Hot spots and gluon field fluctuations as causes of eccentricity in small systems

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The energy density 1- and 2-point functions

In the Color Glass Condensate formalism the energy density 1- and 2-point functions, as functions of transverse coordinates, immediately after a heavy ion collision event at proper time $\tau = 0$, can be expressed as

$$\langle \varepsilon(\mathbf{x}) \rangle = (-ig)^2 (\delta^{ij}\delta^{kl} + \varepsilon^{ij}\varepsilon^{kl}) \frac{1}{2} i f^{abc} i f^{a'b'c} \times \langle \alpha_{\mathbf{x}}^{i,a} \alpha_{\mathbf{x}}^{k,a'} \rangle \langle \beta_{\mathbf{x}}^{j,b} \beta_{\mathbf{x}}^{l,b'} \rangle$$

and

$$\langle \varepsilon(\mathbf{x}) \varepsilon(\mathbf{y}) \rangle = (-ig)^4 (\delta^{ij}\delta^{kl} + \varepsilon^{ij}\varepsilon^{kl})(\delta^{i'j'}\delta^{k'l'} + \varepsilon^{i'j'}\varepsilon^{k'l'}) \times \frac{1}{4} i f^{abe} i f^{cde} i f^{a'b'e'} i f^{c'd'e'} \times \left\langle \alpha_{\mathbf{x}}^{i,a} \alpha_{\mathbf{x}}^{k,c} \alpha_{\mathbf{y}}^{i',a'} \alpha_{\mathbf{y}}^{k',c'} \right\rangle \left\langle \beta_{\mathbf{x}}^{j,b} \beta_{\mathbf{x}}^{l,d} \beta_{\mathbf{y}}^{j',b'} \beta_{\mathbf{y}}^{l',d'} \right\rangle.$$

Here $\alpha_{\mathbf{x}}^{i} = \alpha_{\mathbf{x}}^{i,a} t^a$ with $\alpha_{\mathbf{x}}^{i,a} = \frac{2i}{g} \mathrm{Tr}[t^a U_{\mathbf{x}} \partial^i U_{\mathbf{x}}^{\dagger}]$.

T. Lappi and L. McLerran, Some features of the glasma, Nucl. Phys. A772 (2006) 200

The hot spot average

We have to average over the locations of the hot spots of the proton

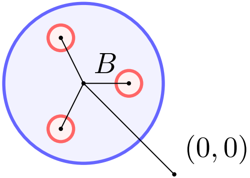
$$\langle\langle\mathcal{O}\rangle\rangle = \left(\frac{2\pi R^2}{N_q}\right) \int \prod_{i=1}^{N_q} \left[d^2\mathbf{b}_i T(\mathbf{b}_i - \mathbf{B})\right]$$

$$\times\, \delta\Big(\frac{1}{N_q}\sum_{i=1}^{N_q}\mathbf{b}_i - \mathbf{B}\Big)\langle\mathcal{O}\rangle_{CGC},$$

where

$$T(\mathbf{b}) = \frac{1}{2\pi R^2}\exp\left[-\frac{\mathbf{b}^2}{2R^2}\right]$$

describes the distributions of the hot spots.



S. Schlichting and B. Schenke, The shape of the proton at high energies, Phys. Lett. B739 (2014) 313.

H. Mäntysaari and B. Schenke, Evidence of strong proton shape fluctuations from incoherent diffraction, Phys. Rev. Lett. 117 (2016) 052301.

Separation of fluctuations

We separate the nucleus and proton parts into their disconnected and connected parts

$$\left\langle \alpha_{\mathbf{x}}^{i,a}\alpha_{\mathbf{x}}^{k,c}\alpha_{\mathbf{y}}^{i',a'}\alpha_{\mathbf{y}}^{k',c'}\right\rangle = \underbrace{\left\langle \alpha_{\mathbf{x}}^{i,a}\alpha_{\mathbf{x}}^{k,c}\right\rangle\left\langle \alpha_{\mathbf{y}}^{i',a'}\alpha_{\mathbf{y}}^{k',c'}\right\rangle}_{\text{Disconnected}}$$
$$+\underbrace{\left(\left\langle \alpha_{\mathbf{x}}^{i,a}\alpha_{\mathbf{x}}^{k,c}\alpha_{\mathbf{y}}^{i',a'}\alpha_{\mathbf{y}}^{k',c'}\right\rangle - \left\langle \alpha_{\mathbf{x}}^{i,a}\alpha_{\mathbf{x}}^{k,c}\right\rangle\left\langle \alpha_{\mathbf{y}}^{i',a'}\alpha_{\mathbf{y}}^{k',c'}\right\rangle\right)}_{\text{Connected}}.$$

This way we can separate the two point function into DC-DC, DC-C, C-DC and C-C contributions. We interpret the connected contributions to be the part describing the fluctuations. The fully connected part is expected to be small.

