Beam Physics Fundamentals. Recap.

AVA School on Precision Studies
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Introduction

❑ Accelerator Physics

Special relativity → Beam dynamics

Electromagnetism → Beam dynamics

Classical mechanics → Beam dynamics

Beam dynamics

Beam dynamics → Longitudinal beam dynamics

Beam dynamics → Transverse beam dynamics

RF design	Non-linear dynamics	Magnet design
Optical design	Collective effects	Operations

\+ many others

Lorentz force

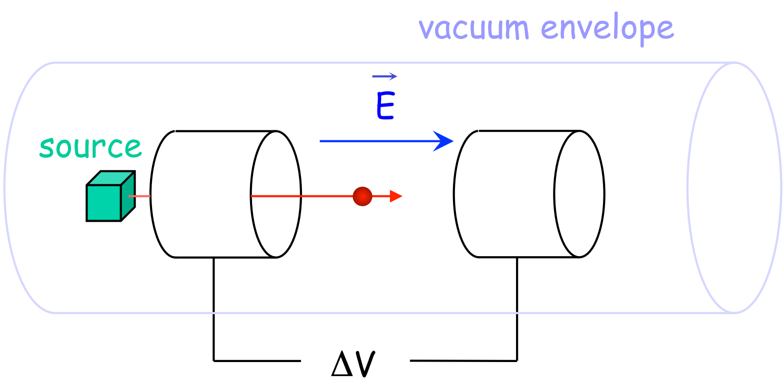
- A charged particle moving with velocity ***v*** through an electro-magnetic field experiences a force

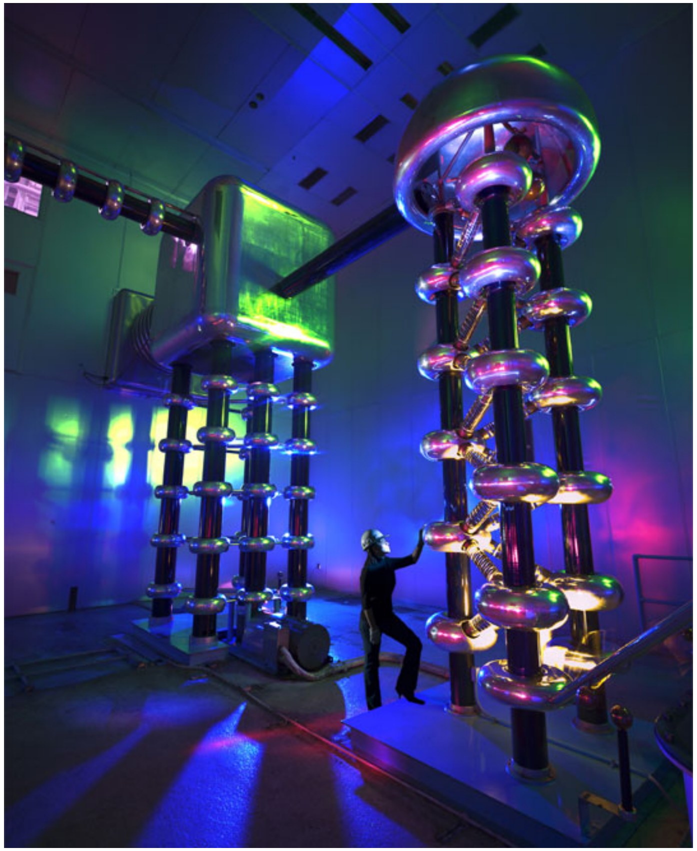
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- The second term is always perpendicular to the direction of motion, so it does not give any longitudinal acceleration and it does not increase the energy of the particle.

- Acceleration has to be done by an electric field E

Electrostatic acceleration





750 kV Cockroft-Walton generator at Fermilab (Proton source)

Electrostatic Field:

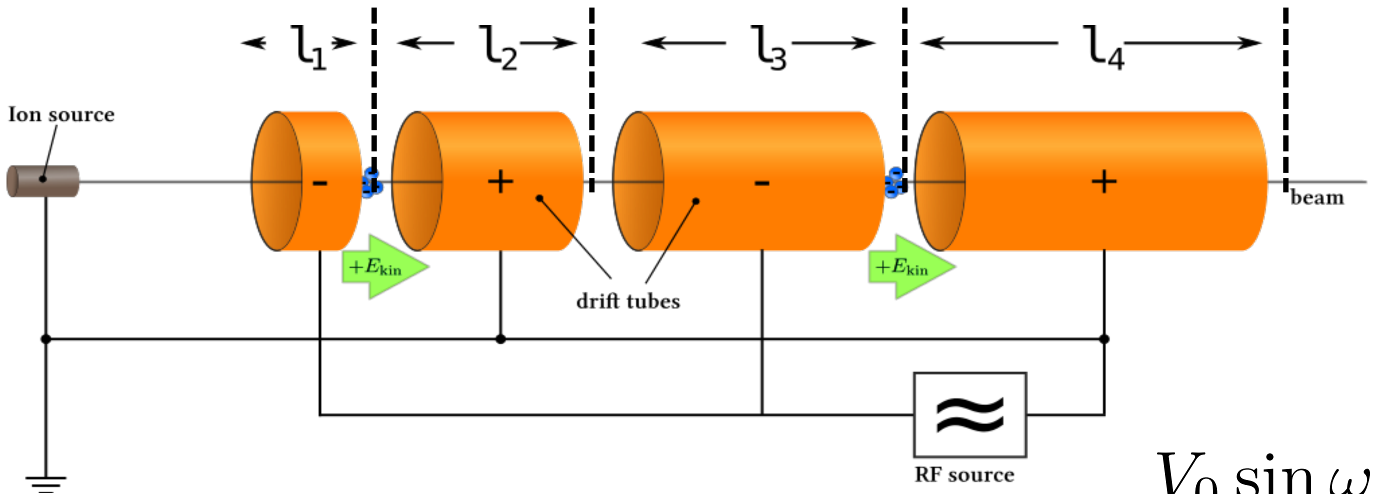
Energy gain: $W = e\,\Delta V$

Limitation: insulation problems
maximum high voltage (~ 10 MV)

Used for first stage of acceleration: particle sources, electron guns, x-ray tubes

Radio-Frequency (RF)

❑ Ising-Wideröe type structure



$$V_0 \sin \omega t$$

R. Wideröe

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

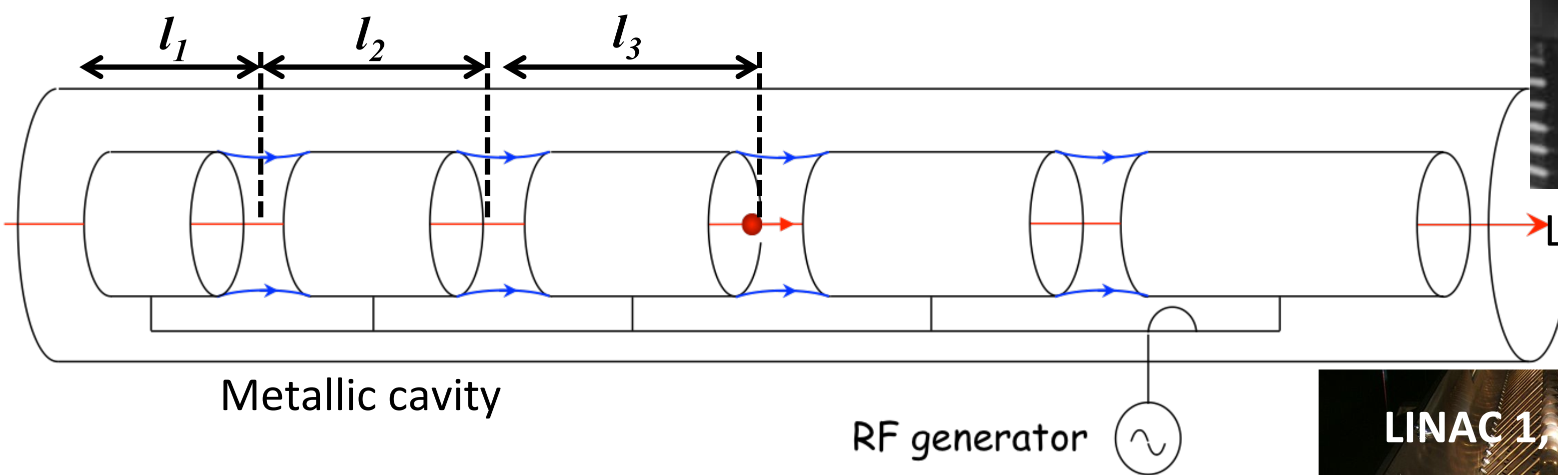
Synchronism condition

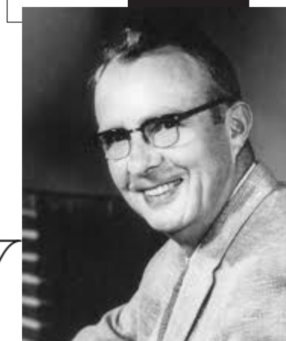
$$\boxed{l_i = \frac{v_i T_{RF}}{2} = \frac{\beta_i \lambda_{RF}}{2}}$$

v_i = particle velocity
T_{RF} = RF period

Radio-Frequency (RF)

❑ Alvarez type structure





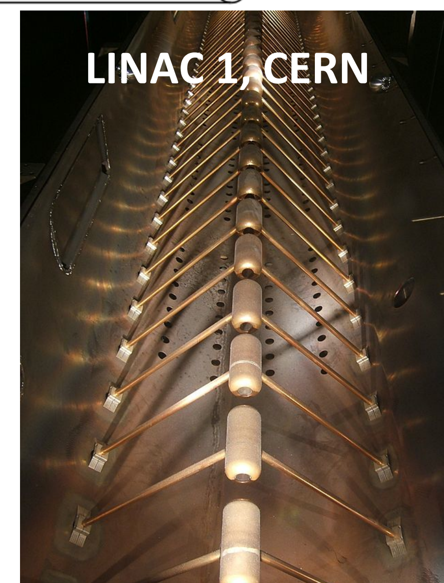
L. W. Alvarez

Synchronism condition

$$\boxed{l_i = v_i T_{RF} = \beta_i \lambda_{RF}}$$

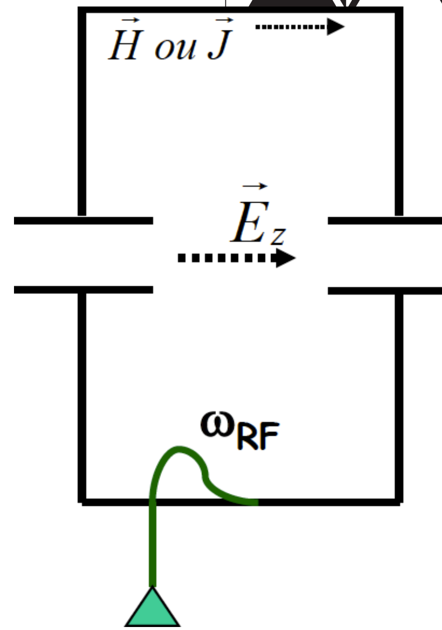
$$\omega_{RF} = 2\pi \frac{v_i}{l_i}$$

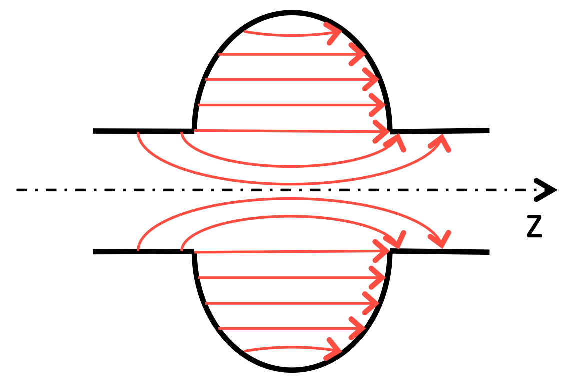
Used for proton and ions (50 – 200 MeV), $f_{RF} \sim 200$ MHz



❑ Resonant Cavities

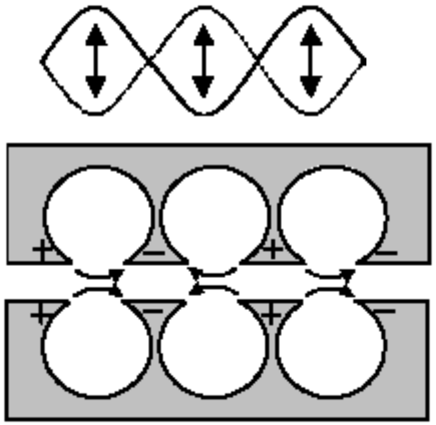
- Higher operating frequency.
- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
- Solution: enclosing the system in a cavity whose resonant frequency matches the RF generator frequency.
- Electromagnetic power constrained in the resonant volume
- Each such cavity can be independently powered from the RF generator
- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)





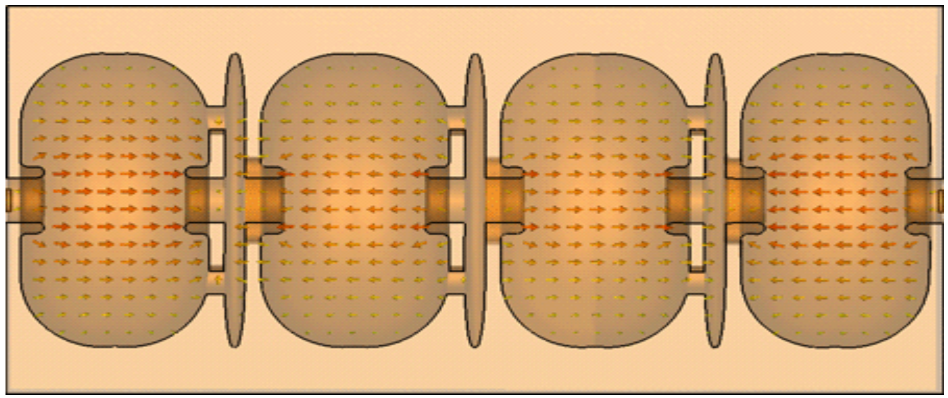
Radio-Frequency (RF)

