A novel Relaxation Time Approximation to the Relativistic Boltzmann Equation

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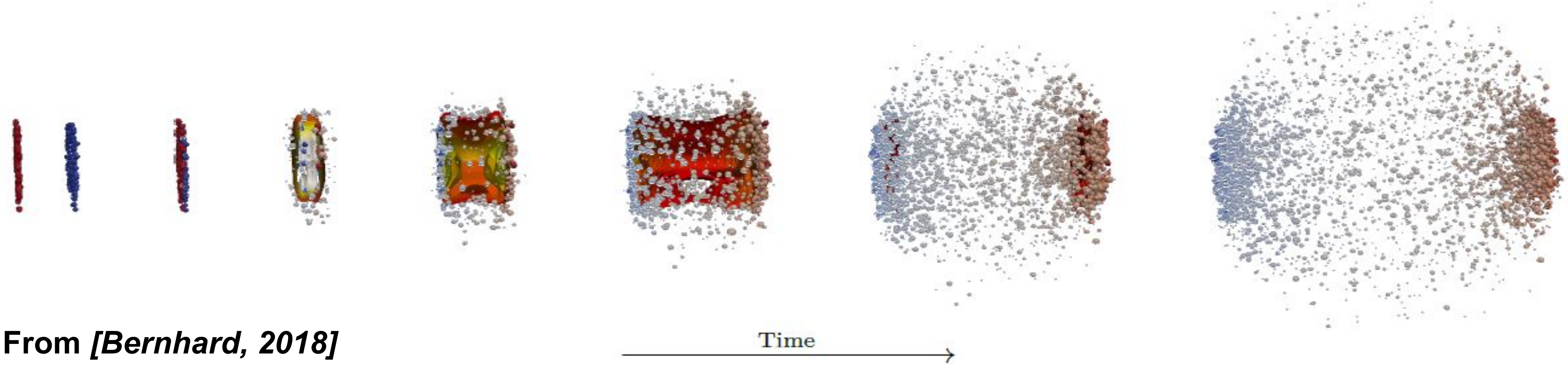
VI International Conference on the Initial Stages of High Energy Nuclear Collisions

January 10-15, 2021

Introduction

- With ultra-relativistic Heavy Ion Collisions, nuclear matter in extreme conditions can be produced and studied;

- In the last decades kinetic theory and hydrodynamics have been essential effective models to understand the evolution of this system;



From *[Bernhard, 2018]*

The relativistic Boltzmann equation

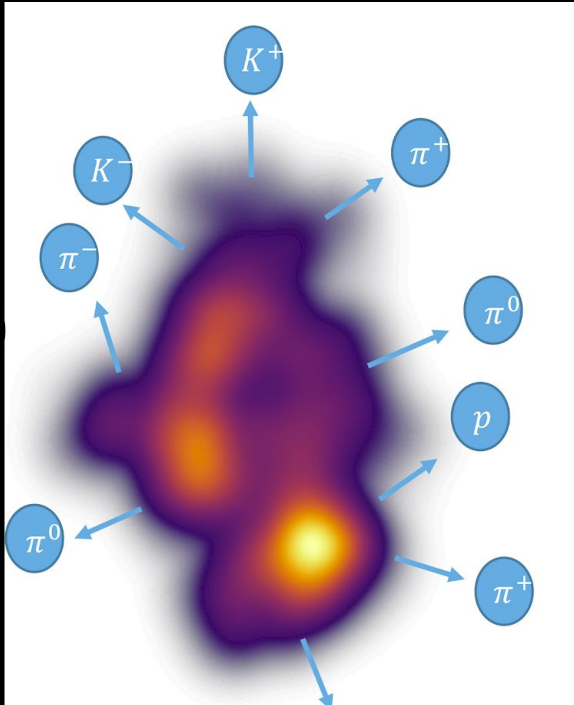
- The main equation from kinetic theory is the relativistic Boltzmann equation

$$p^\mu \partial_\mu f_p = C\,[f] = \int dQ \; dQ' \; dP' \tilde{\mathcal{W}}_{pp' \leftrightarrow qq'} (f_p f_{p'} - f_q f_{q'}),$$