Eccentricities

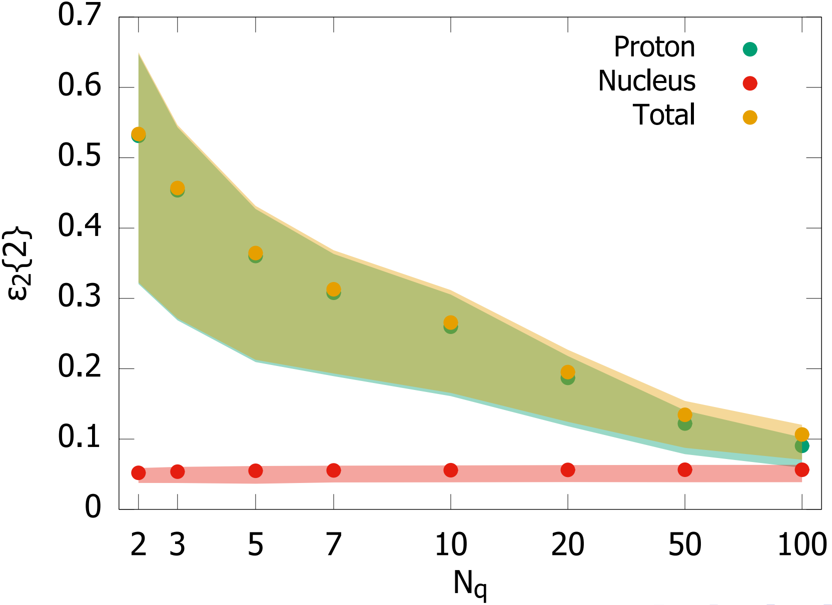
Usually one would compute mean square eccentricities as (Monte Carlo)

$$\varepsilon_n^{'}\{2\}^2 \equiv \langle \varepsilon_n \bar{\varepsilon}_n \rangle$$
$$= \left\langle \frac{\int d^2\mathbf{x} d^2\mathbf{y} |\mathbf{x}-\mathbf{B}|^n |\mathbf{y}-\mathbf{B}|^n \exp\left(in\theta_{\mathbf{x}-\mathbf{B}} - in\theta_{\mathbf{y}-\mathbf{B}}\right) \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})}{\int d^2\mathbf{x} d^2\mathbf{y} |\mathbf{x}-\mathbf{B}|^n |\mathbf{y}-\mathbf{B}|^n \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})} \right\rangle.$$

This is difficult to evaluate analytically. Instead, we compute

$$\varepsilon_n\{2\}^2$$
$$= \frac{\int d^2\mathbf{x} d^2\mathbf{y} |\mathbf{x}-\mathbf{B}|^n |\mathbf{y}-\mathbf{B}|^n \exp\left(in\theta_{\mathbf{x}-\mathbf{B}} - in\theta_{\mathbf{y}-\mathbf{B}}\right) \langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y}) \rangle}{\int d^2\mathbf{x} d^2\mathbf{y} |\mathbf{x}-\mathbf{B}|^n |\mathbf{y}-\mathbf{B}|^n \langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y}) \rangle}.$$

Eccentricity with $m \pm 50\%$ error bars



Conclusions and outlook

- We built a model for computing eccentricities in pA collisions.
  - Averages done analytically.
- Hot spot fluctuations dominate eccentricities.
- Good: Small dependence on UV cutoff (C_0).
- Bad: Large dependence on IR regulator (m).

What next?

- Constrain parameters by comparing to experimental results in DIS.
- Relate eccentricities to anisotrophic flow coefficients.
- Study transverse position-momentum correlations of gluons in the proton.

Backups

Color glass condensate

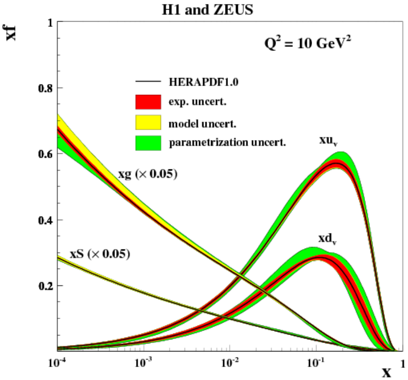


Fig. from J. High Energ. Phys. 2010, 109 (2010).

- Small-x part of a high energy proton/nucleus dominated by gluons.
  $\Longrightarrow$ Treat the small-x gluons as classical fields.
- Treat large-x partons as sources for small-x gluons.

