Multivariate cumulants in flow analyses: The Next Generation

Phys. Rev. C **102**, 024907 (2020), Phys. Rev. C **102**, 024910 (2020), arXiv:2101.05619

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Higher order Symmetric Cumulants

C. Mordasini, AB, D. Karakoç, F. Taghavi, Phys. Rev. C **102**, 024907 (2020)

New paradigm in flow analyses

- General 2-particle cumulant

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

  - Old paradigm: fundamental observable is an angle

$$X_1 \to e^{in\varphi_1}\,, \quad X_2 \to e^{-in\varphi_2}$$

  - New paradigm: fundamental observable is a flow amplitude

$$X_1 \to v_n^2\,, \quad X_2 \to v_m^2$$

- Two approaches yield **accidentally the same results** in the studies of correlated fluctuations of different flow magnitudes (Symmetric Cumulants), but in general results are different
  - Which one is correct?

Generalization: Multi-harmonic SC

- New paradigm:

  1/ Cumulant expansion directly on flow amplitudes:

$$\mathrm{SC}(k,l,m) \equiv \left\langle v_k^2 v_l^2 v_m^2 \right\rangle - \left\langle v_k^2 v_l^2 \right\rangle \left\langle v_m^2 \right\rangle - \left\langle v_k^2 v_m^2 \right\rangle \left\langle v_l^2 \right\rangle - \left\langle v_l^2 v_m^2 \right\rangle \left\langle v_k^2 \right\rangle + 2 \left\langle v_k^2 \right\rangle \left\langle v_l^2 \right\rangle \left\langle v_m^2 \right\rangle$$

  2/ Azimuthal angles are used merely to build an estimator for the above observable:

$$\begin{aligned}
\mathrm{SC}(k,l,m) = & \left\langle\left\langle\cos[k\varphi_1+l\varphi_2+m\varphi_3-k\varphi_4-l\varphi_5-m\varphi_6]\right\rangle\right\rangle \\
& - \left\langle\left\langle\cos[k\varphi_1+l\varphi_2-k\varphi_3-l\varphi_4]\right\rangle\right\rangle \left\langle\left\langle\cos[m(\varphi_5-\varphi_6)]\right\rangle\right\rangle \\
& - \left\langle\left\langle\cos[k\varphi_1+m\varphi_2-k\varphi_5-m\varphi_6]\right\rangle\right\rangle \left\langle\left\langle\cos[l(\varphi_3-\varphi_4)]\right\rangle\right\rangle \\
& - \left\langle\left\langle\cos[l\varphi_3+m\varphi_4-l\varphi_5-m\varphi_6]\right\rangle\right\rangle \left\langle\left\langle\cos[k(\varphi_1-\varphi_2)]\right\rangle\right\rangle \\
& + 2 \left\langle\left\langle\cos[k(\varphi_1-\varphi_2)]\right\rangle\right\rangle \left\langle\left\langle\cos[l(\varphi_3-\varphi_4)]\right\rangle\right\rangle \left\langle\left\langle\cos[m(\varphi_5-\varphi_6)]\right\rangle\right\rangle
\end{aligned}$$

