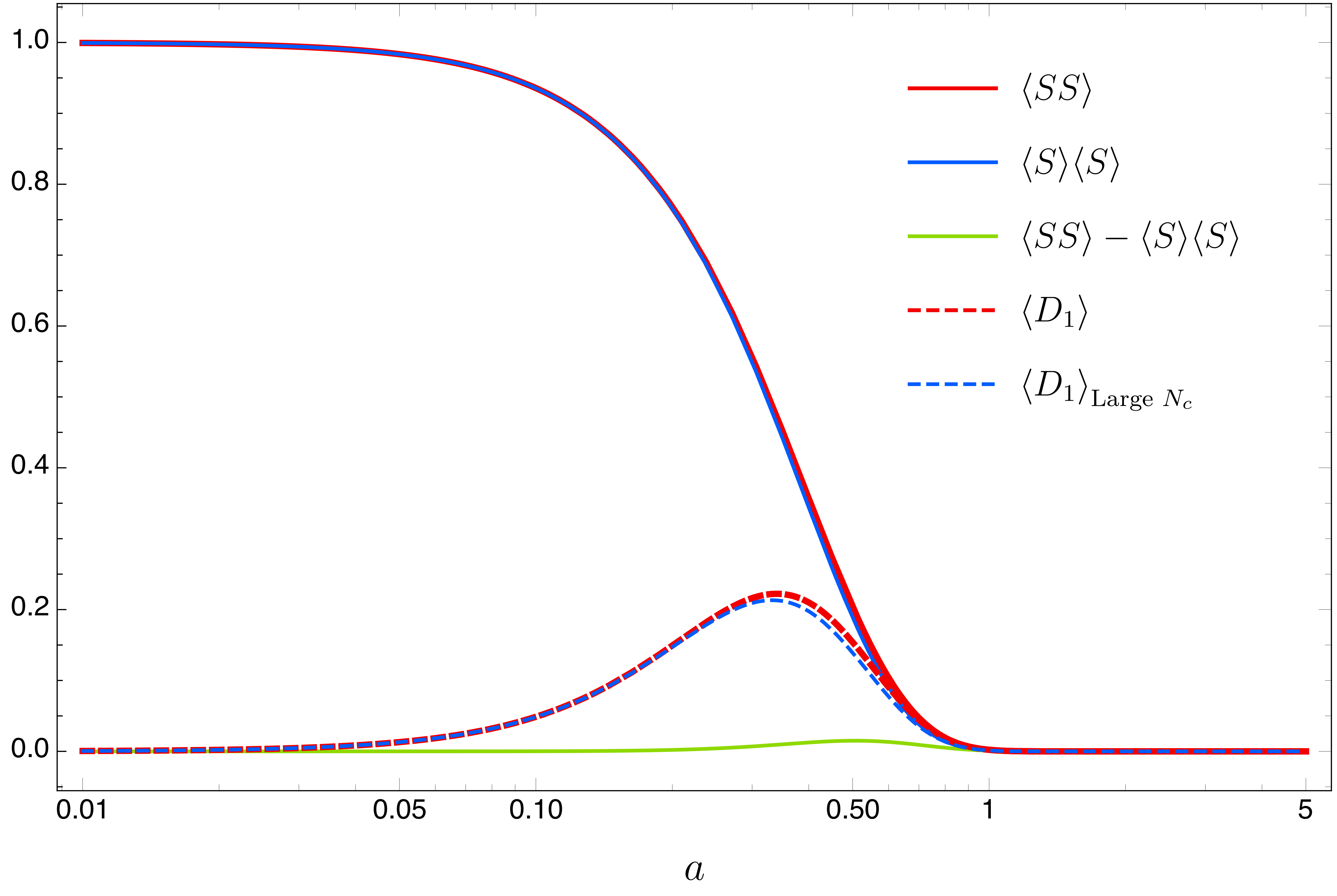
$$\lim_{\substack{\boldsymbol{u}\to\boldsymbol{z}\\ \boldsymbol{v}\to\boldsymbol{z}'}} \left\langle \frac{1}{{N_\mathrm{c}}^3}\right.$$  $$\left.\right\rangle = \left\langle \frac{1}{N_\mathrm{c}} \mathrm{tr}\left\{ U^\dagger_{\boldsymbol{z}} U_{\boldsymbol{z}'} \right\} \frac{1}{N_\mathrm{c}} \mathrm{tr}\left\{ U^\dagger_{\boldsymbol{z}'} U_{\boldsymbol{y}} \right\} \frac{1}{N_\mathrm{c}} \mathrm{tr}\left\{ U^\dagger_{\boldsymbol{x}} U_{\boldsymbol{z}} \right\} \right\rangle$$