- Widely used simplification: Relaxation time approximation (RTA) *[Anderson & Witting, 1974]*;

$$p^\mu \partial_\mu f_p = -\frac{u_\mu p^\mu}{\tau_R}(f_p - f_{0p}), \quad f_{0p} = e^{-\beta_0 u^0_\mu p^\mu + \alpha_0}$$

- Importance in HIC modelling: Conversion from fluid d.o.f's to particles (Cooper-Frye), thermalization of QCD matter [Kamata et al, 2020], among other applications;



From http://jetscape.org/sims/

Severe limitation of RTA

- RTA is inconsistent with the macroscopic conservation laws;

$$N^\mu = \int dP \; p^\mu f_p \qquad T^{\mu\nu} = \int dP \; p^\mu p^\nu f_p,$$

$$\partial_\mu N^\mu = -\int dP \frac{E_p}{\tau_R} \delta f \quad \partial_\mu T^{\mu\nu} = -\int dP \frac{E_p}{\tau_R} p^\nu \delta f,$$

Traditionally, it is assumed that τ_R =cte and one defines (T, u^μ, α) so that the right-hand side is zero

- This happens because an essential property of the collision term was lost:

$$C_{lin}[\mathcal{Q}_p^0] = 0.$$

$\mathcal{Q}_p^0$ **: Microscopically Conserved Quantity**

In the present case: $1, p^\mu$

Our proposal

- To recover the lost properties, we propose schematically

$$C_{lin} \propto -\mathbb{1} + \sum_n |\mathcal{Q}^0_{n,p}\rangle\langle\mathcal{Q}^0_{n,p}|,$$

Traditional RTA

Projector in the subspace of conserved quantities in an ***orthogonal basis***

- Our approximation to the rBE reads

$$p^\mu \partial_\mu f_p = -\frac{E_p}{\tau_R} f_{0p} \left\{ \phi_p - \frac{\langle \phi_p, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle \phi_p, P_1^{(0)}\rangle}{\langle P_1^{(0)}, P_1^{(0)}\rangle} P_1^{(0)} - \frac{\langle \phi_p, p^{\langle\mu\rangle}\rangle}{\frac{1}{3}\langle p^{\langle\nu\rangle}, p_{\langle\nu\rangle}\rangle} p_{\langle\mu\rangle} \right\}$$

Notation: $p^{\langle\mu\rangle} = \Delta^{\mu\nu} p_\nu \;\; \Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu \qquad \langle \psi_p, \phi_p\rangle = \int dP \frac{E_p}{\tau_R} \psi_p \phi_p f_{0p}$

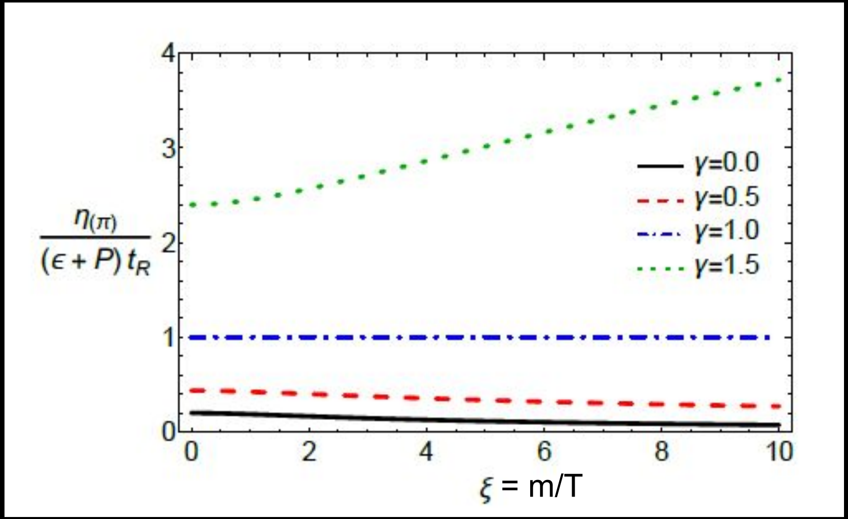
$P_1^{(0)}(E_p) = L_1^{(2-\gamma)}(\beta E_p)$ (Massless limit)

