Hot spots and gluon field fluctuations as causes of eccentricity in small systems

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The energy density 1- and 2-point functions

In the Color Glass Condensate formalism the energy density 1- and 2-point functions, as functions of transverse coordinates, immediately after a heavy ion collision event at proper time \( \tau = 0 \), can be expressed as

\[
\langle \varepsilon(x) \rangle = (-ig)^2 (\delta^{ij} \delta^{kl} + \varepsilon^{ij} \varepsilon^{kl}) \frac{1}{2} i f^{abc} i f^{a'b'c} \\
\times \langle \alpha^{i,a}_x \alpha^{k,a'}_x \rangle \langle \beta^{j,b}_x \beta^{l,b'}_x \rangle
\]

and

\[
\langle \varepsilon(x) \varepsilon(y) \rangle = (-ig)^4 (\delta^{ij} \delta^{kl} + \varepsilon^{ij} \varepsilon^{kl})(\delta^{i'j'} \delta^{k'l'} + \varepsilon^{i'j'} \varepsilon^{k'l'}) \\
\times \frac{1}{4} i f^{abe} i f^{cde} i f^{a'b' e'} i f^{c'd'e'} \\
\times \langle \alpha^{i,a}_x \alpha^{k,c}_x \alpha^{i',a'}_y \alpha^{k',c'}_y \rangle \langle \beta^{j,b}_x \beta^{l,d}_x \beta^{j',b'}_y \beta^{l',d'}_y \rangle.
\]

Here \( \alpha^{i}_x = \alpha^{i,a}_x t^a \) with \( \alpha^{i,a}_x = \frac{2ig}{g} \text{Tr}[t^a U_x \partial^i U_x^\dagger] \).

The hot spot average

We have to average over the locations of the hot spots of the proton

\[
\langle \langle \mathcal{O} \rangle \rangle = \left( \frac{2\pi R^2}{N_q} \right) \int \prod_{i=1}^{N_q} \left[ d^2 b_i T(b_i - B) \right] \\
\times \delta \left( B - \frac{1}{N_q} \sum_{i=1}^{N_q} b_i \right) \langle \mathcal{O} \rangle_{CGC},
\]

where

\[
T(b) = \frac{1}{2\pi R^2} \exp \left[ -\frac{b^2}{2R^2} \right]
\]

describes the distributions of the hot spots.

Separation of fluctuations

We separate the nucleus and proton parts into their disconnected and connected parts

\[
\langle \alpha^{i,a}_x \alpha^{k,c}_x \alpha^{i',a'}_y \alpha^{k',c'}_y \rangle = \langle \alpha^{i,a}_x \alpha^{k,c}_x \rangle \langle \alpha^{i',a'}_y \alpha^{k',c'}_y \rangle
\]

Disconnected

\[
+ (\langle \alpha^{i,a}_x \alpha^{k,c}_x \alpha^{i',a'}_y \alpha^{k',c'}_y \rangle - \langle \alpha^{i,a}_x \alpha^{k,c}_x \rangle \langle \alpha^{i',a'}_y \alpha^{k',c'}_y \rangle).
\]

Connected

This way we can separate the two point function into DC-DC, DC-C, C-DC and C-C contributions. We interpret the connected contributions to be the part describing the fluctuations. The fully connected part is expected to be small.
Eccentricities

Usually one would compute mean square eccentricities as (Monte Carlo)

\[
\varepsilon_n^2 \equiv \langle \varepsilon_n \varepsilon_n \rangle
\]

\[
= \left\langle \frac{\int d^2x d^2y |x - B|^n |y - B|^n \exp (i n \theta x - B - i n \theta y - B) \varepsilon(x) \varepsilon(y)}{\int d^2x d^2y |x - B|^n |y - B|^n \varepsilon(x) \varepsilon(y)} \right\rangle.
\]

This is difficult to evaluate analytically. Instead, we compute

\[
\varepsilon_n^2
\]

\[
= \left\langle \frac{\int d^2x d^2y |x - B|^n |y - B|^n \exp (i n \theta x - B - i n \theta y - B) \varepsilon(x) \varepsilon(y)}{\int d^2x d^2y |x - B|^n |y - B|^n \varepsilon(x) \varepsilon(y)} \right\rangle.
\]
Eccentricity with $m \pm 50\%$ error bars

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Hot spots and gluon field fluctuations as causes of eccentricity in small systems
We built a model for computing eccentricities in pA collisions.
   - Averages done analytically.
- Hot spot fluctuations dominate eccentricities.
- Good: Small dependence on UV cutoff \((C_0)\).
- Bad: Large dependence on IR regulator \((m)\).