$$= \frac{1}{{N_\mathrm{c}}^2} \lim_{\substack{\boldsymbol{u}\to\boldsymbol{z}\\ \boldsymbol{v}\to\boldsymbol{z}'}} \left\langle \frac{1}{N_\mathrm{c}}\right.$$  $$\left.\right\rangle - \frac{2C_\mathrm{F}\sqrt{C_\mathrm{d}}}{\sqrt{{N_\mathrm{c}}^3}} \mathcal{A}_1^{(12)} + \frac{2C_\mathrm{F}}{N_\mathrm{c}} \mathcal{A}_1^{(22)} + \sqrt{8} N_\mathrm{c} C_\mathrm{F} \frac{1}{{N_\mathrm{c}}^3} \mathcal{A}_1^{(23)}$$

Numerical Results: Correlators in LO-like part of NLO BK Equation



Correlators for **LO-like** BK integrand for one particular typical configuration of coordinates



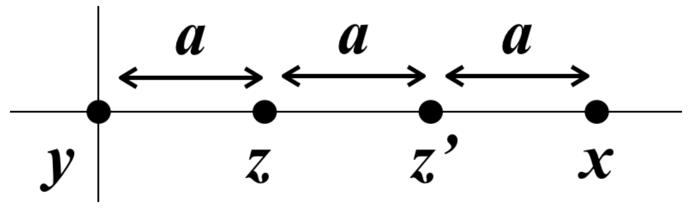
Recall BK equation:

$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle \sim \int K_1^{\text{BC}} \langle D_1 \rangle + \text{NLO-like}$$

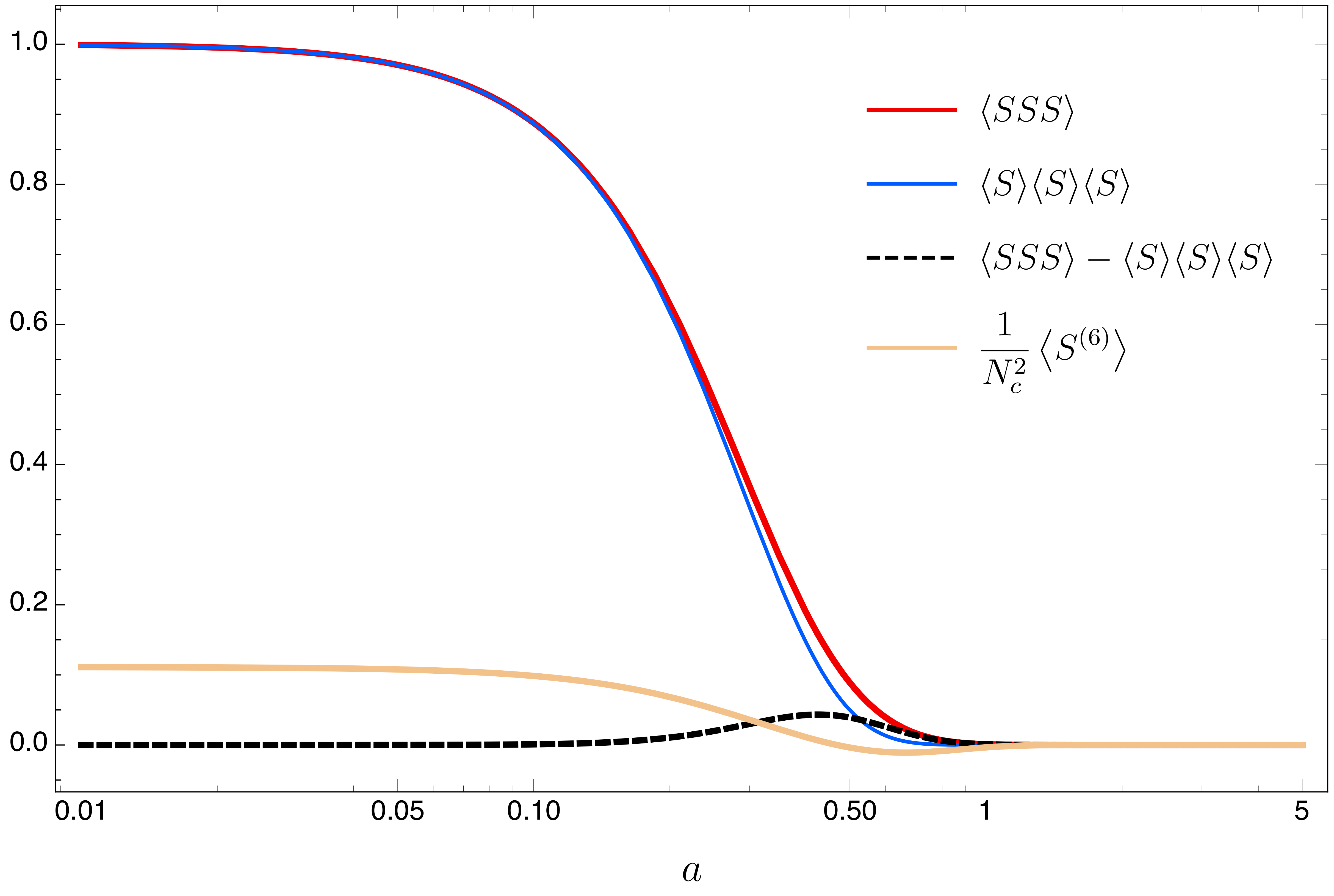
$$\langle D_1 \rangle = \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{y}} \right\rangle - \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle$$