Gluon fields right after the collision

Before the collision the gluon fields of the nuclei can be expressed as

$$\alpha^i_{\mathbf{x}} = \frac{i}{g} U_{\mathbf{x}} \partial^i U^\dagger_{\mathbf{x}}, \qquad \beta^i_{\mathbf{x}} = \frac{i}{g} V_{\mathbf{x}} \partial^i V^\dagger_{\mathbf{x}},$$

where U and V are Wilson lines of the two nuclei.
By requiring the fields to
be continuous in when moving to the future light cone, we get, for proper time $\tau_+ = 0$

$$A^i_{\mathbf{x}} = \alpha^i_{\mathbf{x}} + \beta^i_{\mathbf{x}}, \qquad A^\eta = \frac{ig}{2}[\alpha^i_{\mathbf{x}}, \beta^i_{\mathbf{x}}].$$

$$\tau = \sqrt{2x^+x^-}, \eta = \frac{1}{2}\ln(\frac{x^+}{x^-}),$$
$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3)$$

.

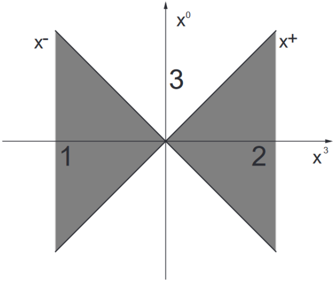
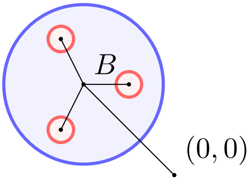


Fig. from Phys. Rev. D52(1995) 3809

The CGC average with Gaussian weighting

For the proton

$$\left\langle \rho^a(\mathbf{x})\rho^b(\mathbf{y})\right\rangle = \sum_{i=1}^{N_q} \mu^2(\mathbf{x}-\mathbf{b}_i)\delta(\mathbf{x}-\mathbf{y})\delta^{ab},$$



where

$$\mu^2(\mathbf{x}) = \frac{\mu_0^2}{2\pi r^2}\exp[-\frac{\mathbf{x}^2}{2r^2}]$$

describes the transverse distribution of color charge in a hot spot.
For the nucleus we have

$$\left\langle \rho^a(\mathbf{x})\rho^b(\mathbf{y})\right\rangle = \mu_A^2\delta(\mathbf{x}-\mathbf{y})\delta^{ab}.$$

Energy density

We decompose the nucleus part into Wilson lines

$$\langle \alpha_{\mathbf{x}}^{i,a} \alpha_{\mathbf{x}}^{k,a'} \rangle = \lim_{\mathbf{x}_i \to \mathbf{x}} \Big\{ -\frac{4}{g^2} \partial_{\mathbf{x}_2}^{i} \partial_{\mathbf{x}_4}^{k} t_{bc}^{a} t_{de}^{a'} \langle [U_{\mathbf{x}_1} U_{\mathbf{x}_2}^{\dagger}]_{cb} [U_{\mathbf{x}_3} U_{\mathbf{x}_4}^{\dagger}]_{ed} \rangle \Big\}.$$

This is doable analytically when assuming Gaussian weight in the CGC average. After some work, we find that this is equal to

$$= \lim_{\mathbf{x}_i \to \mathbf{x}} \left\{ -\frac{4}{g^2} \partial_{\mathbf{x}_2}^{i} \partial_{\mathbf{x}_4}^{k} \begin{bmatrix} \delta^{bc} \delta^{de} \\ \delta^{be} \delta^{cd} \end{bmatrix}^{\mathrm{T}} e^{M_{2\times 2}} \begin{bmatrix} t_{bc}^{a} t_{de}^{a'} \\ 0 \end{bmatrix} \right\}.$$

We use the following identity to make things easier

$$\partial_x e^{M(x)} = \int_0^1 e^{tM} [\partial_x M] e^{(1-t)M} dt.$$

For the Wilson line correlators we use the algorithm presented in Nucl. Phys. A743 (2004) 57.

Energy density

Finally we get

$$\langle \alpha_{\mathbf{x}}^{i,a} \alpha_{\mathbf{x}}^{k,a'} \rangle = \frac{Q_s^2 \delta^{aa'} \delta^{ik}}{2g^2 C_F},$$

where we used the GBW model as input.
Now the energy density takes the form

$$\langle \varepsilon(\mathbf{x}) \rangle = \frac{N Q_s^2}{2C_F} \langle \beta_{\mathbf{x}}^{j,b} \beta_{\mathbf{x}}^{j,b} \rangle$$

We consider the proton to be dilute and we expand the proton Wilson lines

$$V(\mathbf{x}) = \mathcal{P}_+ \exp[-ig \int_{-\infty}^{\infty} dz^+ d^2\mathbf{z} G(\mathbf{x}-\mathbf{z}) \rho_a(z^+, \mathbf{z}) t^a]$$

to the 1st order in source densities.