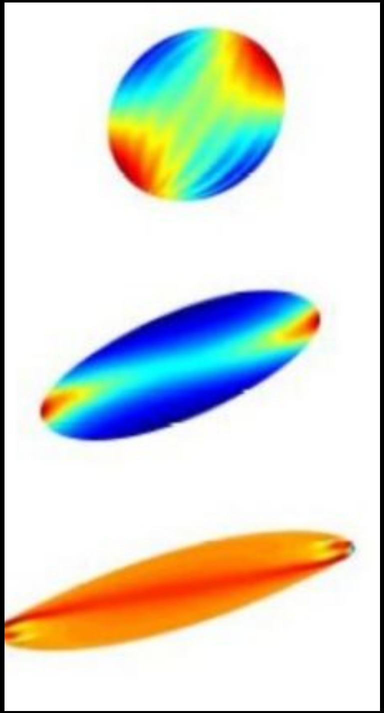
Let's see the effects of the proposal on transport coefficients of Relativistic Navier Stokes

Effects on transport coefficients

- Shear viscosity: resistance to deformation

$$\pi^{\mu\nu} \equiv 2\eta_{(\pi)}\sigma^{\mu\nu} \quad \sigma^{\mu\nu} = \Delta^{\mu\nu}{}_{\alpha\beta}\partial^{\alpha}u^{\beta} \quad \tau_R = t_R\left(\frac{E_p}{T}\right)^{\gamma},$$



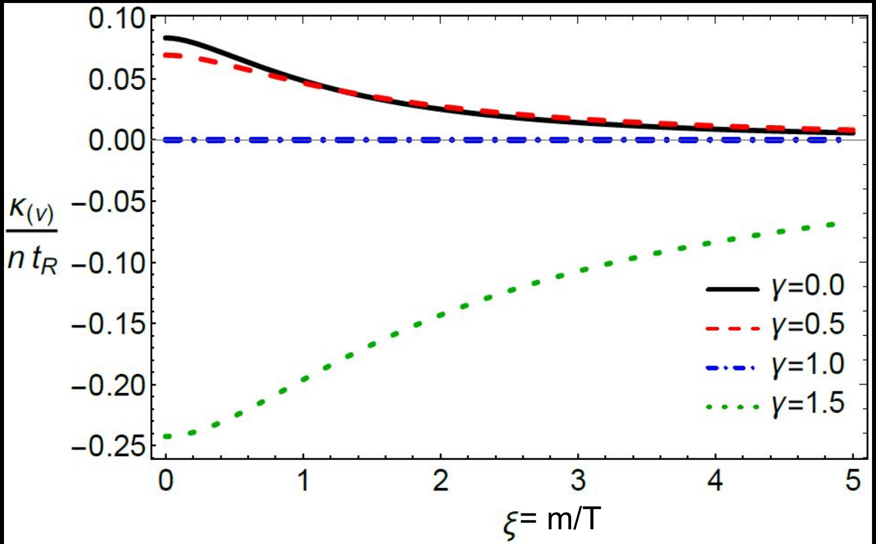


Nothing changes in comparison to traditional RTA besides the energy dependence

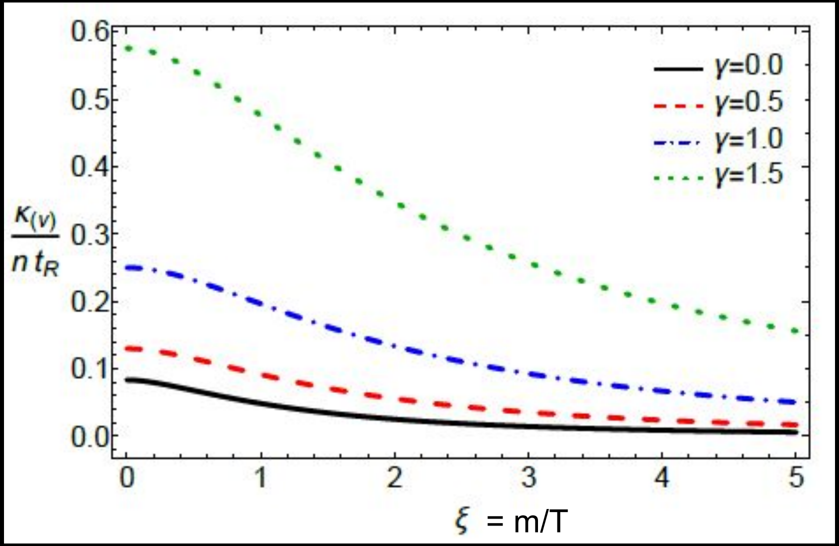
Effects on transport coefficients

- Particle diffusion viscosity

$$\nu^\mu \equiv \kappa_{(\nu)} \nabla^\mu \alpha \quad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu; \quad \tau_R = t_R \left(\frac{E_p}{T}\right)^\gamma,$$