**What next?**
- Constrain parameters by comparing to experimental results in DIS.
- Relate eccentricities to anisotrophic flow coefficients.
- Study transverse position-momentum correlations of gluons in the proton.
Backups
Small-\(x\) part of a high energy proton/nucleus dominated by gluons.
\[\Rightarrow\] Treat the small-\(x\) gluons as classical fields.

Treat large-\(x\) partons as sources for small-\(x\) gluons.

Fig. from J. High Energ. Phys. 2010, 109 (2010).
Gluon fields right after the collision

Before the collision the gluon fields of the nuclei can be expressed as

\[
\alpha^i_x = \frac{i}{g} U_x \partial^i U_x^\dagger, \quad \beta^i_x = \frac{i}{g} V_x \partial^i V_x^\dagger,
\]

where \( U \) and \( V \) are Wilson lines of the two nuclei. By requiring the fields to be continuous in \( \tau_+ = 0 \) when moving to the future light cone, we get, for proper time \( \tau_+ = 0 \)

\[
A^i_x = \alpha^i_x + \beta^i_x, \quad A^\eta = \frac{ig}{2} [\alpha^i_x, \beta^i_x].
\]

\[
\tau = \sqrt{2x^+x^-}, \quad \eta = \frac{1}{2} \ln\left(\frac{x^+}{x^-}\right),
\]

\[
x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3)
\]
The CGC average with Gaussian weighting

For the proton

\[ \langle \rho^a(x) \rho^b(y) \rangle = \sum_{i=1}^{N_q} \mu^2(x - b_i) \delta(x - y) \delta^{ab}, \]

where

\[ \mu^2(x) = \frac{\mu_0^2}{2\pi r^2} \exp\left[-\frac{x^2}{2r^2}\right] \]

describes the transverse distribution of color charge in a hot spot. For the nucleus we have

\[ \langle \rho^a(x) \rho^b(y) \rangle = \mu_A^2 \delta(x - y) \delta^{ab}. \]
Energy density

We decompose the nucleus part into Wilson lines

\[ \langle \alpha_x^i, a \alpha_x^k, a' \rangle = \lim_{x_i \to x} \left\{ -\frac{4}{g^2} \partial_x^i \partial_x^k t_b^a t_{de}^a' \langle [U_{x_1} U_{x_2}^\dagger]_{cb} [U_{x_3} U_{x_4}^\dagger]_{ed} \rangle \right\}. \]

This is doable analytically when assuming Gaussian weight in the CGC average. After some work, we find that this is equal to

\[ \lim_{x_i \to x} \left\{ -\frac{4}{g^2} \partial_x^i \partial_x^k \left[ \delta^{bc} \delta^{de} \right] \left[ \delta^{be} \delta^{cd} \right]^T e^{M_{2 \times 2}} \left[ t_b^a t_{de}^{a'} 0 \right] \right\}. \]

We use the following identity to make things easier

\[ \partial_x e^{M(x)} = \int_0^1 e^{tM} \left[ \partial_x M \right] e^{(1-t)M} dt. \]

Finally we get

$$\langle \alpha^{i,a}_x \alpha^{k,a'}_x \rangle = \frac{Q_s^2 \delta^{aa'} \delta^{ik}}{2g^2 C_F} ,$$

where we used the GBW model as input.

Now the energy density takes the form

$$\langle \varepsilon(x) \rangle = \frac{NQ_s^2}{2C_F} \langle \beta^j,b_x \beta^j,b_x \rangle$$

We consider the proton to be dilute and we expand the proton Wilson lines

$$V(x) = \mathcal{P}_+ \exp[-ig \int_{-\infty}^{\infty} dz^+ d^2z G(x-z) \rho_a(z^+, z) t^a]$$

to the 1st order in source densities.
Energy density

After the averaging procedure we get

\[
\langle \varepsilon(x) \rangle = \frac{N(N^2 - 1)N_Q Q_s^2}{2C_F} \times \int d^2z G^j_x(x - z) G^j_x(x - z) F_1(z, B),
\]

where the Green’s functions derivatives

\[
G^j_x(x - z) \equiv \partial^j_x G_x(x - z)
\]

\[
= -\frac{1}{2\pi} m|x - z| K_1(m|x - z|) \frac{(x - z)^j}{(x - z)^2} \Theta(|x - z| - C_0),
\]

where mass regulates the IR and \(C_0\) regulates the UV divergences.
The hot spot average contribution to the energy density

\[
F_1(z, B) \equiv \langle \mu^2(z - b_i) \rangle_{\text{Hotspot}} \\
= \left( \frac{\mu_0^2}{2\pi r^2} \right) \left( \frac{1}{1 + \left( \frac{N_q - 1}{N_q} \right) \frac{R^2}{r^2}} \right) \exp \left\{ -\frac{1}{2} \frac{(z - B)^2}{r^2} + \left( \frac{N_q - 1}{N_q} \right) \frac{R^2}{r^2} \right\}
\]
Energy density two point function: nucleus part