❑ Resonant Cavities

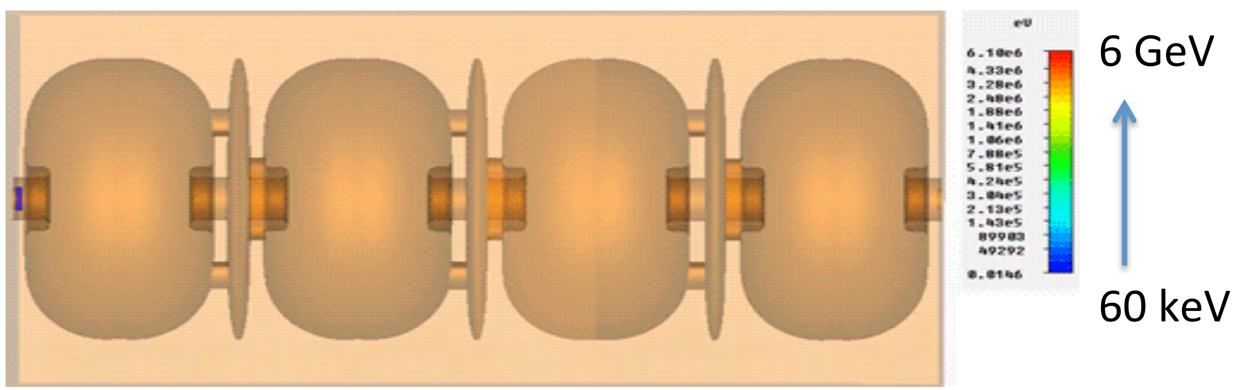


Standing wave linear accelerator

π/2 - mode of the E010 field pattern



Field gradient: 20 MeV/m

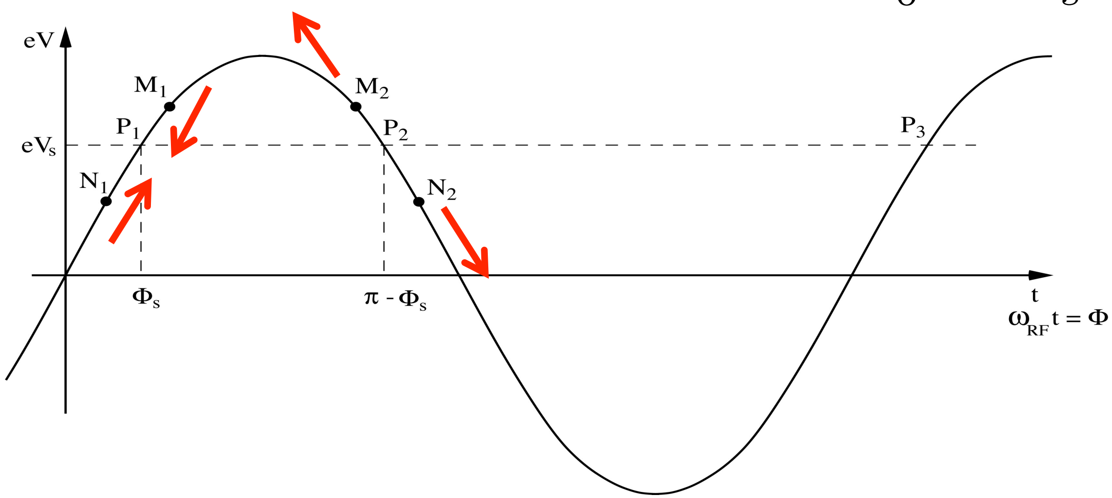


CST® simulation examples: https//www.cst.com/Applications

Synchronous phase

❑ Phase stability (linac)

- Cavity set up so that particle at the centre of bunch (synchronous particle) acquires just the right amount of energy $\Delta E = eV_0 \sin \Phi_s$



Stable phase region

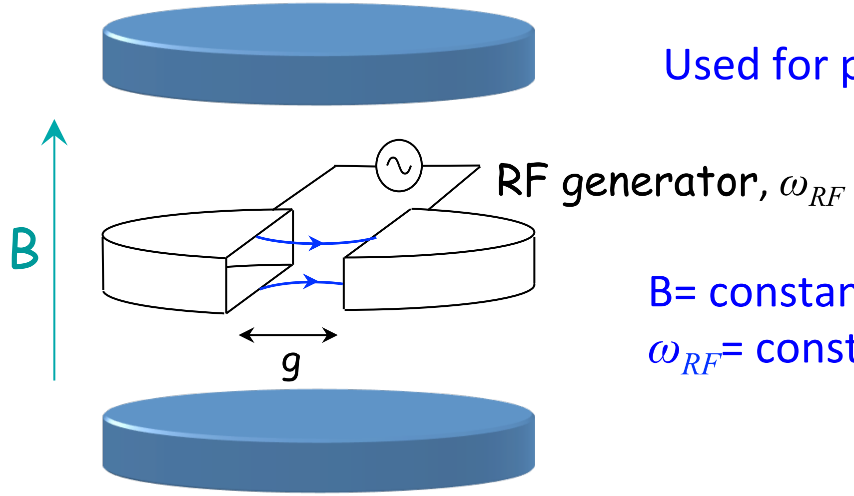
$$0 < \Phi_s < \frac{\pi}{2}$$

- Particles arriving early (N_1) see $\Phi < \Phi_s$ and will gain less energy. In the next gap it will appear closer to particle P_1 (synchronous particle)

- Particles arriving late (M_1) see $\Phi > \Phi_s$ and reduce its delay compared to P_1

Cyclotron


E. Lawrence



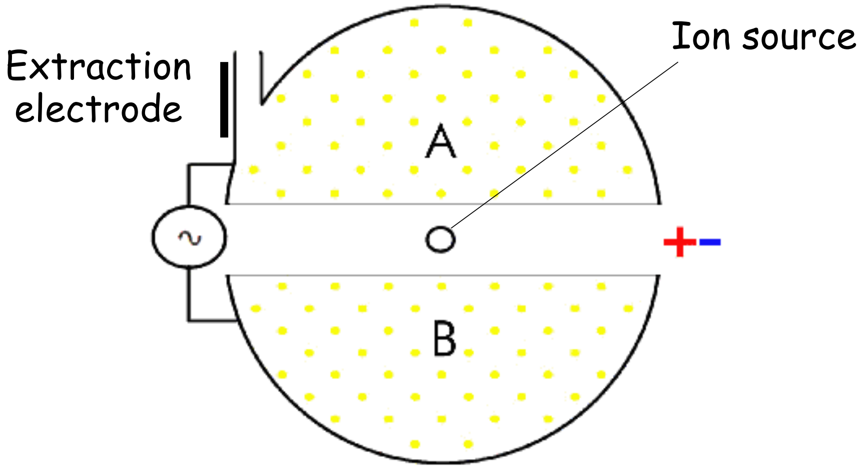
Used for protons, ions

B= constant

ω_{RF}= constant

Synchronism condition

$$\boxed{\begin{aligned}\omega_s &= \omega_{RF} \\ 2\pi\,\rho &= v_s\,T_{RF}\end{aligned}}$$

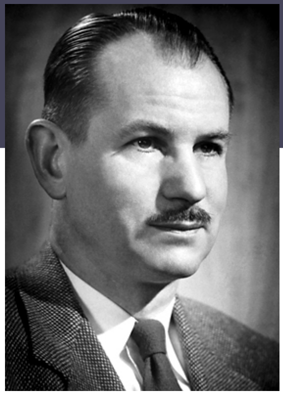


$$\omega_s = \frac{qB}{m_0\gamma}$$

As long as $v_s \ll c$ and $\gamma \approx 1$ the synchronism conditions stays fulfilled, and the revolution frequency does not depend on the radius

For higher energies … ?

Synchrocyclotron



E. McMillan

In order to keep the synchronism at higher energies, one has to decrease ω_{RF} during the acceleration cycle according to the relativistic $\Upsilon(t)$

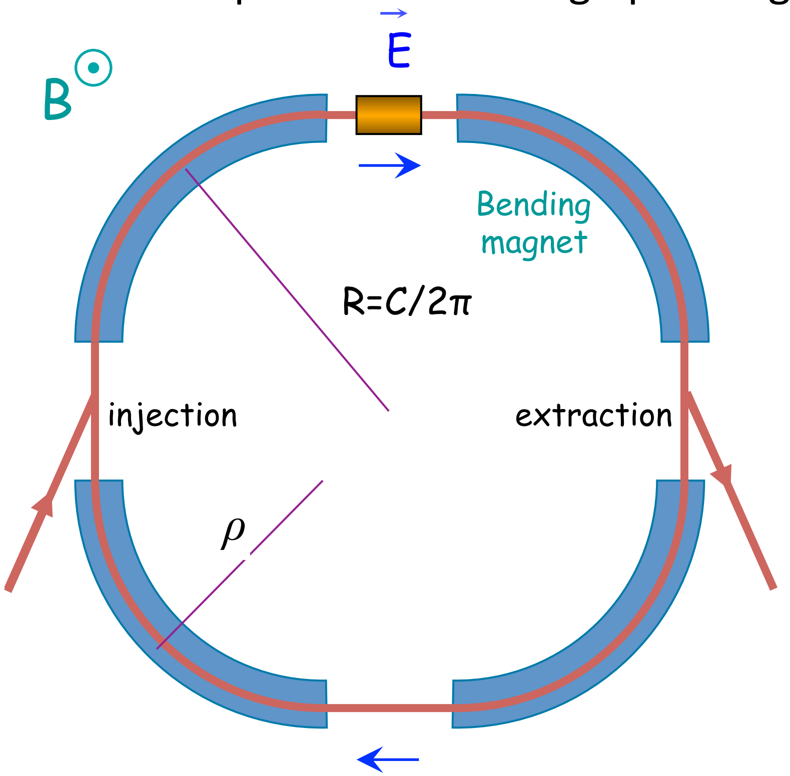
$$\boxed{\omega_{RF}(t) = \omega_s(t) = \frac{qB}{m_0\gamma(t)}}$$

It can accelerate protons up to around 500 MeV.

Limitation due to the size of the magnet

Synchrotron

Synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ **Energy gain per turn**

$\Phi = \Phi_s = cte \longrightarrow$ **Synchronous particle**

$\omega_{RF} = h\omega_r \longrightarrow$ **RF synchronism (h - harmonic number)**

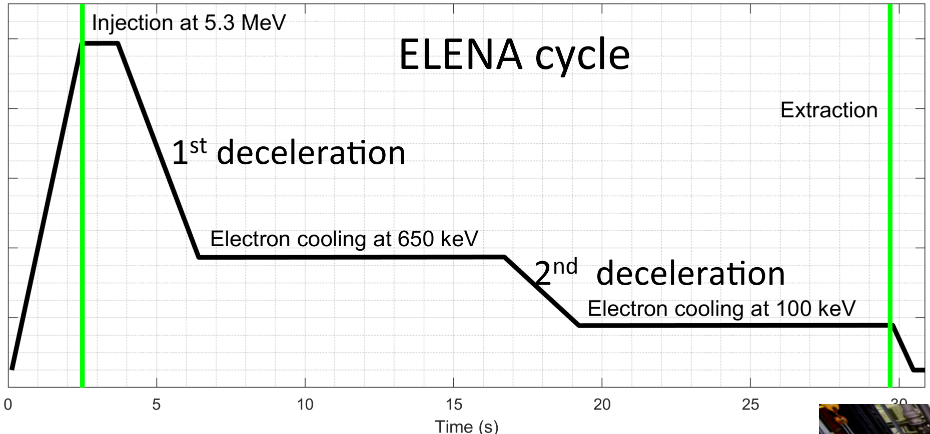
$\rho = cte \quad R = cte \longrightarrow$ **Constant orbit**

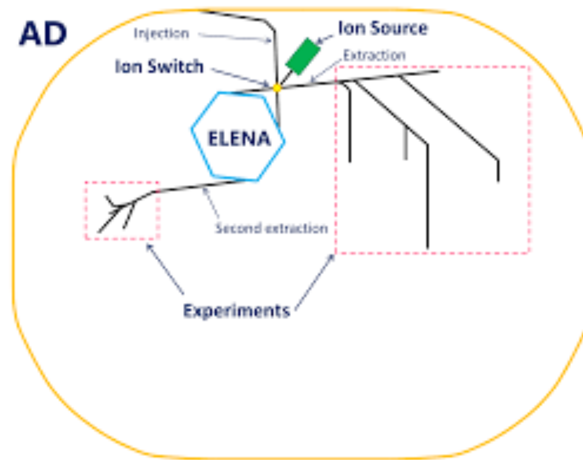
$B\rho = P/e \Rightarrow B \longrightarrow$ **Variable magnetic field**

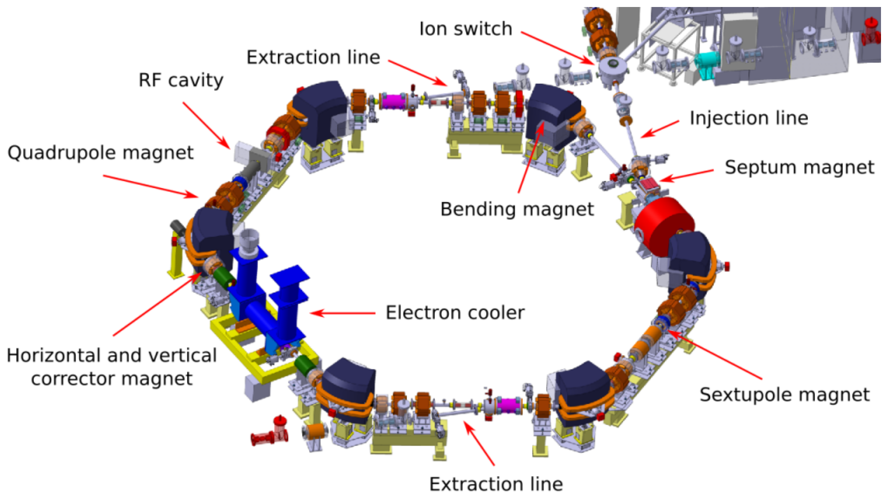
If $v \approx c$, ω_r hence ω_{RF} remain constant (ultra-relativistic regime)