Traditional RTA

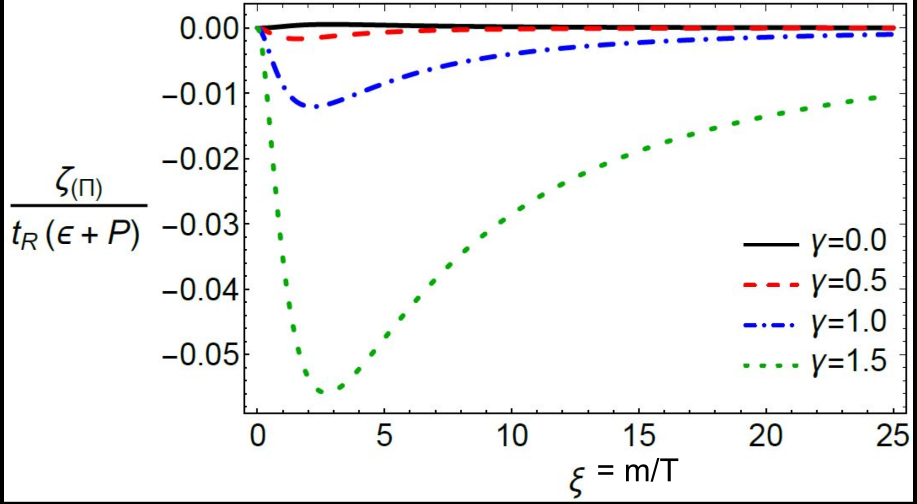


New RTA

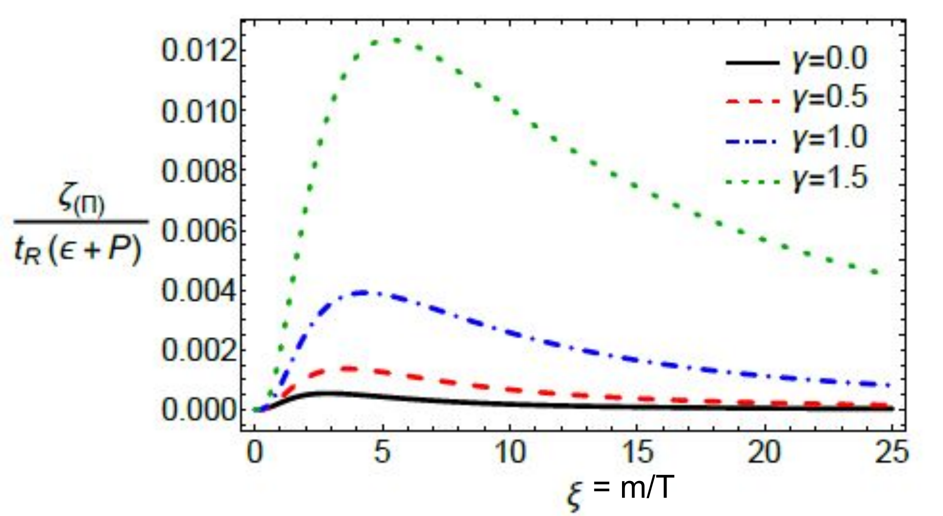
Effects on transport coefficients

- Bulk viscosity: resistance to expansion

$$\Pi \equiv -\zeta_{(\Pi)}\theta \quad \theta \equiv \partial_\mu u^\mu; \; \tau_R = t_R \left(\frac{E_p}{T}\right)^\gamma,$$



Traditional RTA



New RTA

Final remarks and perspectives

- RTA is an extremely important approximation to rBE, however it has some severe limitations that require its reformulation;

- We propose a new RTA which ensures the conservation laws, is consistent with the 2nd law of thermodynamics (see extra slides) and makes it possible to use alternative matching conditions;

- Transport coefficients are computed and they depend drastically on the energy dependence of τ_R ;

- The new RTA can be used in particlization models;

- Prospective works include: generalizations for mixtures; transient dynamics etc.