Correlators in NLO-like part of NLO BK Equation

Correlators for **NLO-like** BK integrand for one particular typical configuration of coordinates



$\rightarrow$ very small correction from including finite-N_c piece



Recall BK equation:

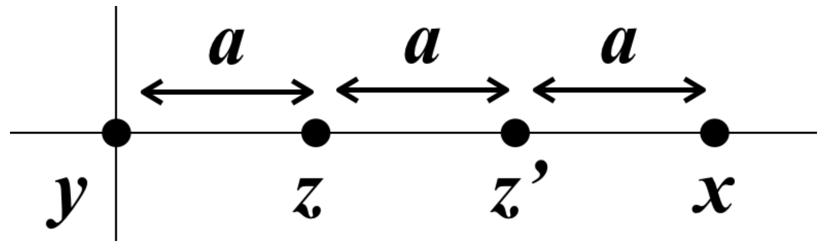
$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle \sim \int K_1^{\text{BC}} \langle D_1 \rangle + \int K_{2,1} \langle D_{2,1} \rangle + \int K_{2,2} \langle D_{2,2} \rangle$$

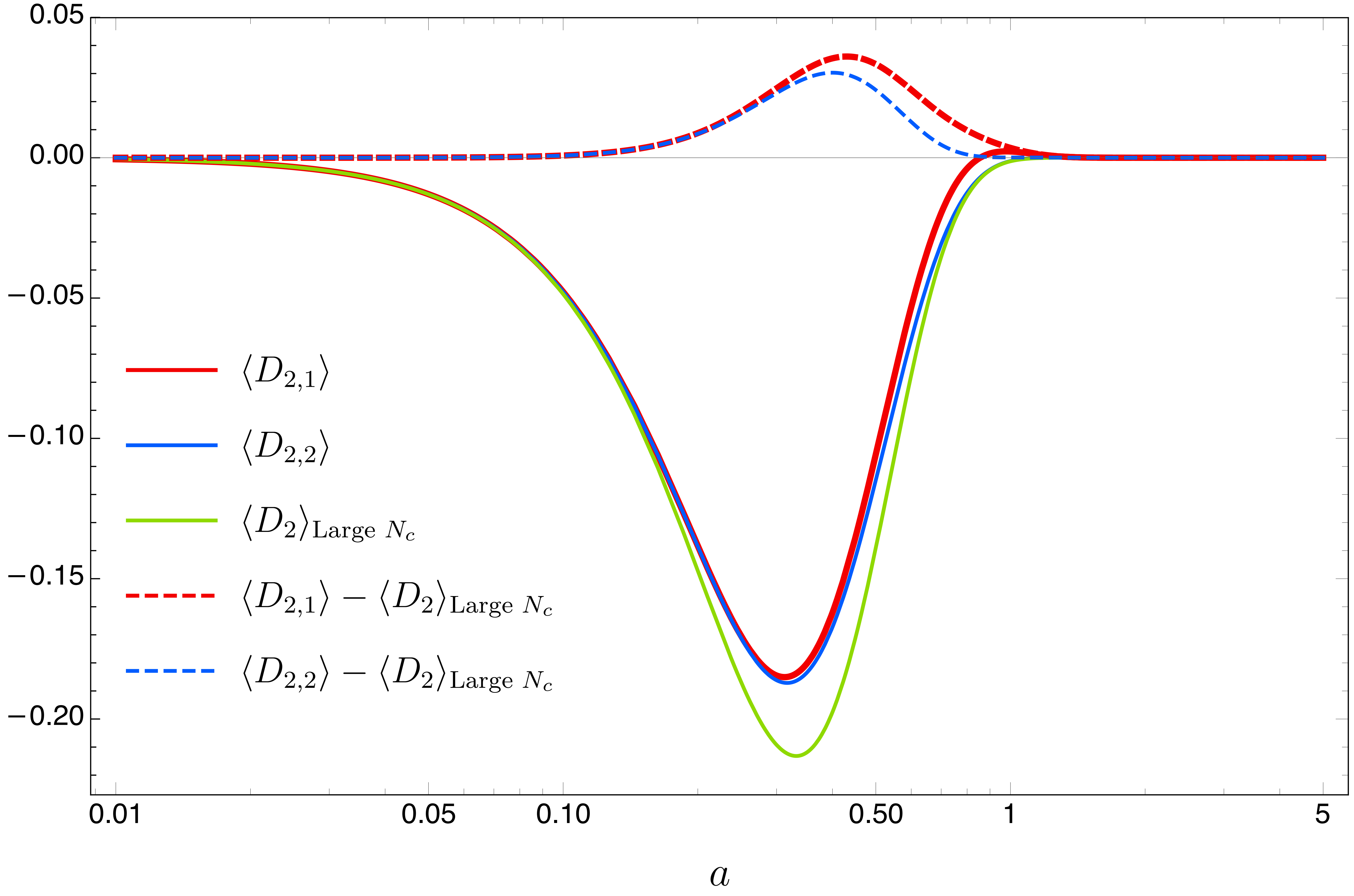
$$\langle D_{2,1} \rangle = \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle - \frac{1}{N_c{}^2} \left\langle S^{(6)}_{\boldsymbol{x},\boldsymbol{z},\boldsymbol{z}',\boldsymbol{y},\boldsymbol{z},\boldsymbol{z}'} \right\rangle - (z' \rightarrow z)$$

$$\langle D_{2,2} \rangle = \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{z}} S^{(2)}_{\boldsymbol{z},\boldsymbol{z}'} S^{(2)}_{\boldsymbol{z}',\boldsymbol{y}} \right\rangle - (z' \rightarrow z)$$