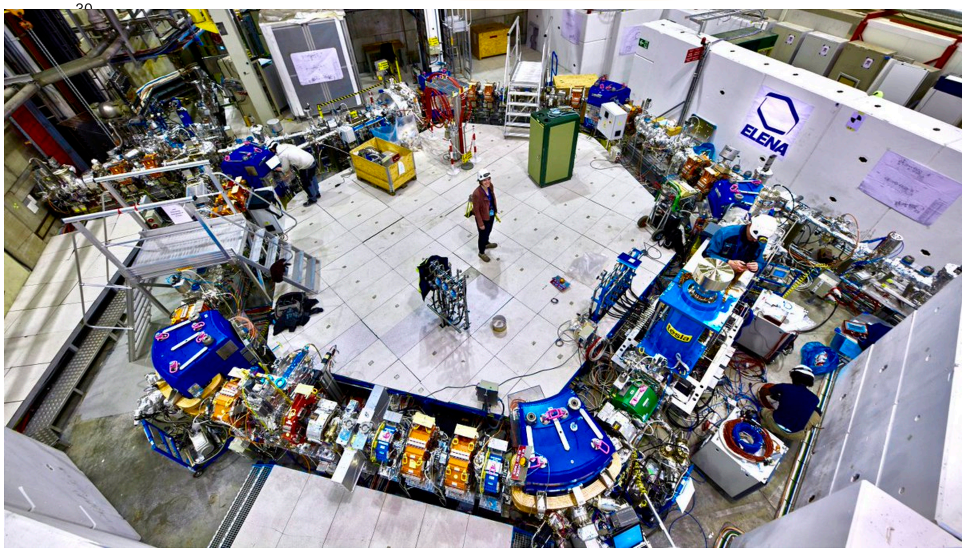
ELENA ring

- A compact synchrotron to decelerate antiprotons









Recap.

❑ Remember the Lorentz force

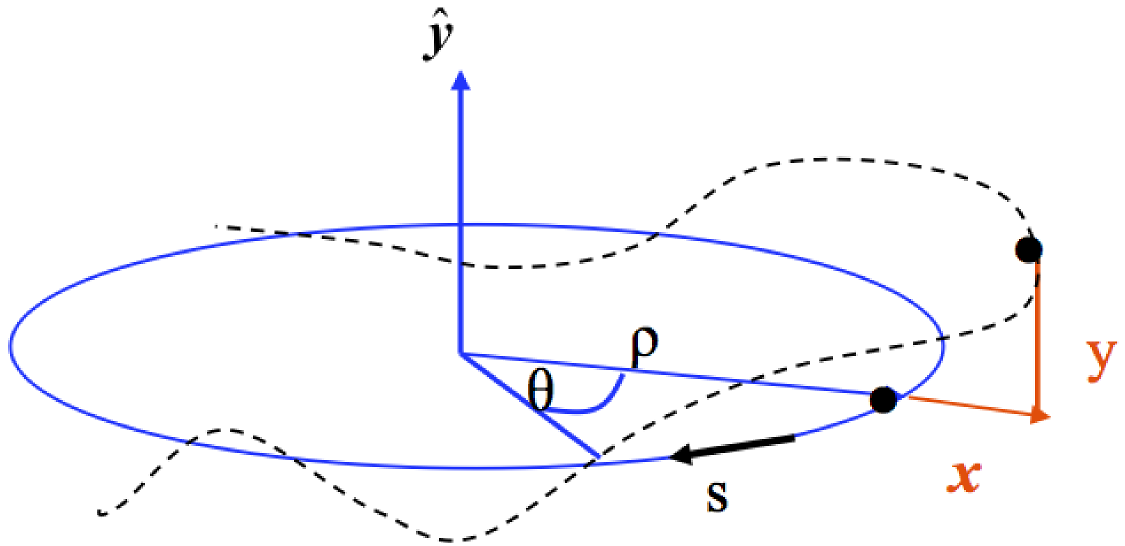
$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- We have used E to accelerate/decelerate particles
- Now we shall use B to deflect particles as we really benefit from the presence of the velocity in the Lorentz force
- At high velocities magnets are used to deflect particles
- Although at low velocities electric fields may be more efficient (e.g. electrostatic elements are used in the ELENA's transfer line to experiments)

Coordinate system

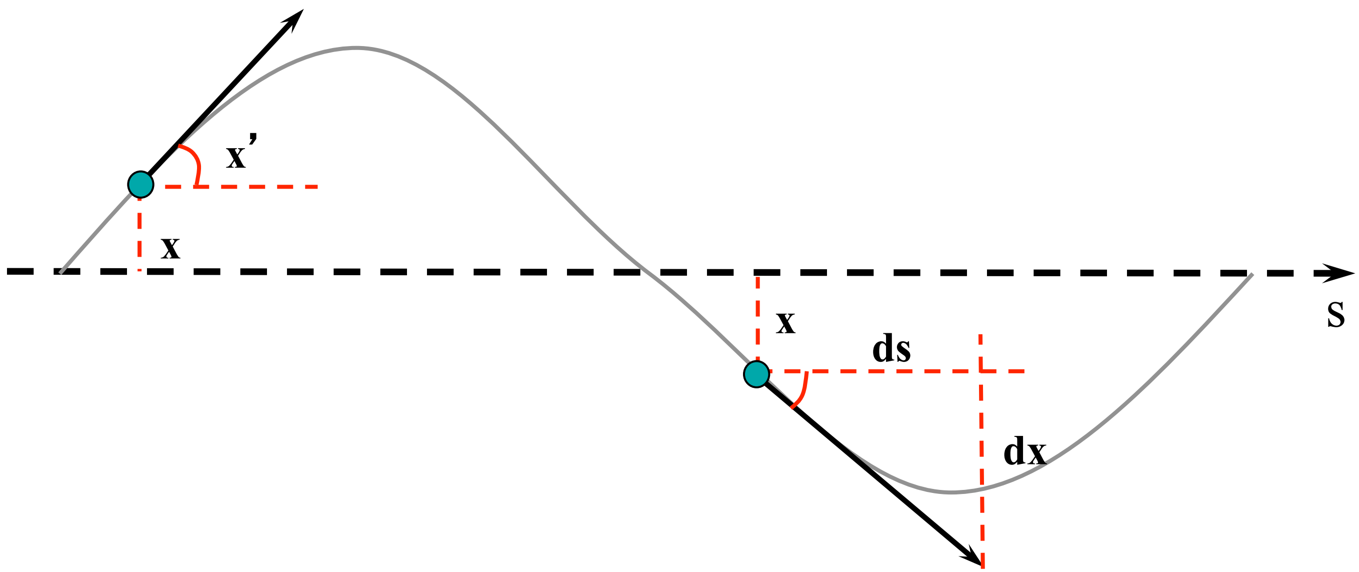
Coordinates w.r.t. the design orbit

Let's consider a local segment of one particle's trajectory



Paraxial approximation: we will assume the deviation of particle coordinates from the design orbit is small , so x, y << bending radius (ρ)

Coordinate system



The state of a particle (phase space) represented with a **6-D vector**

$$(x, x', y, y', z = s - \beta ct, \delta = \Delta p/p_0)$$

with

$$x' = \frac{dx}{ds} = \frac{dxdt}{dtds} = \frac{p_x}{p_z}; \quad y' = \frac{dy}{ds} = \frac{dydt}{dtds} = \frac{p_y}{p_z}$$

Magnetic field

❑ Taylor expansion:

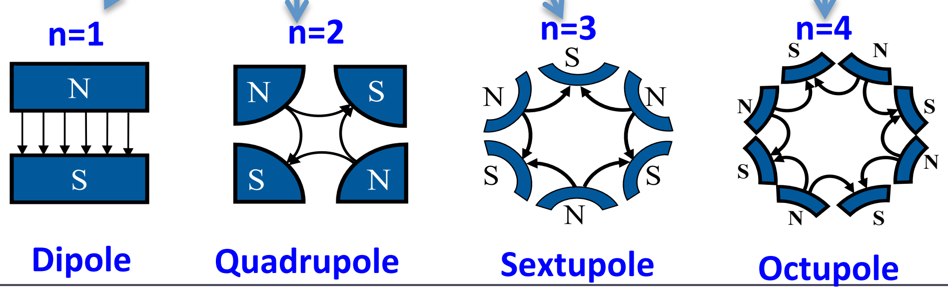
$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}x^3 + \ldots$$

Magnet strengths normalized to the beam rigidity:

$$B_y(x) = B\rho \sum_{n=1}^{\infty} \frac{k_{n-1}x^{n-1}}{(n-1)!}$$

$$\frac{q}{p}B_y(x) = k_0 + k_1 x + \frac{1}{2}k_2 x^2 + \frac{1}{3!}k_3 x^3 + \ldots$$

2n-pole:



Bending

❑ Dipole

If we want to deflect particles

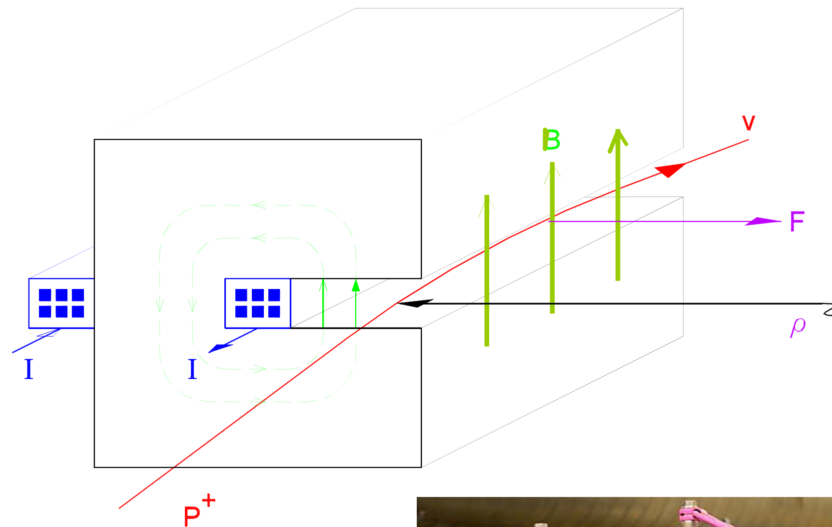
$$F = qvB$$

We equate this to the centripetal force

$$F = \frac{\gamma m v^2}{\rho}$$

$$\boxed{\frac{1}{\rho}[\mathrm{m}^{-1}] = 0.2998 \frac{B[\mathrm{T}]}{p[\mathrm{GeV}/c]}}$$

(for q=e)



Rigidity: $B\rho = \frac{p}{q}$

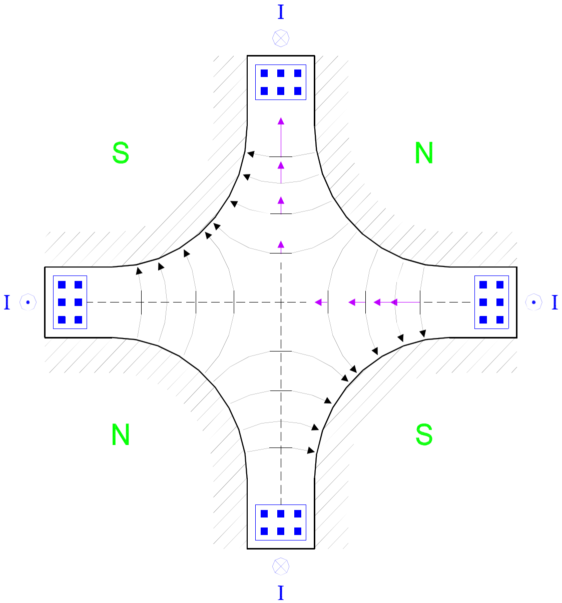
$$\boxed{B\rho[\mathrm{T}\cdot m] = 3.3356 \cdot p[\mathrm{GeV}/c]}$$

ELENA ring dipole

Focusing/defocusing

❑ Quadrupole

- Constant gradient $g = -\dfrac{dB_y}{dx}$
- Focusing forces increase linearly with displacement
- Magnetic lenses are focusing in one plane but are defocusing in the orthogonal plane



Normalised strength:

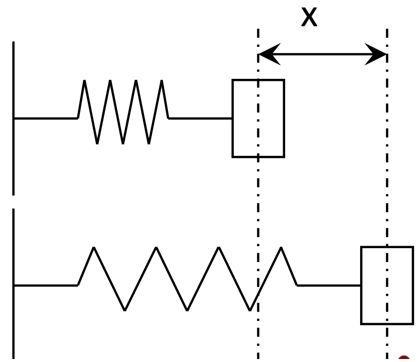
$$\boxed{k_1[\mathrm{m}^{-2}] = \frac{g}{p/e}[\mathrm{m}^{-2}] = 0.2998\,\frac{g[\mathrm{T/m}]}{p[\mathrm{GeV}/c]}}$$