EXTRA SLIDES

Non-equilibrium corrections

- Hydrodynamics: long wavelength/ long timescale effective theory.

- Implementation: Chapman-Enskog expansion *[Chapman,1916], [Enskog, 1921]*

$$\epsilon \sim \frac{\ell_{micro}}{L_{macro}}$$

mean free path

typical macro scale of the fluid

$$\boxed{\epsilon} p^\mu \partial_\mu f_p = -\frac{E_p}{\tau_R} f_{0p} \left\{ \phi_p - \frac{\langle \phi_p, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle \phi_p, P_1^{(0)} \rangle}{\langle P_1^{(0)}, P_1^{(0)} \rangle} P_1^{(0)} - \frac{\langle \phi_p, p^{\langle \mu \rangle} \rangle}{\frac{1}{3} \langle p^{\langle \nu \rangle}, p_{\langle \nu \rangle} \rangle} p_{\langle \mu \rangle} \right\}$$

$$f_p = \sum_{i=0}^{\infty} \epsilon^i f_p^{(i)}.$$

$\mathcal{O}(\epsilon^0) : f_p^{(0)} = f_{eq,p}$ Ideal hydro

We stop here $\mathcal{O}(\epsilon^1) : f_p^{(1)}$ Relativistic Navier Stokes

- This method has its formalization in the theory of asymptotics;
- A similar method is used for WKB semiclassical expansion;

Solution to Chapman-Enskog expansion

$$\phi_p^{(1)} \equiv \frac{f_p^{(1)} - f_{eq,p}}{f_{eq,p}} = F_p^{(0)}\theta + F_p^{(1)} p^{\langle\mu\rangle}\nabla_\mu\alpha + F_p^{(2)} p^{\langle\mu} p^{\nu\rangle}\sigma_{\mu\nu},$$

$$\theta \equiv \partial_\mu u^\mu;$$

$$\nabla^\mu = \Delta^{\mu\nu}\partial_\nu;$$

$$\sigma^{\mu\nu} = \Delta^{\mu\nu}{}_{\alpha\beta}\partial^\alpha u^\beta$$

$$F_p^{(0)} = \boxed{-\frac{\tau_R}{E_p} Q_p^{(0)}} + \left\langle \tau_R E_p^{r-1} Q_p^{(0)} \right\rangle_0 \frac{\left\langle E_p^{s+1}\right\rangle_0 - \left\langle E_p^{s}\right\rangle_0 E_p}{\left\langle E_p^{r}\right\rangle_0 \left\langle E_p^{s+1}\right\rangle_0 - \left\langle E_p^{s}\right\rangle_0 \left\langle E_p^{r+1}\right\rangle_0}$$

$$+ \left\langle \tau_R E_p^{s-1} Q_p^{(0)} \right\rangle_0 \frac{\left\langle E_p^{r}\right\rangle_0 E_p - \left\langle E_p^{r+1}\right\rangle_0}{\left\langle E_p^{r}\right\rangle_0 \left\langle E_p^{s+1}\right\rangle_0 - \left\langle E_p^{s}\right\rangle_0 \left\langle E_p^{r+1}\right\rangle_0}$$

$$Q_p^{(0)} = E_p\Gamma_{(\alpha)} - E_p^2\Gamma_{(\beta)} - \frac{\beta}{3}\Delta^{\lambda\sigma} p_\lambda p_\sigma$$

$$F_p^{(1)} = \boxed{-\frac{\tau_R}{E_p} Q_p^{(1)}} + \frac{\langle\left(\Delta_{\lambda\sigma} p^\lambda p^\sigma\right) \tau_R E_p^{z-1} Q_p^{(1)}\rangle_0}{\langle E_p^z \left(\Delta_{\lambda\sigma} p^\lambda p^\sigma\right)\rangle_0}$$

$$Q_p^{(1)} = 1 - \frac{nE_p}{\varepsilon + P}$$

$$F_p^{(2)} = \boxed{\beta\frac{\tau_R}{E_p}}$$

$$Q_p^{(2)} = -\beta$$

Traditional RTA

$$\Gamma_{(\alpha)}(m,\beta) = -1 + \frac{P\varepsilon}{n\langle E_p^3\rangle_0 - \varepsilon^2}$$

$$\langle\cdots\rangle_0 = \int dP(\cdots) f_{0p}.$$

$$\Gamma_{(\beta)}(m,\beta) = \frac{nP}{n\langle E_p^3\rangle_0 - \varepsilon^2}.$$