Energy density

After the averaging procedure we get

$$\langle \varepsilon(\mathbf{x}) \rangle = \frac{N(N^2-1)N_q Q_s^2}{2C_F} \times \int d^2\mathbf{z} G_{\mathbf{x}}^j(\mathbf{x}-\mathbf{z}) G_{\mathbf{x}}^j(\mathbf{x}-\mathbf{z}) F_1(\mathbf{z},\mathbf{B}),$$

where the Green's functions derivatives

$$G_{\mathbf{x}}^j(\mathbf{x}-\mathbf{z}) \equiv \partial_{\mathbf{x}}^j G_{\mathbf{x}}(\mathbf{x}-\mathbf{z}) = -\frac{1}{2\pi} m|\mathbf{x}-\mathbf{z}| K_1(m|\mathbf{x}-\mathbf{z}|) \frac{(\mathbf{x}-\mathbf{z})^j}{(\mathbf{x}-\mathbf{z})^2} \Theta(|\mathbf{x}-\mathbf{z}| - C_0),$$

where mass regulates the IR and C_0 regulates the UV divergences.

The hot spot average contribution to the energy density

$$F_1(\mathbf{z}, \mathbf{B}) \equiv \langle \mu^2(\mathbf{z} - \mathbf{b}_i) \rangle_{\text{Hotspot}}$$
$$= \left(\frac{\mu_0^2}{2\pi r^2} \right) \left(\frac{1}{1 + \left(\frac{N_q - 1}{N_q} \right) \frac{R^2}{r^2}} \right) \exp \left\{ -\frac{1}{2} \frac{(\mathbf{z} - \mathbf{B})^2}{r^2 + \left(\frac{N_q - 1}{N_q} \right) R^2} \right\}$$

Energy density two point function: nucleus part

We need to compute

$$\langle \alpha_{\mathbf{x}}^{i,a}\alpha_{\mathbf{x}}^{k,c}\alpha_{\mathbf{y}}^{i',a'}\alpha_{\mathbf{y}}^{k',c'}\rangle,$$

which is equal to

$$= \lim_{\substack{x_i\to x\\ y_i\to y}} \Bigg\{ \frac{16}{g^4}\partial^i_{\mathbf{x}_2}\partial^k_{\mathbf{x}_4}\partial^{i'}_{\mathbf{y}_2}\partial^{k'}_{\mathbf{y}_4}$$

$$\times \begin{bmatrix} \delta^{a_1a_2}\delta^{a_3a_4}\delta^{a_5a_6}\delta^{a_7a_8} \\ \vdots \end{bmatrix}^{\mathrm{T}} e^{M_{24\times 24}} \begin{bmatrix} t^a_{a_2a_1}t^c_{a_4a_3}t^{a'}_{a_6a_5}t^{c'}_{a_8a_7} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Bigg\}$$

This 4-α correlator has been computed in a different way before in J. High Energ. Phys. 2019, 73 (2019).

Energy density two point function: proton part

For the proton side, we get

$$
\begin{aligned}
&\langle\langle \beta^{j,b}_{\mathbf{x}} \beta^{l,d}_{\mathbf{x}} \beta^{j',b'}_{\mathbf{y}} \beta^{l',d'}_{\mathbf{y}} \rangle\rangle \\
&= \int d^2\mathbf{a} d^2\mathbf{b} \Big[N_q F_2(\mathbf{a},\mathbf{b},\mathbf{B}) + N_q(N_q-1) F_3(\mathbf{a},\mathbf{b},\mathbf{B}) \Big] \\
&\quad \times \Big[G^j_{\mathbf{x}}(\mathbf{x}-\mathbf{a}) G^l_{\mathbf{x}}(\mathbf{x}-\mathbf{a}) G^{j'}_{\mathbf{y}}(\mathbf{y}-\mathbf{b}) G^{l'}_{\mathbf{y}}(\mathbf{y}-\mathbf{b}) \delta^{bd} \delta^{b'd'} \\
&\quad + G^j_{\mathbf{x}}(\mathbf{x}-\mathbf{a}) G^l_{\mathbf{x}}(\mathbf{x}-\mathbf{b}) G^{j'}_{\mathbf{y}}(\mathbf{y}-\mathbf{a}) G^{l'}_{\mathbf{y}}(\mathbf{y}-\mathbf{b}) \delta^{bb'} \delta^{dd'} \\
&\quad + G^j_{\mathbf{x}}(\mathbf{x}-\mathbf{a}) G^l_{\mathbf{x}}(\mathbf{x}-\mathbf{b}) G^{j'}_{\mathbf{y}}(\mathbf{y}-\mathbf{b}) G^{l'}_{\mathbf{y}}(\mathbf{y}-\mathbf{a}) \delta^{bd'} \delta^{db'} \Big].
\end{aligned}
$$

Note the factor of N_q for when we take the two gluons from the same hot spot and the factor of $N_q(N_q-1)$ for when the gluons are taken from different hot spots.