ELENA ring quadrupole

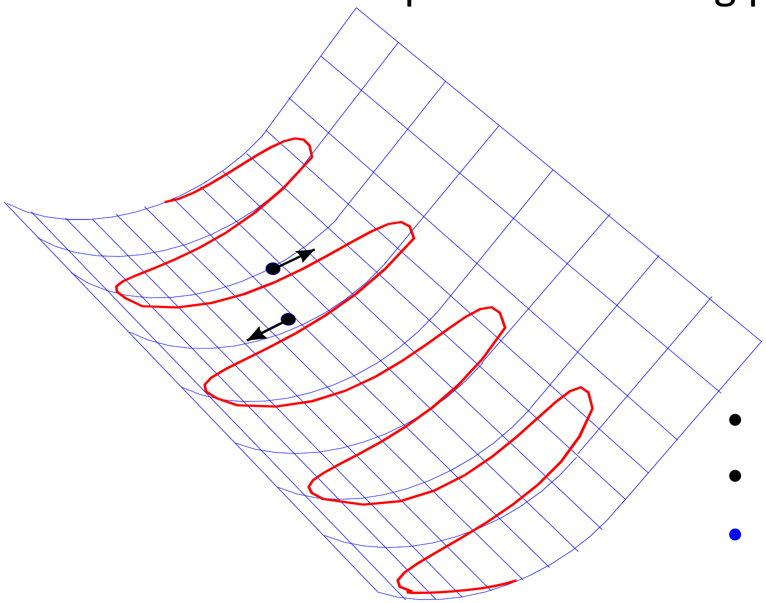
Hill's equation

- **Basic equation of motion for a particle in an accelerator**

$$\boxed{x''(s) + K(s)x(s) = 0}$$



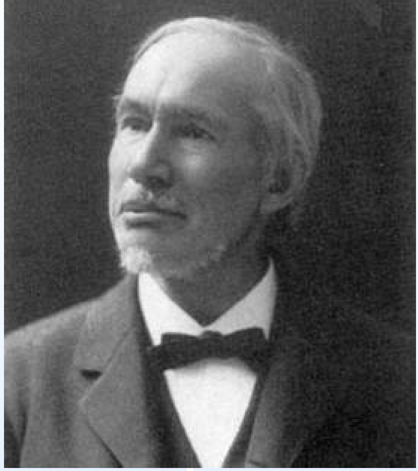
Motion with periodic focusing properties



Compare with a simple harmonic oscillator with restoring force *F=-kx*

- with a restoring force $\neq \mathrm{const}$
- K(s) depends on longitudinal position s
- $K(s+L)=K(s)$ periodic function, where L is the "lattice period"

Hill's Equation


George Hill

- Equation of transverse motion
  - Drift: $x'' = 0, \quad y'' = 0$
  - Solenoid: $x'' + 2k\,y' + k'\,y = 0, \quad y'' - 2k\,x' - k'\,x = 0$
  - Quadrupole: $x'' + k\,x = 0, \quad y'' - k\,y = 0$
  - Dipole: $x'' + \dfrac{1}{\rho^2}x = 0, \quad y'' = 0$
  - Sextupole: $x'' + k\left(x^2 - y^2\right) = 0, \quad y'' - 2kxy = 0$
- Hill's Equation: $x'' + k_x(s)x = 0, \quad y'' + k_y(s)y = 0$

Focusing quadrupole

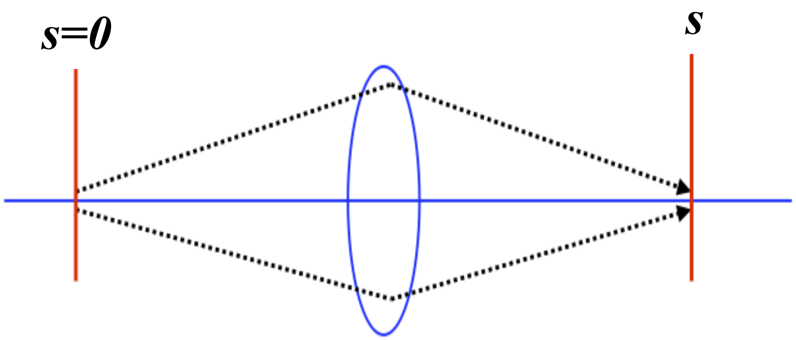
- For similarity with harmonic oscillator, considering K=const.

- Horizontal focusing quadrupole (defocusing in vertical plane), $K > 0$:

$$\boxed{\begin{aligned} x(s) &= x(0)\cos(\sqrt{K}s) + \frac{x'(0)}{\sqrt{K}}\sin(\sqrt{K}s) \\ x'(s) &= -x(0)\sqrt{K}\sin(\sqrt{K}s) + x'(0)\cos(\sqrt{K}s) \end{aligned}}$$

- For convenience we can use a **matrix formalism:**

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$



Defocusing quadrupole

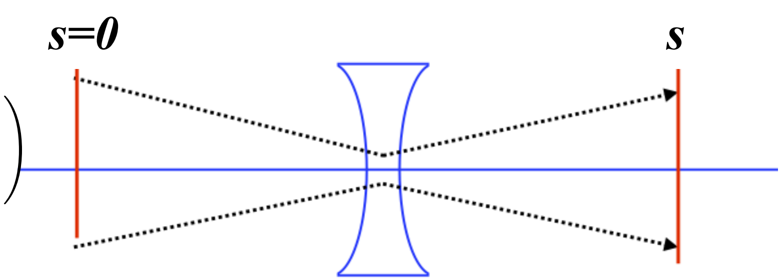
- For similarity with harmonic oscillator, considering K=const.

- Horizontal defocusing quadrupole (focusing in vertical plane), $\boldsymbol{K < 0}$:

$$\boxed{\begin{aligned} x(s) &= x(0)\cosh(\sqrt{|K|}s) + \frac{x'(0)}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ x'(s) &= x(0)\sqrt{|K|}\sinh(\sqrt{|K|}s) + x'(0)\cosh(\sqrt{|K|}s) \end{aligned}}$$

- **Matrix formalism:**

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s) \\ \sqrt{|K|}\sinh(\sqrt{|K|}s) & \cosh(\sqrt{|K|}s) \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix}$$



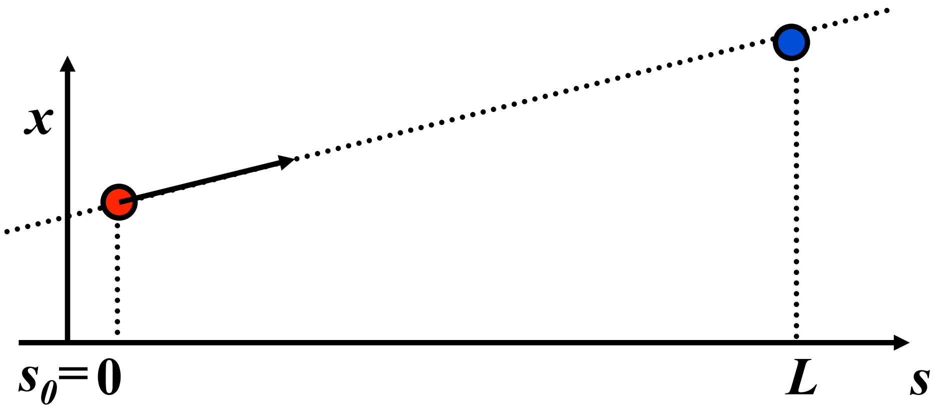
Drift

- For $K=0$, and no further magnetic elements, we have a drift space:

$$x(s) = x(0) + (s - s_0)x'(0) = x(0) + Lx'(0)$$
$$x'(s) = x'(0)$$

- Position changes if particle has a slope which remains unchanged.

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$



Thin lens approximation

- For a focusing quadrupole (K>0)

$$M_{\mathrm{QF}} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix}$$

- For a defocusing quadrupole (K<0)

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix}$$

- In the limit $L \to 0$, and KL=const.

Focal length

$$M_{\mathrm{QF,QD}} = \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

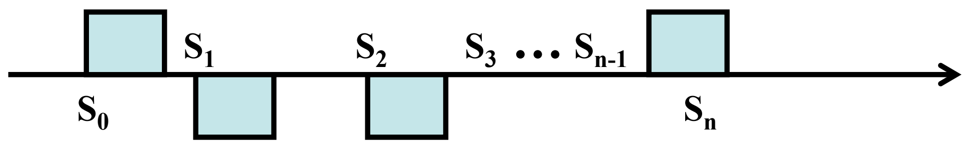
$$f = \frac{1}{KL}$$

- **Note** that the **sign** of K or f is now absorbed inside the symbol
- In the other plane, focusing becomes defocusing and vice versa

Joining elements

For an arbitrary number of transport elements, each with a constant, but different, K_n, we have

$$M(s_n|s_0) = \underbrace{M(s_n|s_{n-1}) \ldots \underbrace{M(s_3|s_2) \cdot \underbrace{M(s_2|s_1) \cdot \underbrace{M(s_1|s_0)}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_1}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_2}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_3}}}_{\text{from } \mathbf{s_0} \text{ to } \mathbf{s_n}}$$



$$\boxed{\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = M(s_n|s_0) \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}}$$

Thus by breaking up the parameter $K(s)$ into piecewise constant chunks, $K(s)=\{K1, K2, \ldots Kn\}$, we have found a useful method for finding the particle transport equation through a long section of beamline with many elements.

Betatron motion

- **General solution of Hill's eq. for on-momentum linear motion of a particle:**

$$\boxed{\begin{aligned} x(s) &= \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\phi_0) \\ x'(s) &= \sqrt{\frac{\epsilon}{\beta(s)}}(\sin(\psi(s)+\phi_0)+\alpha(s)\cos(\psi(s)+\phi_0)) \end{aligned}}$$

Twiss parameters: $\beta(s), \quad \alpha(s) = -\frac{\beta'(s)}{2}, \quad \gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$

Betatron phase: $\psi(s) = \int \frac{ds}{\beta(s)}$

Tune (in a ring): number of betatron oscillations per turn, or phase advance per turn in units of 2π $\quad Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$

Phase space

- The twiss parameters *α, β, γ* have a geometric meaning

Phase space ellipse

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = \epsilon$$

$$\text{Area of ellipse} = \pi\epsilon$$

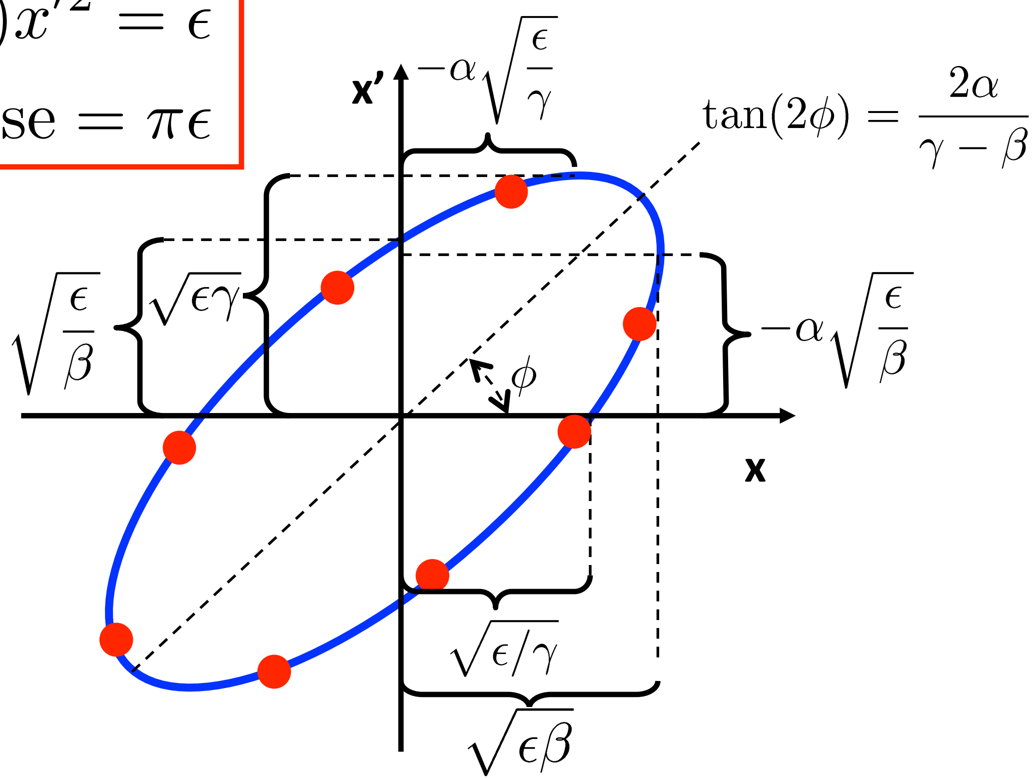
Courant-Snyder invariant: ϵ

Beam envelope

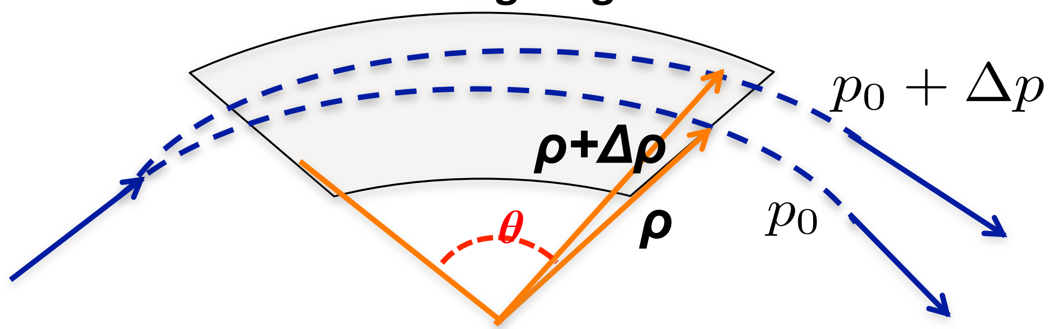
$$E(s) = \sqrt{\epsilon\beta(s)}$$

Beam divergence

$$A(s) = \sqrt{\epsilon\gamma(s)}$$



Off-momentum particles



Recall that the **magnetic rigidity** is $B\rho = \dfrac{p}{q}$ and for off-momentum particles:

$$B(\rho + \Delta\rho) = \frac{p_0 + \Delta p}{q} \Rightarrow \frac{\Delta\rho}{\rho} = \frac{\Delta p}{p_0}$$