We need to compute
\[ \langle \alpha_i^a, x \alpha_k^c, y \alpha_i'^{a'}, x \alpha_k'^{c'} \rangle, \]
which is equal to
\[ = \lim_{x_i \to x} \lim_{y_i \to y} \left\{ \frac{16}{g^4} \partial^{i x_2} \partial^{k y_2} \partial^{i' x_4} \partial^{k' y_4} \delta^{a_1 a_2} \delta^{a_3 a_4} \delta^{a_5 a_6} \delta^{a_7 a_8} \right\}^T e^{M_{24 \times 24}} \begin{pmatrix} t_{a_2 a_1} & t_{a_4 a_3} & t_{a_6 a_5} & t_{a_8 a_7} \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \}

This 4-\(\alpha\) correlator has been computed in a different way before in J. High Energ. Phys. 2019, 73 (2019).
Energy density two point function: proton part

For the proton side, we get

\[
\langle \langle \beta_{x}^{j,b} \beta_{x}^{l,d} | \beta_{y}^{j',b'} \beta_{y}^{l',d'} \rangle \rangle \\
= \int d^{2}a d^{2}b \left[ N_{q} F_{2}(a, b, B) + N_{q}(N_{q} - 1) F_{3}(a, b, B) \right] \\
\times \left[ G_{x}^{i}(x - a) G_{x}^{l}(x - a) G_{y}^{j'}(y - b) G_{y}^{l'}(y - b) \delta^{bd} \delta^{b'd'} \right. \\
+ G_{x}^{i}(x - a) G_{x}^{l}(x - b) G_{y}^{j'}(y - a) G_{y}^{l'}(y - b) \delta^{bb'} \delta^{dd'} \\
+ G_{x}^{i}(x - a) G_{x}^{l}(x - b) G_{y}^{j'}(y - b) G_{y}^{l'}(y - a) \delta^{bd'} \delta^{db'} \left. \right].
\]

Note the factor of \( N_{q} \) for when we take the two gluons from the same hot spot and the factor of \( N_{q}(N_{q} - 1) \) for when the gluons are taken from different hot spots.
\[ F_2(a, b, B) \equiv \langle \mu^2(a - b_i)\mu^2(b - b_i) \rangle_{\text{Hotspot}} \]
\[
= \left( \frac{\mu_0^2}{2\pi r^2} \right)^2 \left( \frac{1}{1 + 2 \left( \frac{N_q - 1}{N_q} \right) \frac{R^2}{r^2}} \right) \times \exp \left\{ -\frac{(a + b - 2B)^2}{4r^2 \left( 1 + 2 \left( \frac{N_q - 1}{N_q} \right) \frac{R^2}{r^2} \right)} - \frac{(a - b)^2}{4r^2} \right\}
\]

\[ F_3(a, b, B) \equiv \langle \mu^2(a - b_i)\mu^2(b - b_j) \rangle_{\text{Hotspot}} \]
\[
= \left( \frac{\mu_0^4}{(2\pi)^2(R^2 + r^2)} \right) \left( \frac{1}{r^2 + \left( \frac{N_q - 2}{N_q} \right) R^2} \right) \times \exp \left\{ -\frac{(a + b - 2B)^2}{4 \left( r^2 + \left( \frac{N_q - 2}{N_q} \right) R^2 \right)} - \frac{(a - b)^2}{4(R^2 + r^2)} \right\} \]
Fully disconnected and proton fluctuation parts

Disconnected part:
\[
\langle \varepsilon(x)\varepsilon(y) \rangle_{\text{DC,DC}} = \langle \varepsilon(x) \rangle \langle \varepsilon(y) \rangle.
\]