$$F_2(\mathbf{a},\mathbf{b},\mathbf{B}) \equiv \langle \mu^2(\mathbf{a}-\mathbf{b}_i)\mu^2(\mathbf{b}-\mathbf{b}_i)\rangle_{\text{Hotspot}}$$

$$= \left(\frac{\mu_0^2}{2\pi r^2}\right)^2 \left(\frac{1}{1+2\left(\frac{N_q-1}{N_q}\right)\frac{R^2}{r^2}}\right)$$

$$\times \exp\left\{-\frac{(\mathbf{a}+\mathbf{b}-2\mathbf{B})^2}{4r^2\left(1+2\left(\frac{N_q-1}{N_q}\right)\frac{R^2}{r^2}\right)} - \frac{(\mathbf{a}-\mathbf{b})^2}{4r^2}\right\}$$

$$F_3(\mathbf{a},\mathbf{b},\mathbf{B}) \equiv \langle \mu^2(\mathbf{a}-\mathbf{b}_i)\mu^2(\mathbf{b}-\mathbf{b}_j)\rangle_{\text{Hotspot}}$$

$$= \left(\frac{\mu_0^4}{(2\pi)^2(R^2+r^2)}\right)\left(\frac{1}{r^2+\left(\frac{N_q-2}{N_q}\right)R^2}\right)$$

$$\times \exp\left\{-\frac{(\mathbf{a}+\mathbf{b}-2\mathbf{B})^2}{4\left(r^2+\left(\frac{N_q-2}{N_q}\right)R^2\right)} - \frac{(\mathbf{a}-\mathbf{b})^2}{4(R^2+r^2)}\right\}$$

Fully disconnected and proton fluctuation parts

Disconnected part:

$$\langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle_{\mathrm{DC,DC}} = \langle \varepsilon(\mathbf{x})\rangle\langle \varepsilon(\mathbf{y})\rangle.$$

Proton fluctuations:

$$\begin{aligned}
&\langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle_{\mathrm{DC,C}} \\
&= \frac{Q_s^4 N^2}{4C_F^2}\int d^2\mathbf{a}d^2\mathbf{b}\Big[N_q F_2(\mathbf{a},\mathbf{b},\mathbf{B}) + N_q(N_q-1)F_3(\mathbf{a},\mathbf{b},\mathbf{B})\Big] \\
&\times \Big\{(N^2-1)^2 \mathbf{G}^i_{\mathbf{x}}(\mathbf{x}-\mathbf{a})\mathbf{G}^i_{\mathbf{x}}(\mathbf{x}-\mathbf{a})\mathbf{G}^j_{\mathbf{y}}(\mathbf{y}-\mathbf{b})\mathbf{G}^j_{\mathbf{y}}(\mathbf{y}-\mathbf{b}) \\
&+ 2(N^2-1)\mathbf{G}^i_{\mathbf{x}}(\mathbf{x}-\mathbf{a})\mathbf{G}^i_{\mathbf{x}}(\mathbf{x}-\mathbf{b})\mathbf{G}^j_{\mathbf{y}}(\mathbf{y}-\mathbf{a})\mathbf{G}^j_{\mathbf{y}}(\mathbf{y}-\mathbf{b})\Big\} \\
&- \langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle_{\mathrm{DC,DC}}
\end{aligned}$$

Nucleus fluctuations part

Nucleus fluctuations:

$$
\begin{aligned}
&\langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle_{\mathrm{C,DC}} \\
&= \frac{N^2 N_q^2}{(\mathbf{x}-\mathbf{y})^4}\left\{ \frac{8(N^2-1)}{N^2}\exp[-\frac{N^2 Q_s^2(\mathbf{x}-\mathbf{y})^2}{2(N^2-1)}] \right. \\
&\left. + N^2 Q_s^4(\mathbf{x}-\mathbf{y})^4 + 4Q_s^2(\mathbf{x}-\mathbf{y})^2 + \frac{8(1-N^2)}{N^2} \right\} \\
&\times \int d^2\mathbf{a} d^2\mathbf{b} \mathbf{G}^i_{\mathbf{x}}(\mathbf{x}-\mathbf{a})\mathbf{G}^i_{\mathbf{x}}(\mathbf{x}-\mathbf{a})\mathbf{G}^j_{\mathbf{x}}(\mathbf{y}-\mathbf{b})\mathbf{G}^j_{\mathbf{x}}(\mathbf{y}-\mathbf{b}) \\
&\times F_1(\mathbf{a},\mathbf{B})F_1(\mathbf{b},\mathbf{B}) - \langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle_{\mathrm{DC,DC}}
\end{aligned}
$$