Consider the effective length of the dipole unchanged:

$$\theta\rho = l_{\text{eff}} = \text{const.} \Rightarrow \rho\Delta\theta + \theta\Delta\rho = 0 \Rightarrow \frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = -\frac{\Delta p}{p_0}$$

Off-momentum particles get different deflection (different orbit):

$$\boxed{\Delta\theta = -\theta\frac{\Delta p}{p_0}}$$

Dispersion

❑ Inhomogeneous Hill's equation

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

The solution is a sum of the **homogeneous** equation (on-momentum) and the **inhomogeneous** (off-momentum):

$$x(s) = x_\beta(s) + D(s)\frac{\Delta p}{p_0}$$

Dispersion function: $D(s)$

Dispersion equation: $D''(s) + K(s)D(s) = \frac{1}{\rho}$

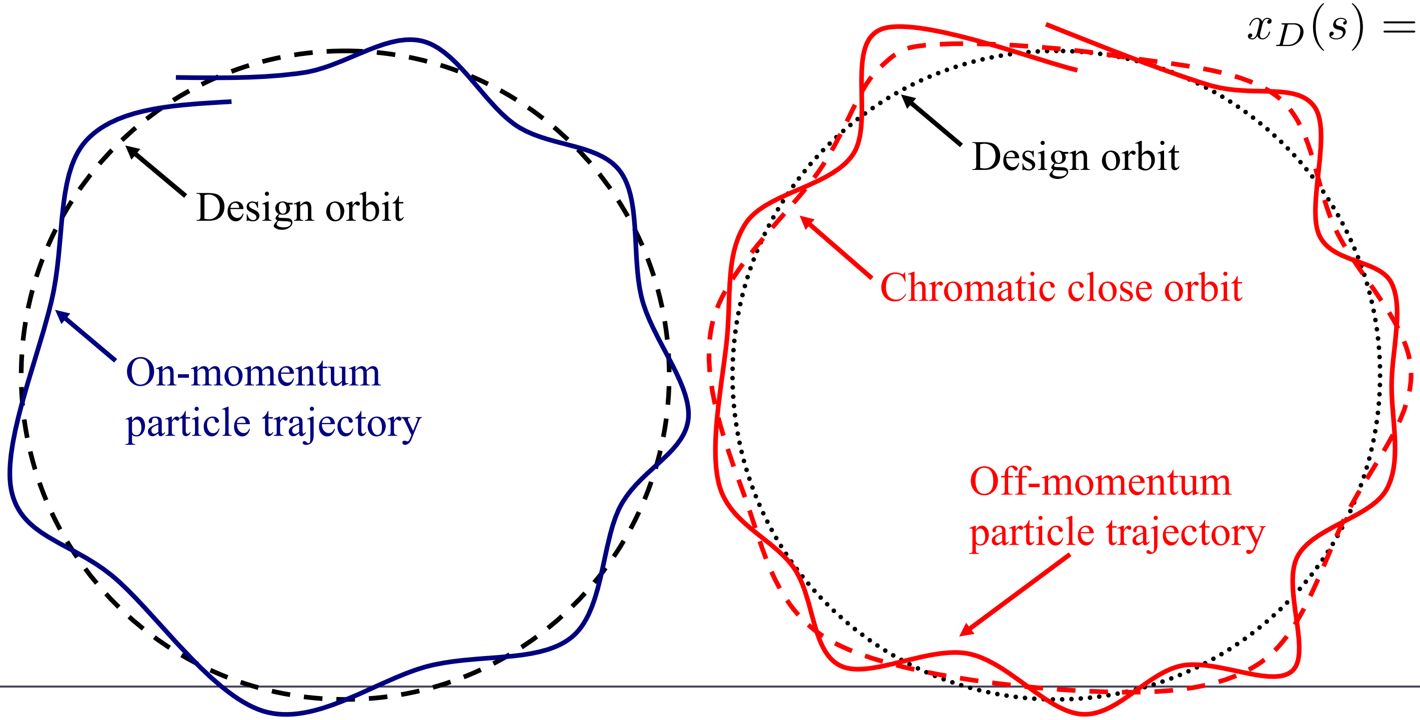
Dispersion and orbit

Meaning:

$x_D(s)$ describes the deviation of the closed orbit for an off-momentum particle with $p=p_0 + \Delta p$

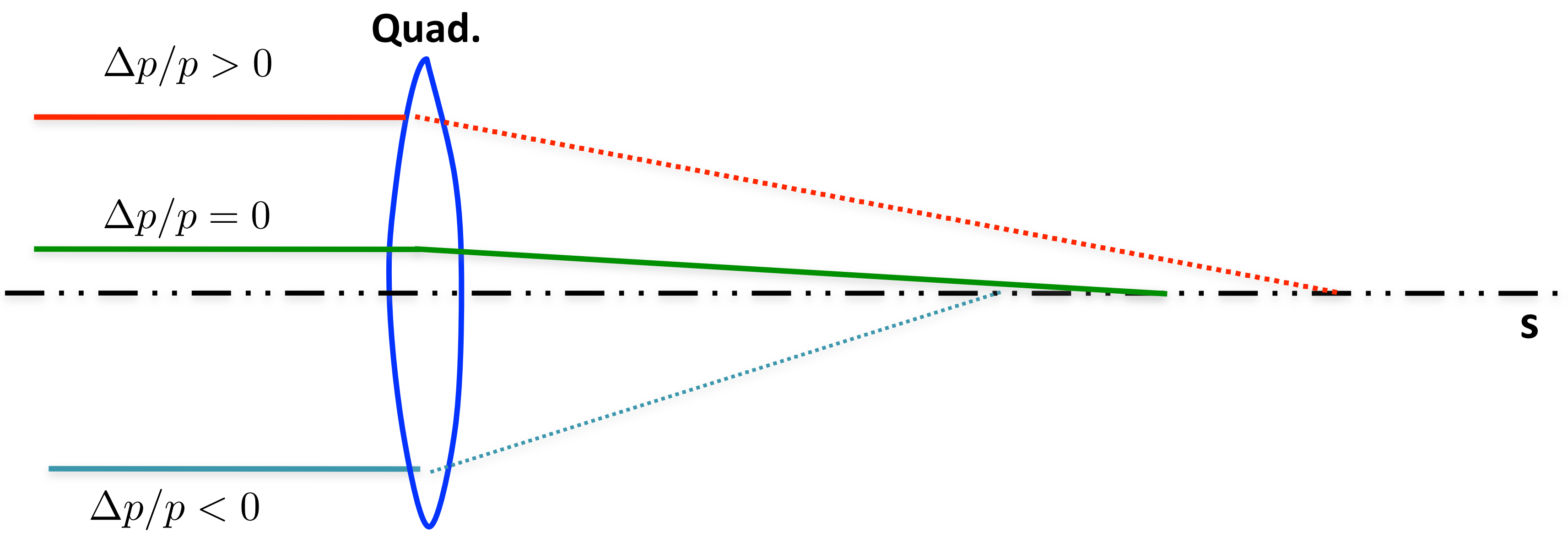
x_β describes the betatron oscillation around the new chromatic close orbit

$$x_D(s) = D(s)\frac{\Delta p}{p_0}$$



Chromaticity

- Off-momentum particle gets different focusing

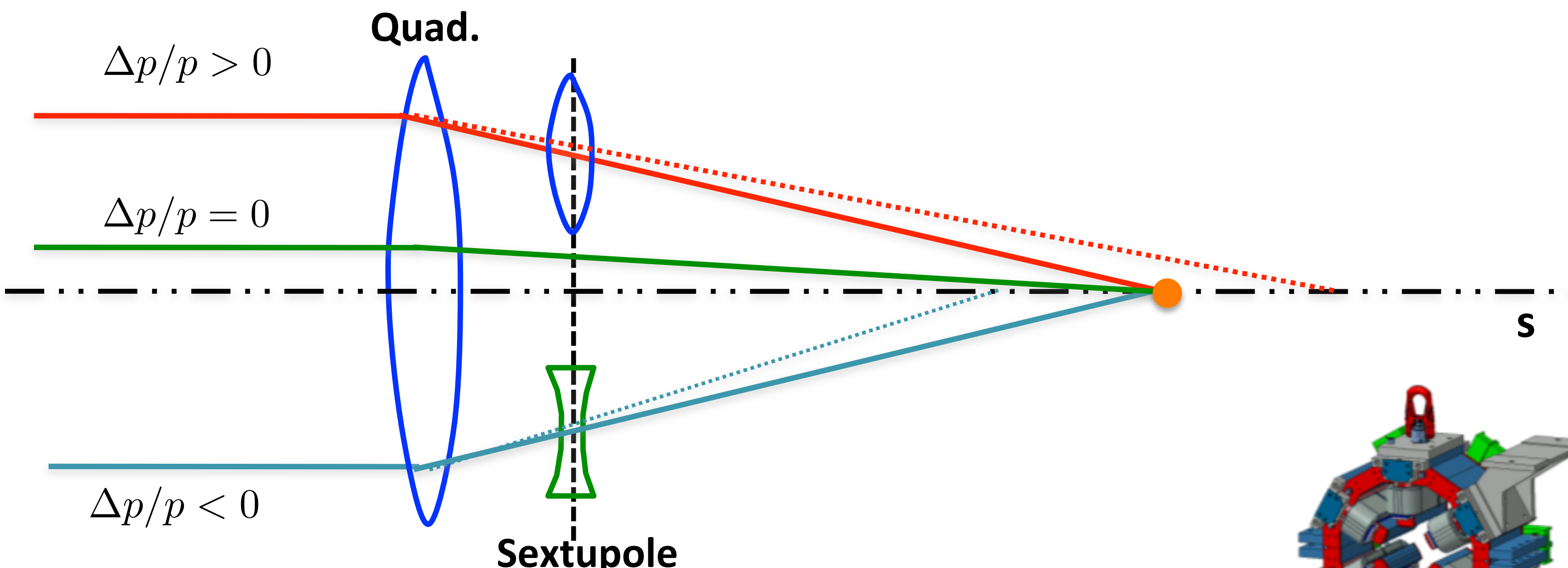


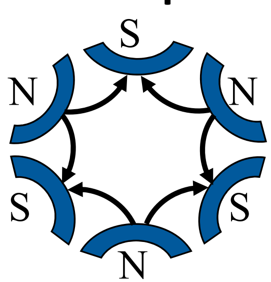
- Chromaticity acts like a quadrupole error (optical aberration), $\Delta K = -K\dfrac{\Delta p}{p}$ and leads to a tune spread:

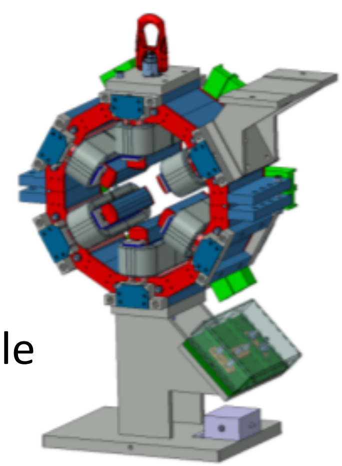
$$\boxed{\xi = \frac{\Delta Q}{\frac{\Delta p}{p}}, \ \text{first order chromaticity}}$$

Chromaticity

- How to correct chromaticity?







ELENA ring sextupole

ELENA ring optics

- **Betatron and dispersion functions**

Good tunability in the range $2 < Q_x < 2.5$ and $1 < Q_y < 1.5$

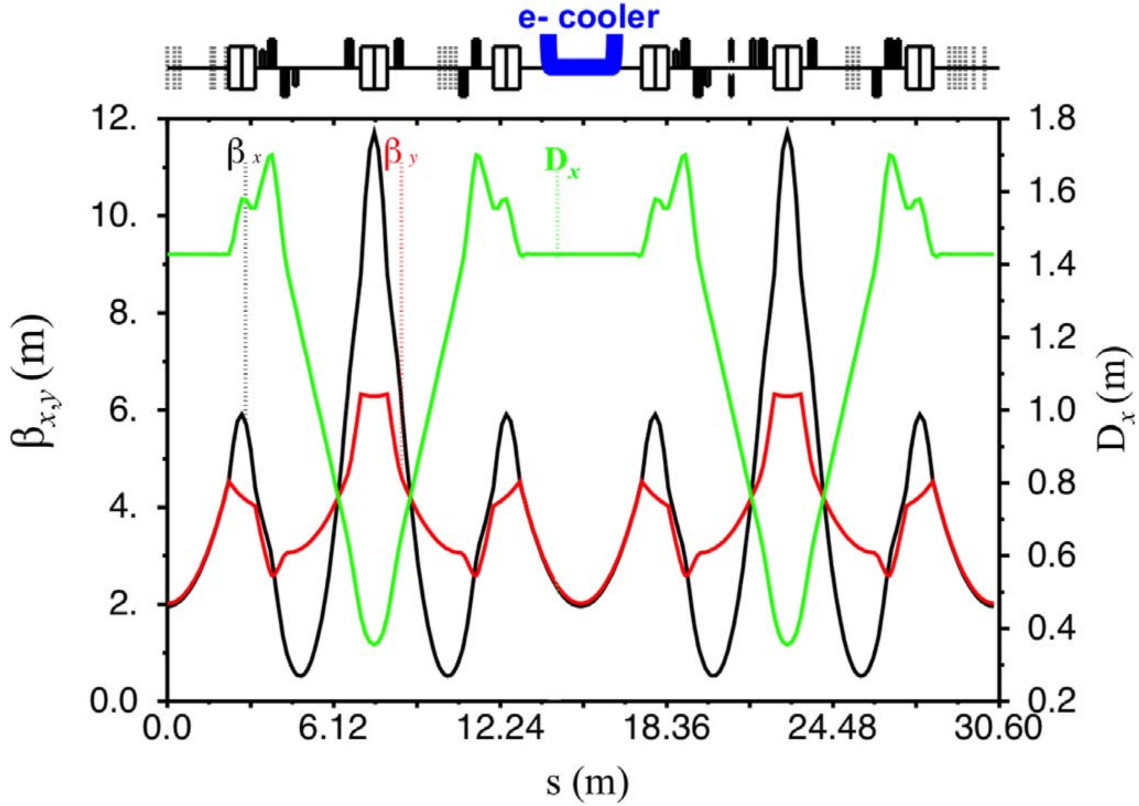
Hexagonal lattice

Periodicity of two:
- e-cooling section + 2 standard sections
- Injection section + 2 standard sections