C. Mordasini, AB, D. Karakoç, F. Taghavi: *'Higher order Symmetric Cumulants'*, Phys. Rev. C **102**, 024907 (2020)

Generalization: Multi-harmonic SC

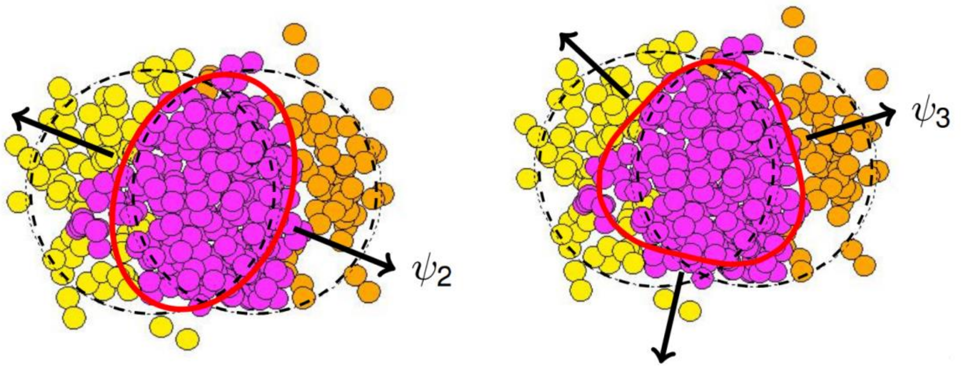
- **The main conclusion:** One cannot perform cumulant expansion in one set of stochastic observables, then in the resulting expression perform the transformation to some new set of observables, and then claim that the cumulant properties are preserved in the new set of observables
  - After such transformation, the fundamental properties of cumulants are lost in general

- Flow amplitudes $v_k, v_l, \ldots$ are stochastic observables in a sense that there exists underlying multivariate p.d.f. $f(v_k, v_l, \ldots)$ from which they are sampled event-by-event
  - The formalism of cumulants can be applied directly on flow amplitudes

C. Mordasini, AB, D. Karakoç, F. Taghavi: *'Higher order Symmetric Cumulants'*, Phys. Rev. C **102**, 024907 (2020)

New estimator for symmetry plane correlations

AB, M. Lesch, F. Taghavi, Phys. Rev. C **102**, 024910 (2020)



Symmetry plane correlations (SPC)

- Definition of SPC:

$$\left\langle e^{i(a_1 n_1 \Psi_{n_1} + \ldots + a_k n_k \Psi_{n_k})} \right\rangle, \qquad \sum_i a_i n_i = 0$$

- n_i are flow harmonics
- a_i are positive integers: the number of appearances of harmonic n_i associated with different azimuthal angles on the RHS in the analytic expression below:

$$v_{n_1}^{a_1} \ldots v_{n_k}^{a_k} \, e^{i(a_1 n_1 \Psi_{n_1} + \ldots + a_k n_k \Psi_{n_k})} = \langle e^{i(a_1 n_1 \varphi_1 + \ldots + a_k n_k \varphi_k)} \rangle$$

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, PRC 84 034910 (2011)

However…

- In all current SPC measurements, there is a very bold assumption that flow amplitudes v_n are independent
  - Can really the denominator be fully factorised in the Scalar Product method?

$$\left\langle \cos\left(a_1 n_1 \Psi_{n_1} + \cdots + a_k n_k \Psi_{n_k}\right)\right\rangle_{\mathrm{SP}} = \frac{\left\langle v_{n_1}^{a_1} \ \ldots \ v_{n_k}^{a_k} \cos\left(a_1 n_1 \Psi_{n_1} + \cdots + a_k n_k \Psi_{n_k}\right)\right\rangle}{\sqrt{\left\langle v_{n_1}^{2a_1}\right\rangle \ldots \left\langle v_{n_k}^{2a_k}\right\rangle}}$$

- Experimentally, we know that is not true, i.e. different flow amplitudes are definitely strongly correlated!
  - Can we improve the SPC measurements by taking this correlation into account?