Pointlike energy density

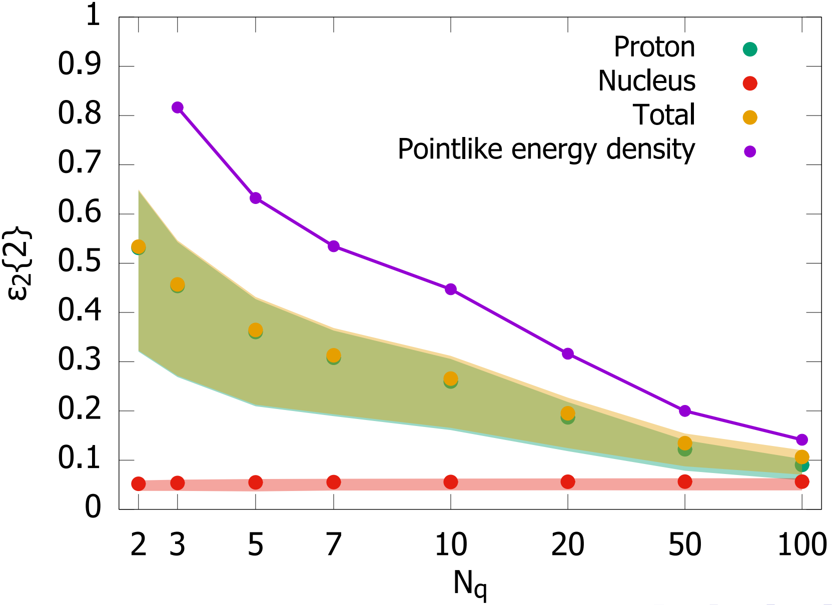
We expect to get the limiting behavior of our model by considering an extremely localized model of the energy density

$$\varepsilon(\mathbf{x}) = \varepsilon_0 \sum_{i=1}^{N_q} \delta^2(\mathbf{x} - \mathbf{b}_i).$$

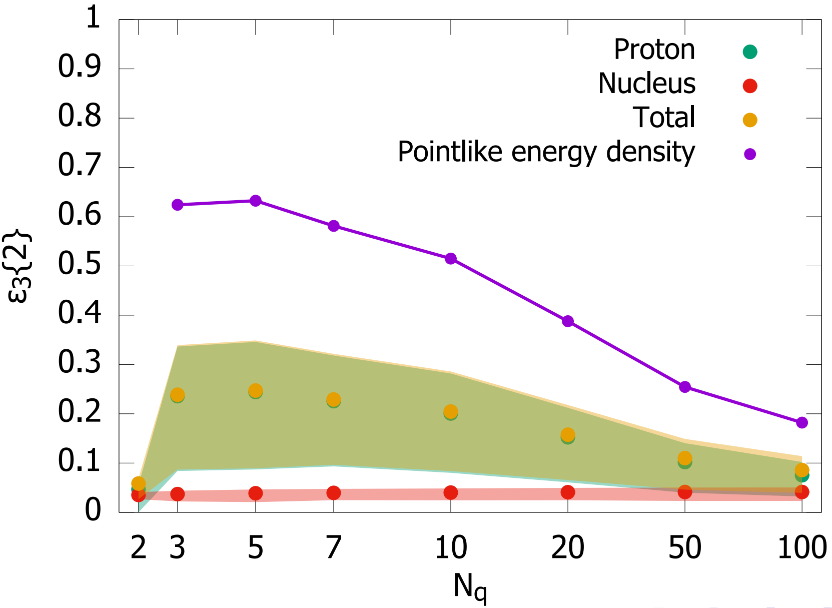
With this model, the eccentricity gets the form

$$\varepsilon_n\{2\} = \sqrt{\frac{(-2)^n (N_q-1) N_q^{1-n} n\Gamma(n) + N_q \left(2 - \frac{2}{N_q}\right)^n n\Gamma(n)}{2^n N_q^2 \left(\frac{N_q-2}{N_q-1}\right)^{n+1} \Gamma\left(\frac{n}{2}+1\right)^2 \times {}_2F_1\left(\frac{n+2}{2}, \frac{n+2}{2}; 1; \frac{1}{(N_q-1)^2}\right) + N_q \left(2 - \frac{2}{N_q}\right)^n n\Gamma(n)}}$$