NLO Integrand

One particular typical configuration of coordinates

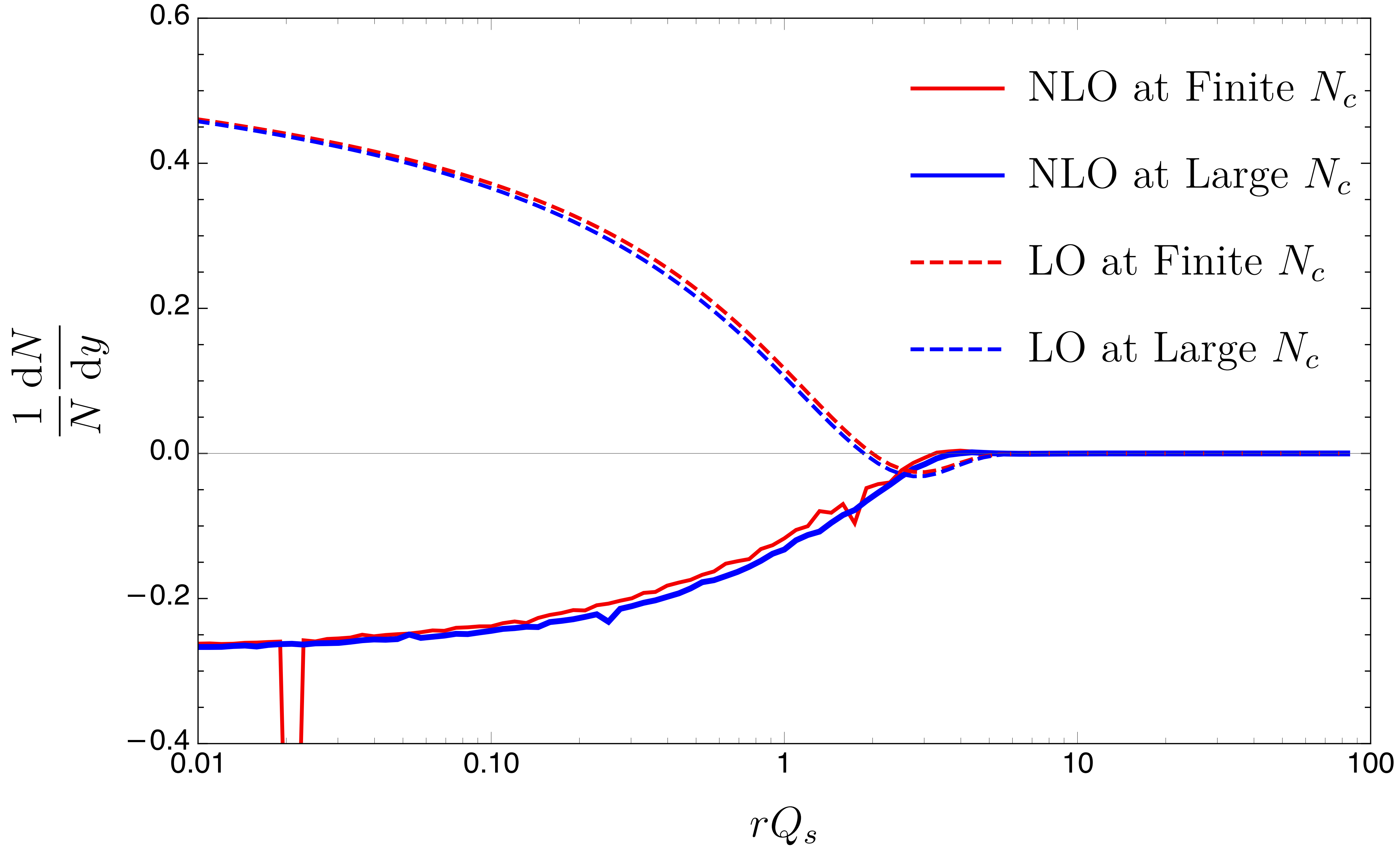




Recall BK equation:

$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle \sim \int K_1^{\text{BC}} \langle D_1 \rangle + \int K_{2,1} \langle D_{2,1} \rangle + \int K_{2,2} \langle D_{2,2} \rangle$$