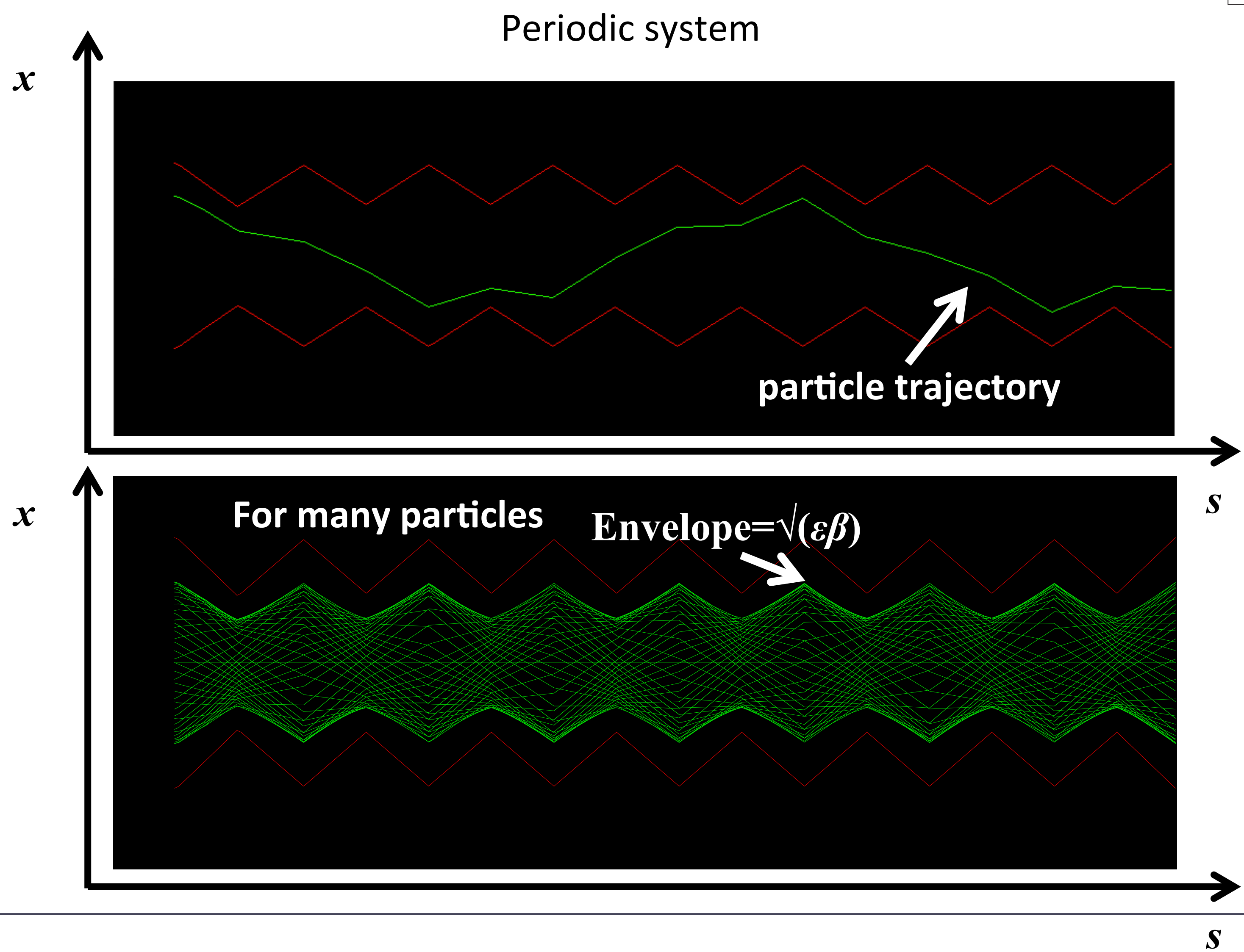
3 families of quadrupoles (each of 4 members)

2 skew quadrupoles

2 families of sextupoles (each of two members)



Beam


Periodic system
x
particle trajectory
s
x
For many particles
Envelope=√(εβ)
s

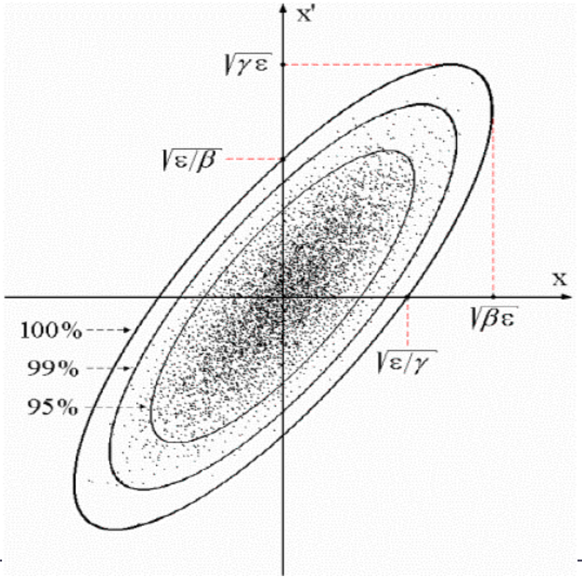
Beam

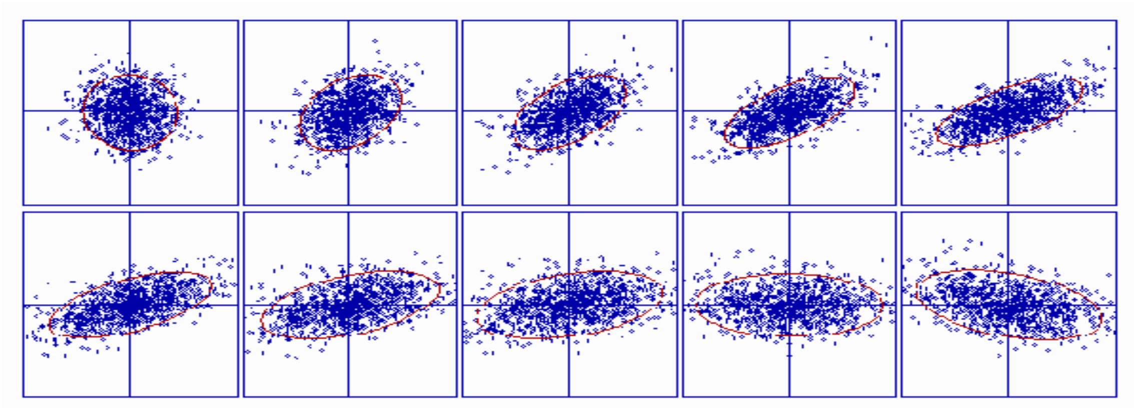
❑ Ensemble of particles

Beam is a set of millions/billions of particles (N)

For example, a Gaussian transverse distribution has a Gaussian density profile in phase space

$$f(x,x',y,y') = \frac{N}{A} exp\left(-\frac{\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2}{2\epsilon_{x,\mathrm{rms}}} + \frac{\gamma_y y^2 + 2\alpha_y y y' + \beta_y y'^2}{2\epsilon_{y,\mathrm{rms}}}\right)$$





Emittance

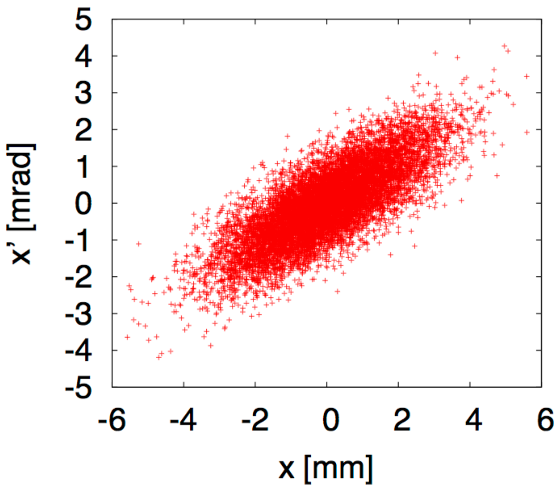
❑ Statistical definition

Let's consider first a 2D $x - x'$ beam phase space for an ensemble of particles. We need to characterise the spread of particles in phase space

Emittance is a measure of the phase space area occupied by a beam

$$\boxed{\epsilon_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}}$$

$$\begin{cases} \langle x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} (x_i - \langle x \rangle)^2 \\ \langle x'^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} (x'_i - \langle x' \rangle)^2 \\ \langle x x' \rangle = \frac{1}{N} \sum_{i \neq j}^{N} \sum_{j=1}^{N} (x_i - \langle x \rangle)(x'_j - \langle x' \rangle) \end{cases}$$



Emittance

❑ Emittance and Courant-Snyder Invariant

Particle orbit in terms of lattice Twiss parameters

$$(i = 1 \text{ to } N) \begin{cases} x_i(s) = \sqrt{\epsilon_i \beta(s)} \cos(\psi(s) + \phi_i) \\ x_i'(s) = -\sqrt{\epsilon_i/\beta(s)} [\alpha(s) \cos(\psi(s) + \phi_i) + \sin(\psi(s) + \phi_i)] \end{cases}$$

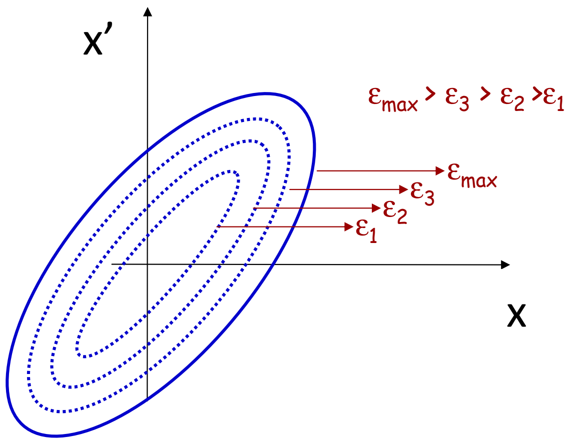
For a matched beam, for each ε_i, particles are uniformly distributed around the ellipse

$$\langle x^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \epsilon_i \beta(s) \cos^2(\psi(s) + \phi_i) = \beta(s) \langle \epsilon \rangle$$

$$\langle x'^2 \rangle = \gamma(s) \langle \epsilon \rangle$$

$$\langle x x' \rangle = \alpha(s) \langle \epsilon \rangle$$

$$\boxed{\epsilon_{rms} = \frac{1}{N} \sum_{i=1}^{N} \epsilon_i = \langle \epsilon \rangle}$$



Courant-Snyder Invariant can be seen as a single particle emittance

Emittance (other definitions)

- **Normalised emittance:**

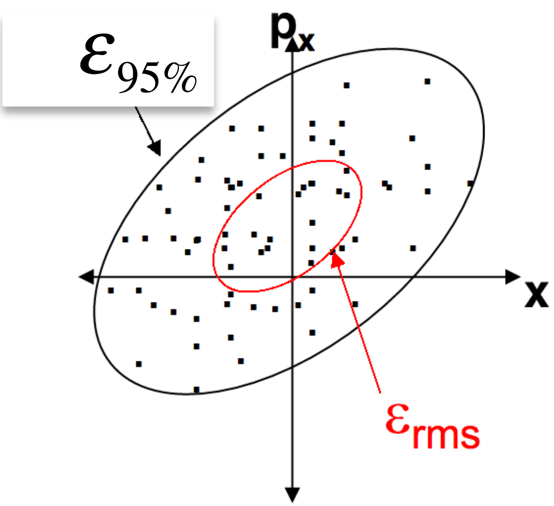
$$\text{Area enclosed} = \iint dx\,dx' = \frac{1}{p}\iint dx\,dp_x = \pi\epsilon$$

The geometric emittance, as defined before, decreases as $1/p$ or $1/(\beta\gamma)$, so when a beam is accelerated, it is not ϵ that is conserved, but the quantity

$$\epsilon_n = \beta_{rel}\gamma_{rel}\epsilon$$
$$\beta_{rel} = v/c$$
$$\gamma_{rel}: \quad \text{Lorentz factor}$$

- **Percentile emittance:**

e.g. 95% emittance, which defines the area of ellipse that contains 95% of the beam. It is a convenient definition when we have to deal with irregular beam distributions



Liouville's theorem

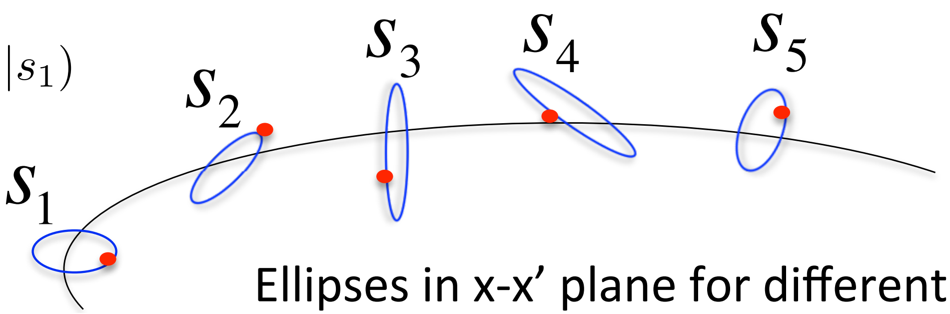
Joseph Liouville

In a Hamiltonian system, i.e. in the absence of collisions or dissipative processes, the density in phase space along the trajectory is invariant