Proton fluctuations:
\[
\langle \varepsilon(x)\varepsilon(y) \rangle_{\text{DC,C}} = \frac{Q_s^4 N^2}{4 C_F^2} \int d^2a d^2b \left[ N_q F_2(a, b, B) + N_q(N_q - 1) F_3(a, b, B) \right]
\times \left\{ (N^2 - 1)^2 G_x^i (x - a) G_x^i (x - a) G_y^j (y - b) G_y^j (y - b) \right.
\left. + 2(N^2 - 1) G_x^i (x - a) G_x^i (x - b) G_y^j (y - a) G_y^j (y - b) \right\}
\]
\[
- \langle \varepsilon(x)\varepsilon(y) \rangle_{\text{DC,DC}}
\]
Nucleus fluctuations:

\[ \langle \varepsilon(x)\varepsilon(y) \rangle_{C,DC} \]

\[ = \frac{N^2 N_q^2}{(x - y)^4} \left\{ \frac{8(N^2 - 1)}{N^2} \exp\left[ -\frac{N^2 Q_s^2(x - y)^2}{2(N^2 - 1)} \right] \right. \]

\[ + N^2 Q_s^4(x - y)^4 + 4Q_s^2(x - y)^2 + \frac{8(1 - N^2)}{N^2} \left. \right\} \]

\[ \times \int d^2a d^2b G^i_x(x - a)G^i_x(x - a)G^j_x(y - b)G^j_x(y - b) \]

\[ \times F_1(a, B) F_1(b, B) - \langle \varepsilon(x)\varepsilon(y) \rangle_{DC,DC} \]
Pointlike energy density

We expect to get the limiting behavior of our model by considering an extremely localized model of the energy density

\[ \varepsilon(x) = \varepsilon_0 \sum_{i=1}^{N_q} \delta^2(x - b_i). \]

With this model, the eccentricity gets the form

\[
\varepsilon_n\{2\} = \sqrt{\frac{(-2)^n (N_q-1) N_q^{1-n} n \Gamma(n) + N_q \left(2 - \frac{2}{N_q}\right)^n}{2^n N_q^2 \left(\frac{N_q-2}{N_q-1}\right)^{n+1} \Gamma\left(\frac{n}{2} + 1\right)^2 \times {}_2F_1\left(\frac{n+2}{2}, \frac{n+2}{2}; 1; \frac{1}{(N_q-1)^2}\right) + N_q \left(2 - \frac{2}{N_q}\right)^n \Gamma(n)}}
\]
Eccentricity with $m \pm 50\%$ error bars

Pointlike energy density

$\varepsilon_2\{2\}$ vs $N_q$

Proton
Nucleus
Total

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Eccentricity with $m \pm 50\%$ error bars

Pointlike energy density

Proton
Nucleus
Total

$\epsilon_{3\{2\}}$

$N_q$

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Eccentricity with $m \pm 50\%$ error bars

Pointlike energy density

\[ \varepsilon_{4\{2\}} \]

Proton
Nucleus
Total

$N_q$

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Eccentricity with $C_0 \pm 50\%$ error bars

**Graph Description:**

- **x-axis:** $N_q$
- **y-axis:** $\varepsilon_2^{\{2\}}$
- **Legend:**
  - Proton
  - Nucleus
  - Total

**Data Points:**

- Proton
- Nucleus
- Total

**Graph Analysis:**

The graph illustrates the relationship between $N_q$ (number of quarks) and $\varepsilon_2^{\{2\}}$ (eccentricity). The data points for Proton, Nucleus, and Total are plotted with different markers and colors for easy distinction. The error bars indicate the uncertainty in the eccentricity measurements.
Eccentricity with $m \pm 50\%$ error bars. R is kept constant.
In the following we plot the energy density two-point function \( \langle \varepsilon(x)\varepsilon(y) \rangle \) by taking a straight line through the center of our proton and letting the two coordinates \( x, y \) move along the line, at the same rate, in different directions.

\[ \text{Diagram:} \quad x \quad \text{B} \quad y \]
The two-point function parts with $C_0 \pm 50\%$ error bars.

$N_q = 3$

\begin{align*}
\langle \varepsilon(x)\varepsilon(y) \rangle / \mu &\sim Q^4 \\
N_q = 3
\end{align*}
The two-point function parts with $C_0 \pm 50\%$ error bars

$N_q = 10$
The two-point function parts with $C_0 \pm 50\%$ error bars

$N_q = 100$

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The two-point function parts on a log scale with \( m \pm 50\% \) error bars. \( N_q = 3 \)
In GeV:

\[ \mu_0 = \frac{1}{\sqrt{N_q}} \]
\[ Q_s = 2.0 \]
\[ C_0 = 0.05 \]
\[ m = 0.22 \]
\[ R = \sqrt{3.3} \approx 1.82 \]
\[ r = \sqrt{0.7} \approx 0.84 \]