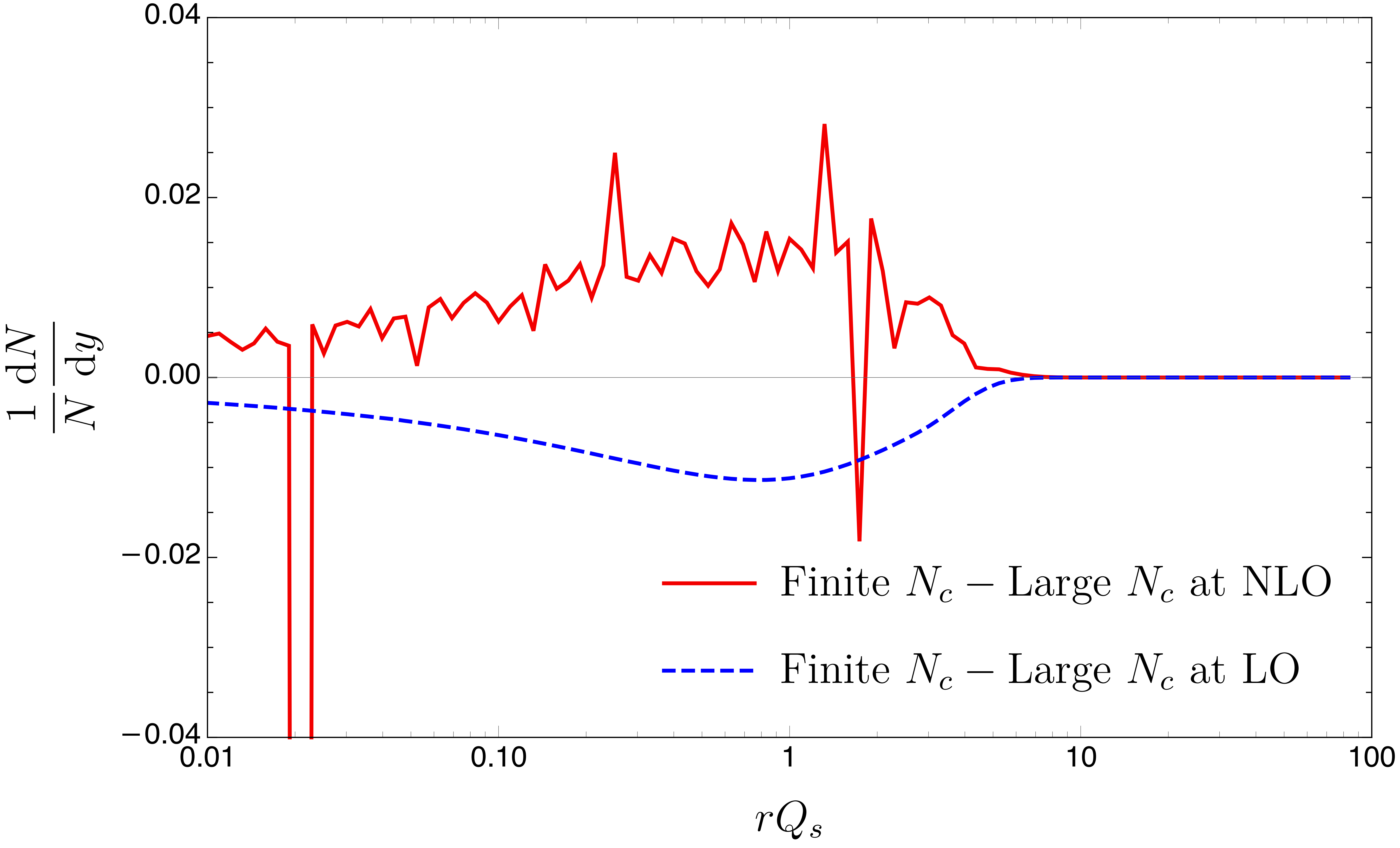
Finite- vs Large-N_c dN/dy



$$N_{\boldsymbol{x},\boldsymbol{y}} := 1 - S^{(2)}_{\boldsymbol{x},\boldsymbol{y}}$$

$$\partial_Y \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle \sim \int K_1^{\mathrm{BC}} \langle D_1 \rangle + \int \Big(K_{2,1} \langle D_{2,1} \rangle + K_{2,2} \langle D_{2,2} \rangle \Big) + \mathcal{O}(n_f)$$

Finite- vs Large-N_c Difference in dN/dy



$$N_{\boldsymbol{x},\boldsymbol{y}} = 1 - \left\langle S^{(2)}_{\boldsymbol{x},\boldsymbol{y}} \right\rangle$$

LO means only $\sim \int K_1^{\text{BC}} \langle D_1 \rangle$

NLO means only

$\int K_{2,1} \langle D_{2,1} \rangle + \int K_{2,2} \langle D_{2,2} \rangle$

Summary

- Studied high energy evolution of Wilson line correlators within CGC framework
- Infinite hierarchy of evolution equations can be truncated using Gaussian approximation to parametrise correlators
- NLO BK requires 6-point correlators

  $\rightarrow$ better basis choice leads to simplified calculation: 6×6 matrix equation block diagonalises in particular coincidence limits
- We have purely analytical parametrisations for correlators
- Used parametric equations for numerical studies of NLO BK and found very small difference between large-N_c and finite-N_c results
- Naive expectation before calculation: finite-N_c corrections at NLO are $\frac{1}{{N_c}^2} \sim \mathcal{O}(10\%)$

  $\rightarrow$ but numerics show much smaller correction, $\sim \mathcal{O}(1\%)$ (similar to LO-like case)