New estimator for symmetry plane correlations

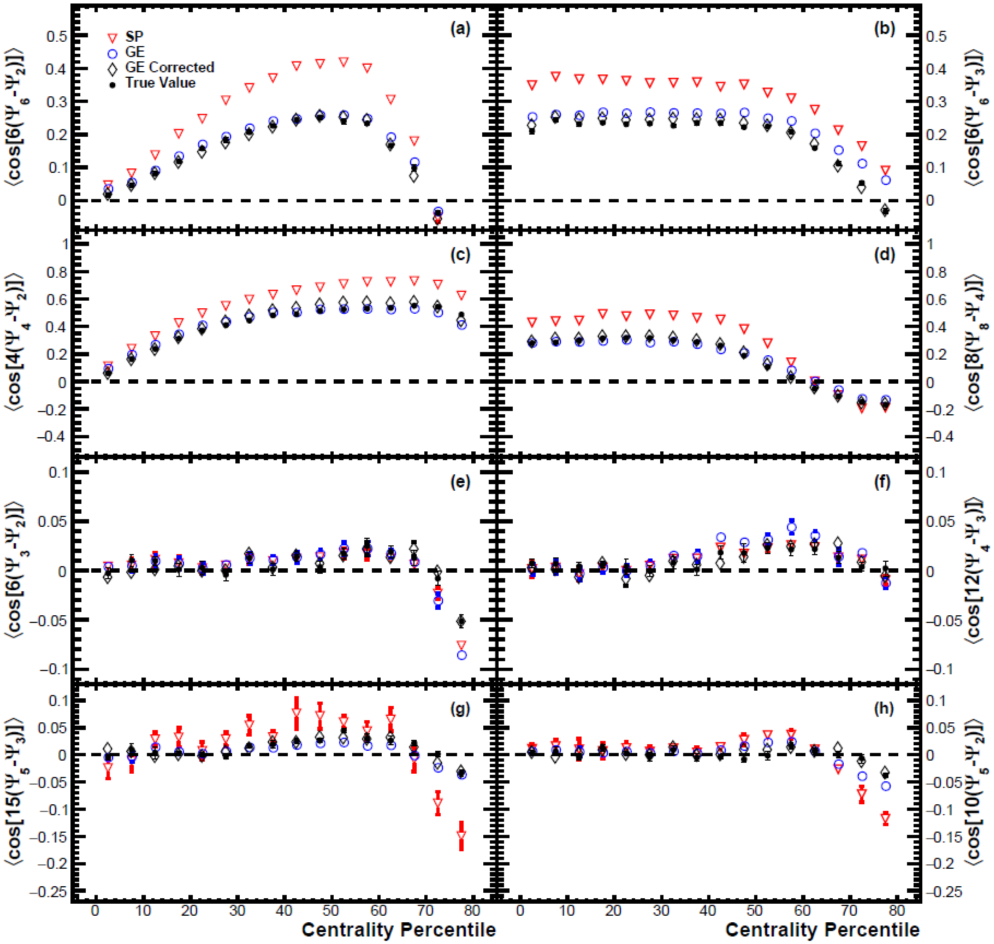
- The new estimator, which correctly accounts for the biases from correlations of flow amplitudes:

$$\left\langle \cos\left(a_1 n_1 \Psi_{n_1} + \cdots + a_k n_k \Psi_{n_k}\right)\right\rangle_{\mathrm{GE}} \simeq \sqrt{\frac{\pi}{4}}\,\frac{\left\langle v_{n_1}^{a_1} \ldots\ v_{n_k}^{a_k} \cos\left(a_1 n_1 \Psi_{n_1} + \ldots + a_k n_k \Psi_{n_k}\right)\right\rangle}{\sqrt{\left\langle v_{n_1}^{2a_1} \ldots\ v_{n_k}^{2a_k}\right\rangle}}$$

- The new denominator can be estimated with suitable chosen multiparticle azimuthal correlators
  - How important is the difference?

AB, M. Lesch, F. Taghavi: *'New estimator for symmetry plane correlations in anisotropic flow analyses',* Phys. Rev. C **102**, 024910 (2020)

New estimator for symmetry plane correlations



- Clear improvement over other existing estimators (e.g. the one based on traditional Scalar Product (SP) method)
- For centralities in which SP estimator (red markers) fails to reproduce the true values (black markers), our new estimator is still doing a great job!
- Works also if different flow amplitudes are correlated

AB, M. Lesch, F. Taghavi: *'New estimator for symmetry plane correlations in anisotropic flow analyses'*, Phys. Rev. C **102**, 024910 (2020)

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AB, M. Lesch, C. Mordasini, F. Taghavi, https://arxiv.org/abs/2101.05619