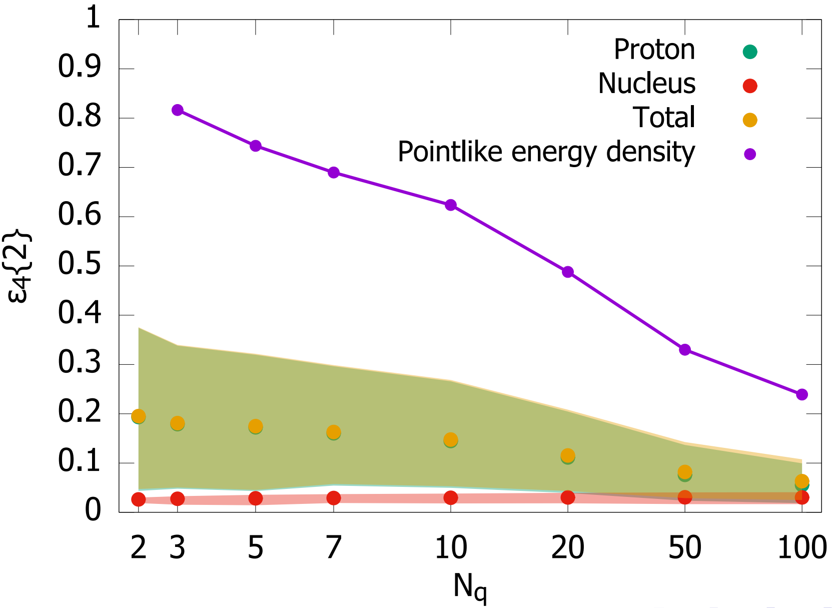
Eccentricity with $m \pm 50\%$ error bars



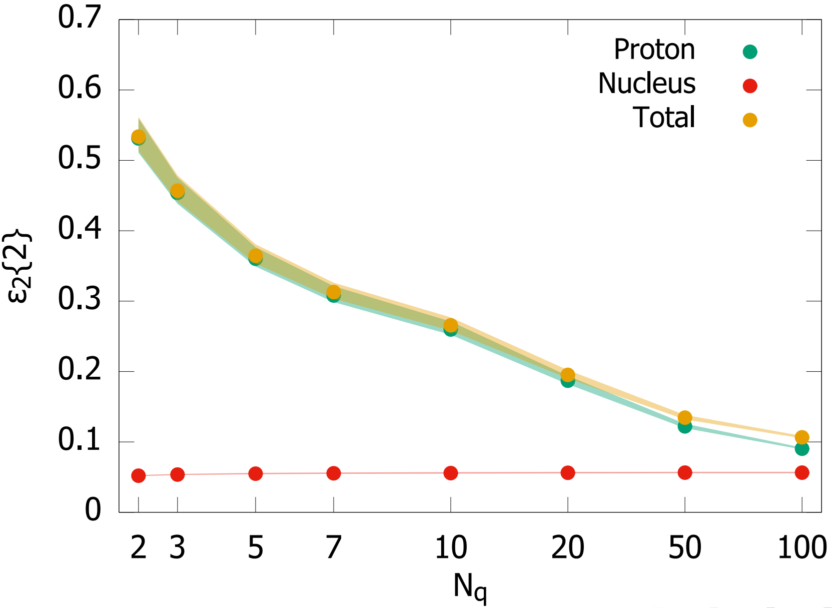
Eccentricity with $m \pm 50\%$ error bars



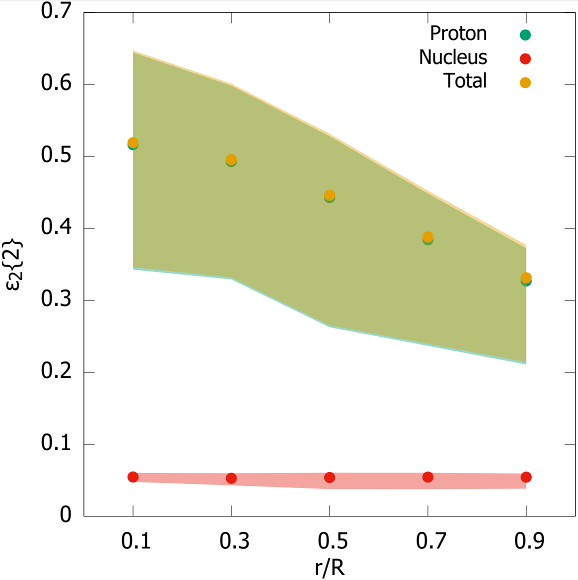
Eccentricity with $m \pm 50\%$ error bars



Eccentricity with $C_0 \pm 50\%$ error bars

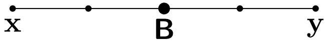


Eccentricity with $m \pm 50\%$ error bars. R is kept constant.