As particle moves along the orbit the shape and position of the phase space ellipse change according to $\beta(s)$, but the area remains constant.

$$M(s_5|s_1) = M(s_5|s_4)\cdots M(s_3|s_2)M(s_2|s_1)$$

$$\det(M) = 1$$



Ellipses in x-x' plane for different s

Poincare invariant:

$$\text{Area enclosed} = \iint dx dx' = \frac{1}{p} \iint dx dp_x = \pi\epsilon$$

(in 2-D. It can be extended to the beam volume)

Javier Resta Lopez

Emittance dilution

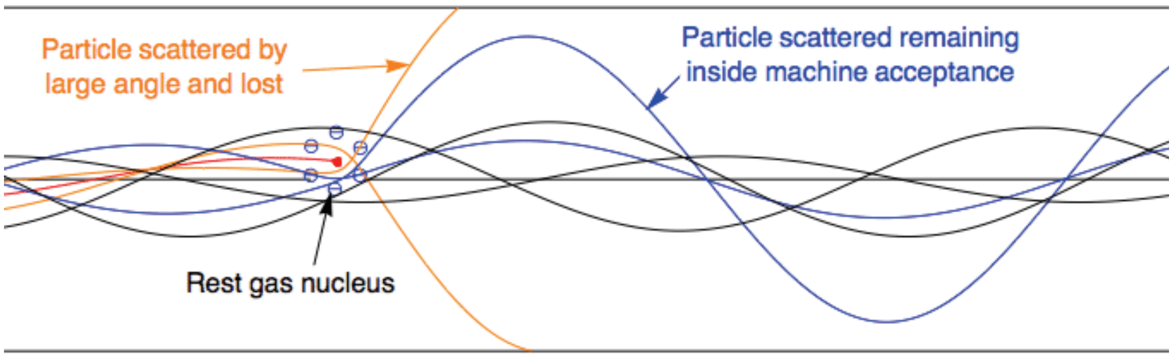
Hamiltonian

- Accelerator system
  - Beam mismatch
  - Nonlinear optics
  - Errors,misalignments
- Collective effects
  - Space charge
  - Coherent synchrotron radiation (CSR)
  - Wakefield (impedance)
- Two-beam effects
  - Beam-beam
  - Electron cloud for positively charged beam
  - Ion effects for electron beams

Non-Hamiltonian

- Synchrotron radiation
- Scattering
  - Residual gas scattering
  - Intrabeam scattering
  - Touschek scattering

Rest gas scattering



- Emittance growth caused mainly by Multiple Coulomb Scattering

$$\frac{d\epsilon_{rms}}{dt} = 2\pi\langle\beta_\perp\rangle n_{ms}\ln\left(\frac{280}{\alpha}\right) r_e^2 \frac{(m_e c^2)^2}{\beta c p^2}$$

Relativistic velocity factor: $\beta = v/c$

Average betatron function: $\langle\beta_\perp\rangle = 1/2(\langle\beta_x\rangle + \langle\beta_y\rangle)$

Multiple scattering density: $n_{ms} = \sum_i n_i \frac{Z^2 \ln\left(280/\left(\alpha (AZ)^{1/3}\right)\right)}{\ln(280/\alpha)}$

N. Madsen, CERN/PS/DI Note 99-06, (1999)

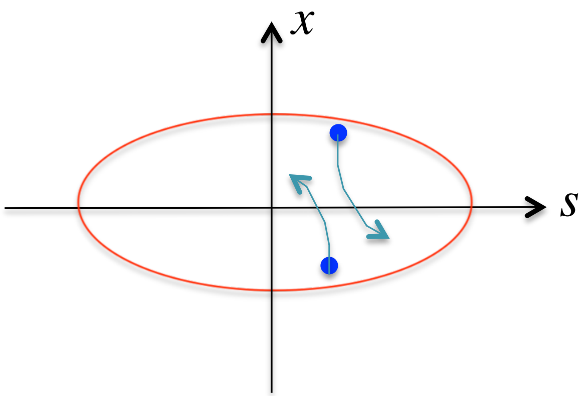
IBS

- Multiple small-angle Coulomb scatterings of charged particles within the beam itself
- Exchange of energy between the transverse and longitudinal degrees of freedom, leading to the growth of the beam phase dimensions
- **Emittance growth rates:**

$$\boxed{\frac{1}{\tau_x}, \frac{1}{\tau_y}, \frac{1}{\tau_p} \propto \frac{r_p^2 c}{32\pi\sqrt{\pi}\beta^3\gamma^4\epsilon_x\epsilon_y\sigma_p}\lambda}$$

$$\lambda = \begin{cases} N/C & \text{for coasting beams} \\ N_b/(2\sqrt{\pi}\sigma_s) & \text{for bunched beams} \end{cases}$$

where N is the number of particles in the beam,
and N_b is the number of particles per bunch;
C is the circumference of the ring, σ_s the bunch length, and
σ_p the momentum spread



Beam cooling

- Beam cooling means reduction of beam temperature

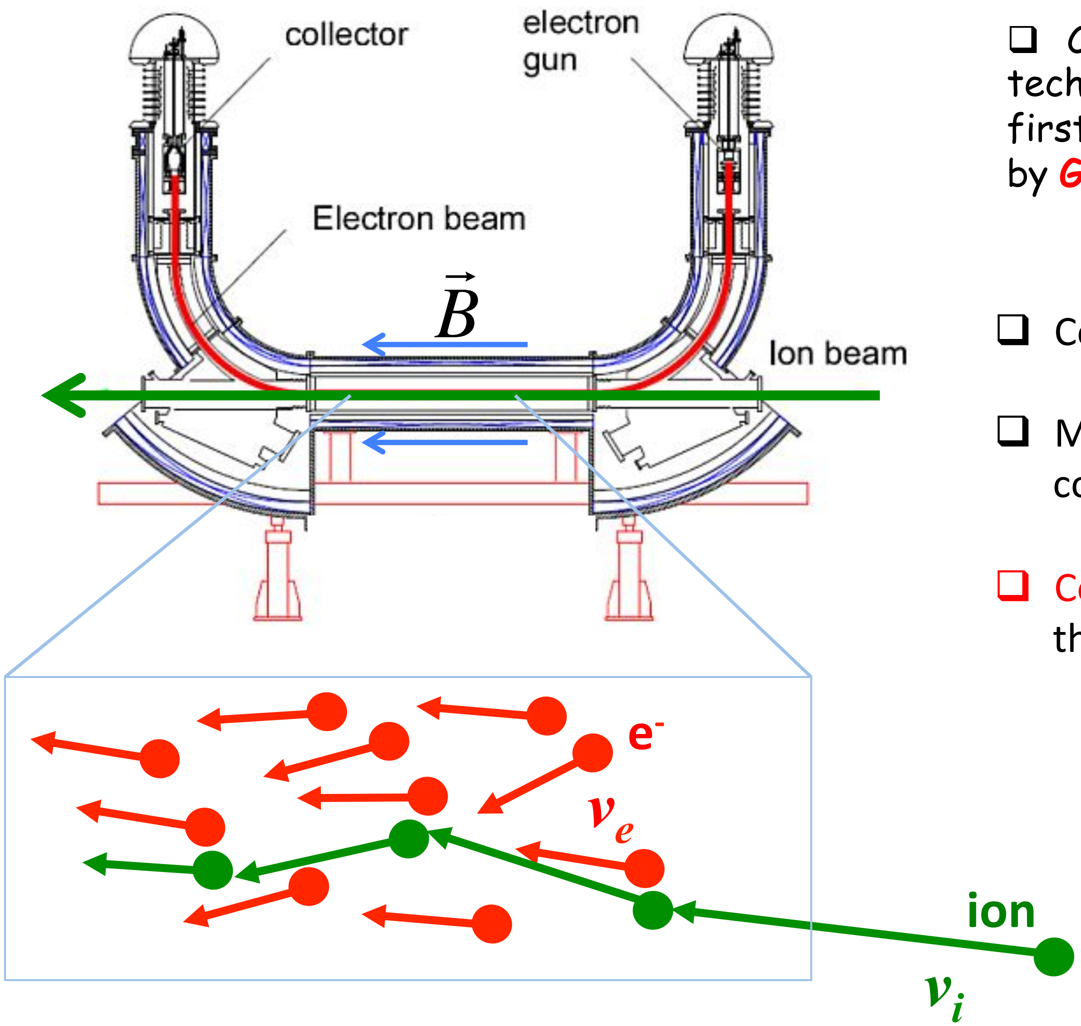
- In this context, temperature is equivalent to terms as phase space volume, emittance and momentum spread

- Beam cooling techniques are non-Liouvillean, i.e. violate the assumption of conservative forces

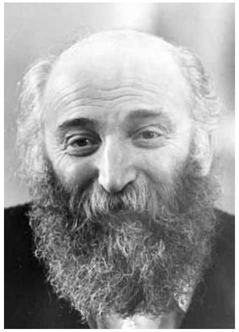
Benefits of beam cooling

- Improved beam quality
  - Precision experiments
  - Luminosity increase (in colliders)
  - Increase lifetime
- Compensation of heating
  - Experiments with internal target
  - Colliding beams
  - Other scattering effects, e.g. IBS
- Intensity increase by accumulation
  - Weak beams from the source can be enhanced
  - Secondary beams (antiprotons, rare isotopes)

Electron cooling



- Consolidated technique, first proposed by **G. Budker** in 1966



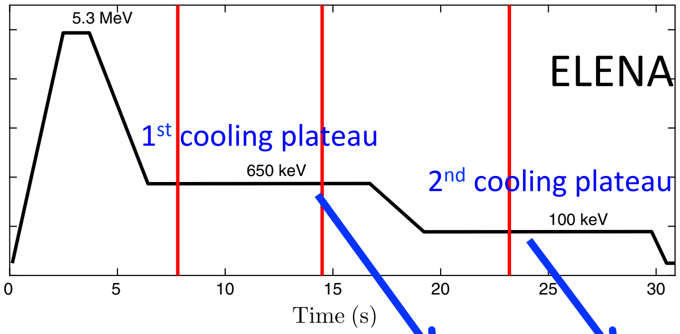
- Cold electrons interacting with hot ions
- Momentum transfer by Coulomb collisions
- Cooling force results from energy loss in the co-moving gas of free electrons

Equilibrium:

$$k_B T_i = k_B T_e \quad \text{or} \quad m_i v_i^2 = m_e v_e^2$$

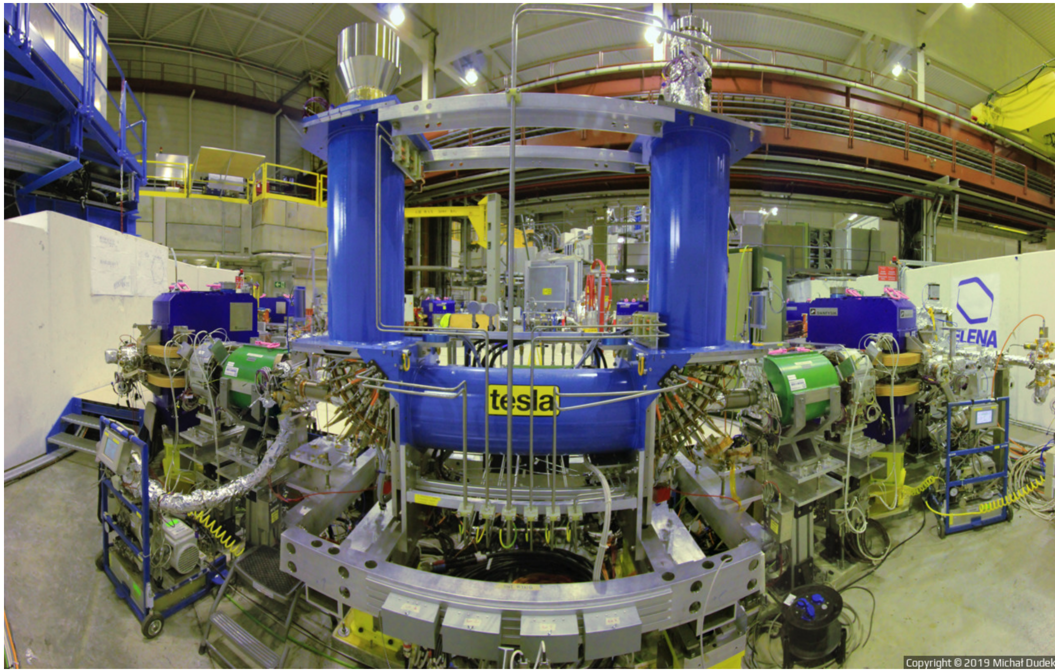
$$v_i = v_e \sqrt{\frac{m_e}{m_i}}$$

Electron cooling in ELENA



ELENA magnetic cycle

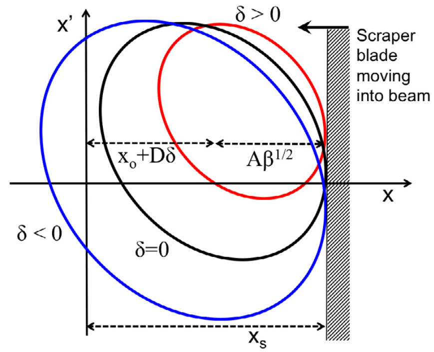
pbar kinetic energy	650 keV	100 keV
Electron beam energy (eV)	355	55
Electron current (mA)	5	2
Electron beam density (m^{-3})	1.38×10^{12}	1.41×10^{12}
B_{gun} (G)	1000	
B_{drift} (G)	100	
Expansion factor	10	
Cathode radius (mm)	8	
Electron beam radius (mm)	25	
β_x, β_y, D_x (m)	2.103, 2.186, 1.498	
Flange-to-flange length (mm)	2330	
Drift solenoid length (mm)	1000	
Effective length (mm)	700	