Reconciliation

- For the first time the strict mathematical formalism of multivariate cumulants is reconciled with the usage of cumulants in flow analyses
- From the fundamental properties of cumulants, the following two-steps recipe can be established:

1. We take temporarily that in the definition of $\lambda(X_1,\ldots,X_N)$ all observables $X_1,\ldots,X_N$ are statistically independent and factorize all multivariate averages into the product of single averages $\Rightarrow$ the resulting expression must reduce identically to 0;

2. We set temporarily in the definition of $\lambda(X_1,\ldots,X_N)$ all observables $X_1,\ldots,X_N$ to be the same and equal to X $\Rightarrow$ for the resulting expression it must hold that

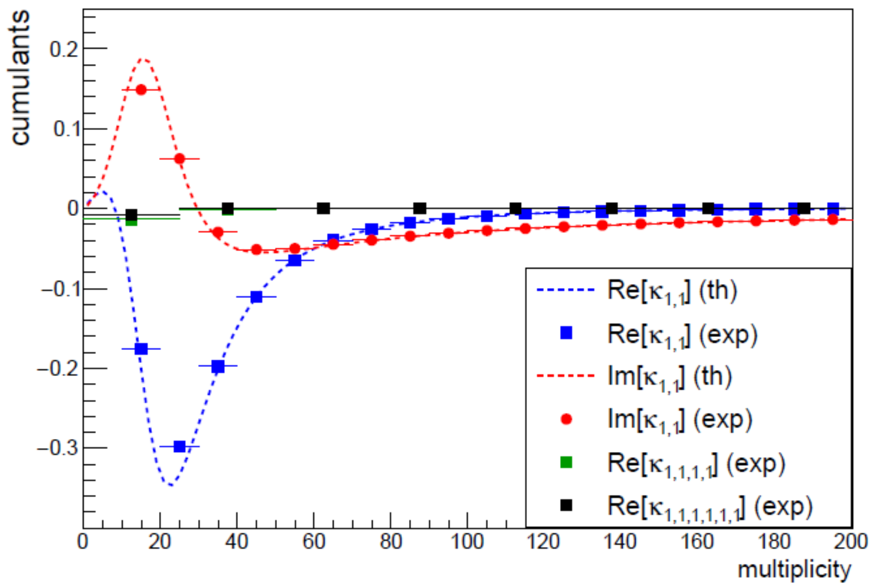
$$\lambda(aX+b) = a^N\lambda(X)\,, \tag{23}$$

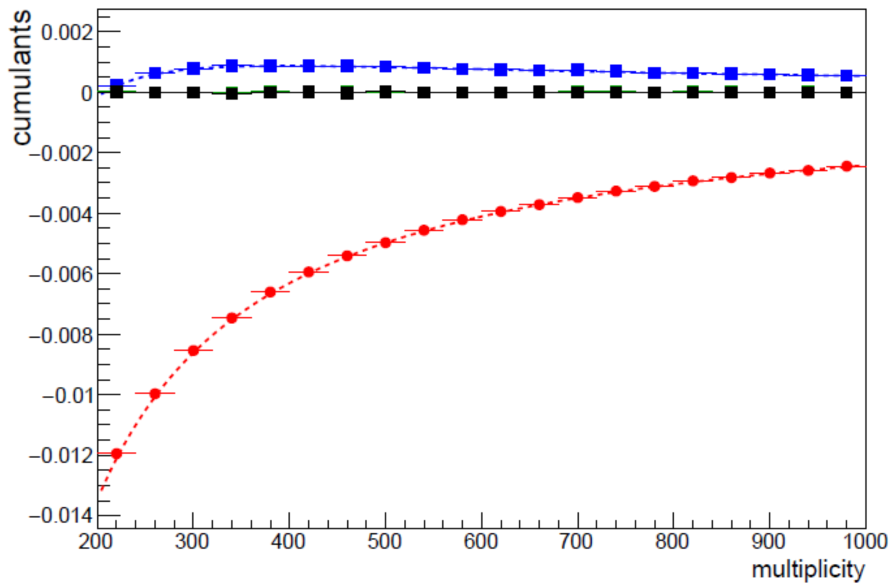
where a and b are arbitrary constants, and N is the number of observables in the starting definition of $\lambda(X_1,\ldots,X_N)$.