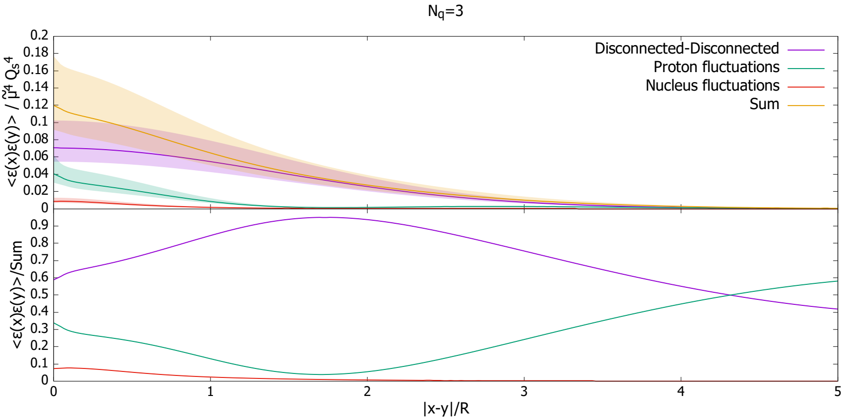


Energy density two-point function plot coordinate system.

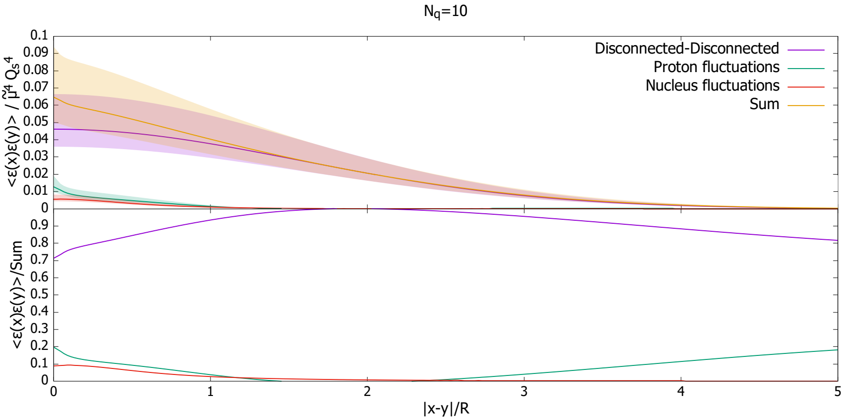
In the following we plot the energy density two-point function $\langle \varepsilon(\mathbf{x})\varepsilon(\mathbf{y})\rangle$ by taking a straight line through the center of our proton and letting the two coordinates $\mathbf{x}, \mathbf{y}$ move along the line, at the same rate, in different directions.



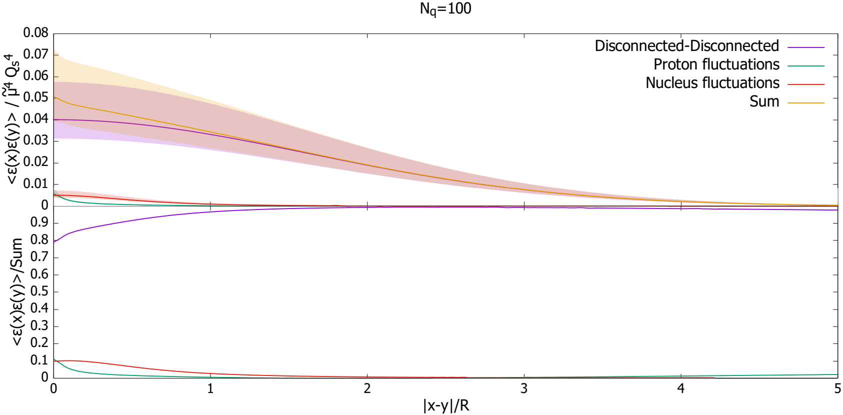
The two-point function parts with $C_0 \pm 50\%$ error bars. $N_q = 3$



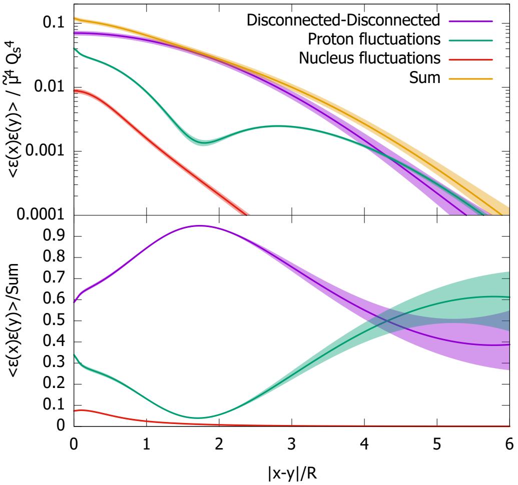
The two-point function parts with $C_0 \pm 50\%$ error bars $N_q = 10$



The two-point function parts with $C_0 \pm 50\%$ error bars $N_q = 100$



The two-point function parts on a log scale with $m \pm 50\%$ error bars. $N_q = 3$



Default parameters

In GeV:

$$\mu_0 = \frac{1}{\sqrt{N_q}}$$
$$Q_s = 2.0$$
$$C_0 = 0.05$$
$$m = 0.22$$
$$R = \sqrt{3.3} \approx 1.82$$
$$r = \sqrt{0.7} \approx 0.84$$