The irreducible basis

- The basis $\{P_n^{(\ell)}(E_p)p^{\langle\mu_1}...p^{\mu_\ell\rangle}\}_{\ell,n}$ is irreducible with respect to the little group;

- $p^{\langle\mu_1}...p^{\mu_\ell\rangle} = \Delta^{\mu_1\cdots\mu_\ell}{}_{\nu_1\ldots\nu_\ell}p^{\nu_1}...p^{\nu_\ell}$ $\Delta^{\mu_1\cdots\mu_\ell}{}_{\nu_1\ldots\nu_\ell}$*: Symmetric and traceless projector*

$$\Delta^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$$

$$\Delta^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}$$

$$\int dP \frac{E_p}{\tau_{R,p}}\left(\Delta_{\mu\nu}p^\mu p^\nu\right)^\ell P_n^{(\ell)}P_m^{(\ell)}f_{0p} = A_n^{(\ell)}\delta_{mn}.$$

- Basis of microscopically conserved quantities

$$\{\mathcal{Q}^0_{n,p}\} = \{1, P_1^{(0)}(E_p), p^{\langle\mu\rangle}\} \subset \{P_n^{(\ell)}(E_p)p^{\langle\mu_1}...p^{\mu_\ell\rangle}\}_{\ell,n}$$

Matching and frame conditions

- In hydro the QCD EM-tensor/currents are effectively represented in terms of hydro fields (T, u^μ, α) which must be defined out of equilibrium;

- However, out of equilibrium, many different definitions of hydro fields can give the same EM-tensor, which is the physical object *[Kovtun, 2012]*;

- *EX.: Landau matching and frame conditions [Landau, 1959]:*

$$\int dP\ E_p f_p = n_0, \int dP\ E_p^2 f_p = \varepsilon_0,\ T^\mu_{\ \nu} u^\nu = \varepsilon u^\mu$$

- Moreover, recent studies on uniqueness and causality of first order hydro, e.g. *[Bemfica et al, 2018; Bemfica et al 2019]* lead to the use of alternative matching conditions

$$N^\mu = (n_0 + \underline{\delta n})u^\mu + \underline{\nu^\mu}$$
$$T^{\mu\nu} = (\varepsilon_0 + \underline{\delta\varepsilon})u^\mu u^\nu - (P_0 + \underline{\Pi})\Delta^{\mu\nu} + \underline{h^\mu} u^\nu + \underline{h^\nu} u^\mu + \underline{\pi^{\mu\nu}},$$

Non-eql. corrections

General irreducible decomposition

Entropy production

- We have shown that

$$\partial_\mu S^\mu = \beta\zeta_s\theta^2 - \kappa_s\nabla^\mu\alpha\nabla_\mu\alpha + 2\eta_s\beta\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$$

$$\beta\zeta_s = \beta\zeta_{(\Pi)} + \Gamma_{(\beta)}\zeta_{(\delta\varepsilon)} + \Gamma_{(\alpha)}\zeta_{(\delta n)} \qquad \zeta_s = \left\langle \frac{\tau_R}{E_p}[Q_p^{(0)}]^2 \right\rangle_0 \qquad \langle \cdots \rangle_0 = \int dP(\cdots) f_{0p}.$$

$$\kappa_s = \kappa_{(\nu)} + \frac{n}{\varepsilon + P}\kappa_{(h)}, \qquad \kappa_s = -\frac{1}{3}\left\langle (\Delta^{\mu\nu}p_\mu p_\nu)\frac{\tau_R}{E_p}[Q_p^{(1)}]^2 \right\rangle_0$$

- It does not matter if one coefficient is negative, the sum will always be non-negative
- For usual (Landau) matching conditions, $\zeta(\delta n) = \zeta(\delta\varepsilon) = \kappa(h) = 0$

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[slide 2 picture] https://www.lavision.de/en/products/fluidmaster/mixing-fluids/index.phpd

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