Multivariate observable is a multivariate cumulant only if it satisfies both above requirements

Example: New cumulants of azimuthal angles

- Defined event-by-event and by keeping all non-isotropic terms in the cumulant expansion





- Toy Monte Carlo study: From the sampled angles we can recover the theoretical values for cumulants (dashed lines)

Asymmetric Cumulants (AC)

- Further generalization of Symmetric Cumulants for the case when flow amplitudes are raised to different powers

$$\mathrm{AC}_{2,1}(m,n) = \langle v_m^4 v_n^2\rangle - \langle v_m^4\rangle\langle v_n^2\rangle - 2\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle + 2\langle v_m^2\rangle^2\langle v_n^2\rangle,$$

$$\begin{aligned}\mathrm{AC}_{3,1}(m,n) = {} & \langle v_m^6 v_n^2\rangle - \langle v_m^6\rangle\langle v_n^2\rangle - 3\langle v_m^2 v_n^2\rangle\langle v_m^4\rangle - 3\langle v_m^4 v_n^2\rangle\langle v_m^2\rangle \\ & + 6\langle v_m^4\rangle\langle v_m^2\rangle\langle v_n^2\rangle + 6\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle^2 - 6\langle v_m^2\rangle^3\langle v_n^2\rangle,\end{aligned}$$

$$\begin{aligned}\mathrm{AC}_{4,1}(m,n) = {} & \langle v_m^8 v_n^2\rangle - \langle v_m^8\rangle\langle v_n^2\rangle - 4\langle v_m^2 v_n^2\rangle\langle v_m^6\rangle - 6\langle v_m^4 v_n^2\rangle\langle v_m^4\rangle \\ & + 6\langle v_m^4\rangle^2\langle v_n^2\rangle - 4\langle v_m^6 v_n^2\rangle\langle v_m^2\rangle + 8\langle v_m^6\rangle\langle v_m^2\rangle\langle v_n^2\rangle \\ & + 24\langle v_m^2 v_n^2\rangle\langle v_m^4\rangle\langle v_m^2\rangle + 12\langle v_m^4 v_n^2\rangle\langle v_m^2\rangle^2 \\ & - 36\langle v_m^4\rangle\langle v_m^2\rangle^2\langle v_n^2\rangle - 24\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle^3 + 24\langle v_m^2\rangle^4\langle v_n^2\rangle,\end{aligned}$$

$$\begin{aligned}\mathrm{AC}_{2,1,1}(k,l,m) = {} & \langle v_k^4 v_l^2 v_m^2\rangle - \langle v_k^4 v_l^2\rangle\langle v_m^2\rangle - \langle v_k^4 v_m^2\rangle\langle v_l^2\rangle - \langle v_k^4\rangle\langle v_l^2 v_m^2\rangle \\ & + 2\langle v_k^4\rangle\langle v_l^2\rangle\langle v_m^2\rangle - 2\langle v_k^2 v_l^2\rangle\langle v_k^2 v_m^2\rangle - 2\langle v_k^2 v_l^2 v_m^2\rangle\langle v_k^2\rangle \\ & + 4\langle v_k^2 v_l^2\rangle\langle v_k^2\rangle\langle v_m^2\rangle + 4\langle v_k^2 v_m^2\rangle\langle v_k^2\rangle\langle v_l^2\rangle \\ & + 2\langle v_k^2\rangle^2\langle v_l^2 v_m^2\rangle - 6\langle v_k^2\rangle^2\langle v_l^2\rangle\langle v_m^2\rangle.\end{aligned}$$

These are unique combinations which satisfy all fundamental mathematical properties of multivariate cumulants

Thanks!

Backup slides