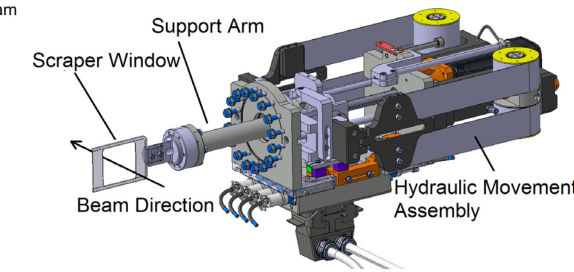


More details about cooling in L. V. Joergenson's talk, this workshop

Electron cooling in ELENA

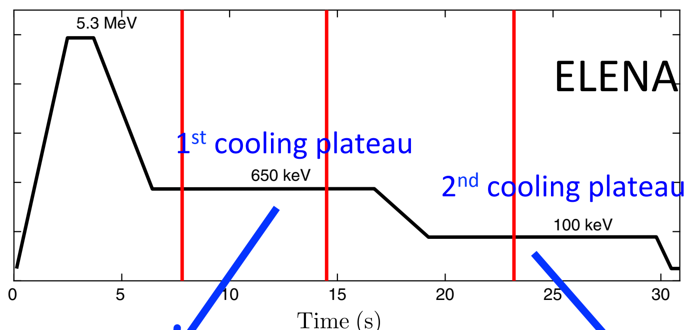
- **Emittance measurement by scraping**





ELENA scraper

First evidence of significant electron cooling in ELENA



ELENA magnetic cycle

J. Hunt et al., NIMA 896 (2018) 139
J. Hunt et al., PRST-AB 23 (2020) 032802

	$t = 7.8$ s	$t = 14.5$ s	Change
ϵ_y (mm mrad)	1.59 ± 0.02	1.15 ± 0.02	$28\% \pm 2\%$
y_0 (mm)	-2.88 ± 0.03	-2.89 ± 0.03	-0.01 ± 0.06
ϵ_x (mm mrad)	3.6 ± 0.27	0.70 ± 0.05	$81\% \pm 10\%$
x_0 (mm)	-4.05 ± 0.04	-4.22 ± 0.04	-0.17 ± 0.08

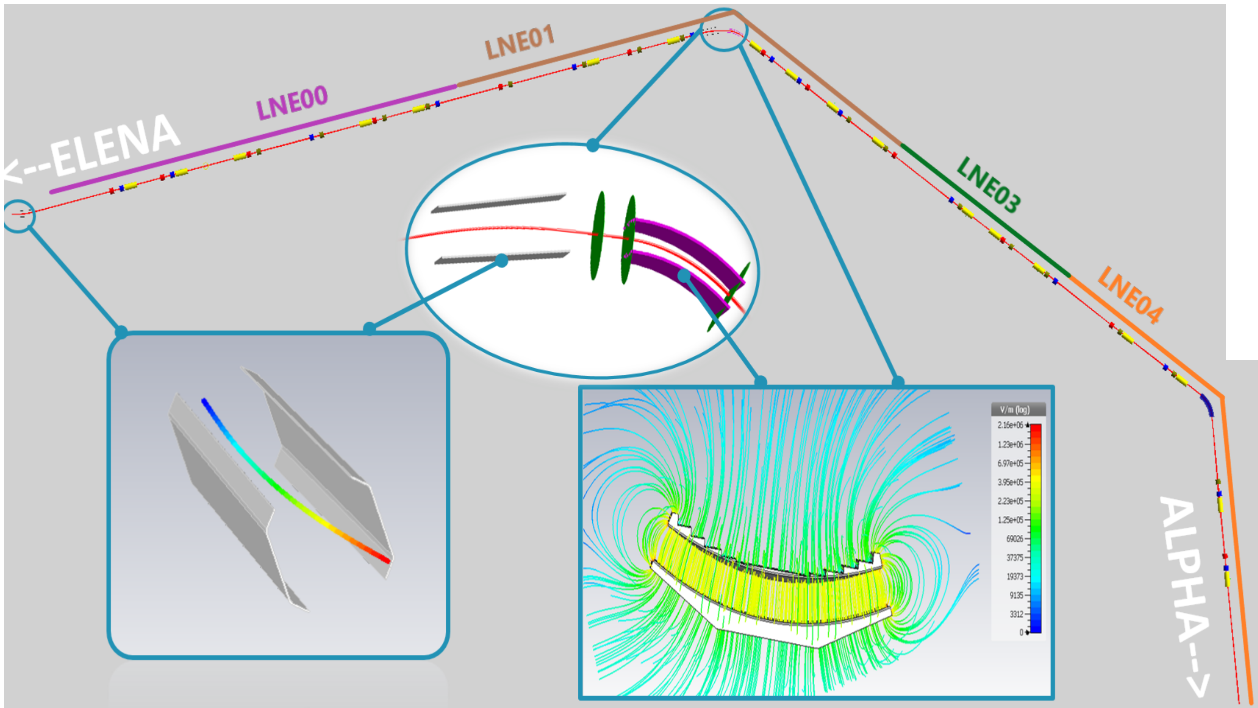
	Off	On	Change
ϵ_y (mm mrad)	2.55 ± 0.03	0.53 ± 0.01	$79\% \pm 2\%$
y_0 (mm)	-2.08 ± 0.03	-2.03 ± 0.03	0.05 ± 0.06
ϵ_x (mm mrad)	2.5 ± 0.20	0.55 ± 0.04	$78\% \pm 10\%$
x_0 (mm)	-3.67 ± 0.04	-3.91 ± 0.04	-0.24 ± 0.08

Electrostatic transfer lines

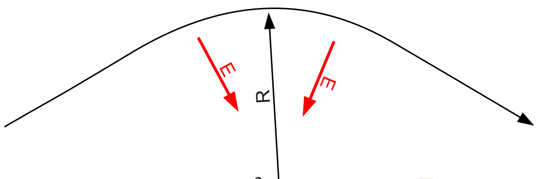
- Electrostatic or magnetic beamline?
- At low velocities electric fields may be more efficient
- 100 keV is still in the reachable range for electrostatic elements
- Advantages of electrostatic:
  - No hysteresis
  - Better stability. Easy field shaping
  - Low power consumption
  - Cheap power supplies
  - Good magnetic shielding possibilities
- Disadvantages:
  - Safety with high-voltage
  - Interlocking against sparks

Electrostatic deflectors

G4beamline model of TL from ELENA to ALPHA



Fast deflector   Bend + fringe field effects

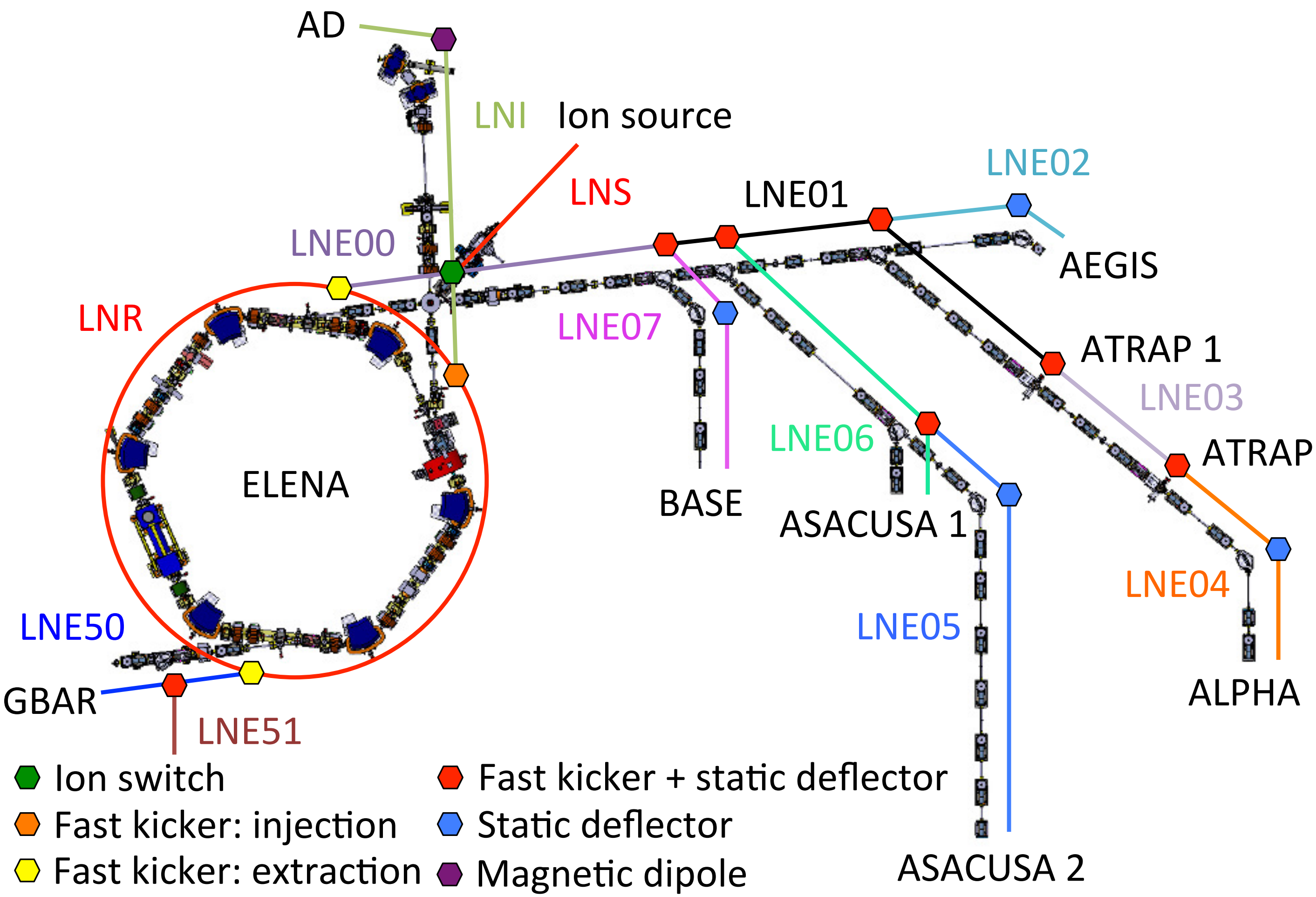


$$F = q\,E = m\,a = m\frac{v^2}{R} \rightarrow R = \frac{2}{E}\frac{E_{kin}}{q}$$

Bending radius depends only on the $\mathbf{E_{kin}/q}$ of the particle. Independent of mass.

Courtesy of V. Rodin

ELENA Electrostatic transfer lines



Experiments, next talks …

Summary

- Basic definitions (briefly) reviewed for both linear longitudinal and transverse beam dynamics
  - Starting from Lorentz equation
  - Techniques to accelerate particles (linacs, circular accelerators)
  - Magnets to guide the particle trajectory (dipoles, quadrupoles)
  - Single-particle beam dynamics
  - Hill's equation and betatron motion
  - Matrix description (maps)
  - Off momentum particles: Dispersion function, Chromaticity
  - Multiparticle-beam
  - Transverse phase space: C-S invariant, emittance, Liouville's Theorem
  - Emittance dilution mechanism
  - Beam cooling; electron cooling
  - Electrostatic elements
- New accelerator facilities, such as ELENA, play a key role in antimatter research

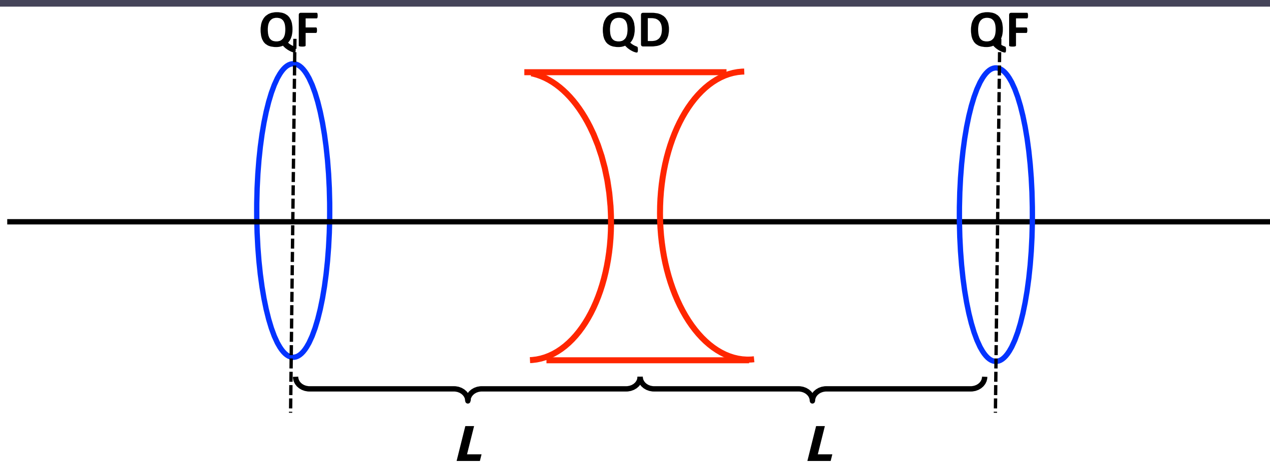
Thank you!

Normalised emittance

- Apply some acceleration along *z* to all particles in the bunch
  – P_x is constant
  – P_z increases
  – $x' = P_x/P_z$ decreases!
- So the bunch emittance decreases
  – This is an example of something called *Liouville's Theorem*
  – ~"Emittance is conserved in (x, P_x) space"
- Define normalised emittance

$$\varepsilon_n = \frac{<p_z>}{m}\varepsilon$$

Basic blocks: FODO Cell



Symmetric transfer matrix from center to center of focusing quads

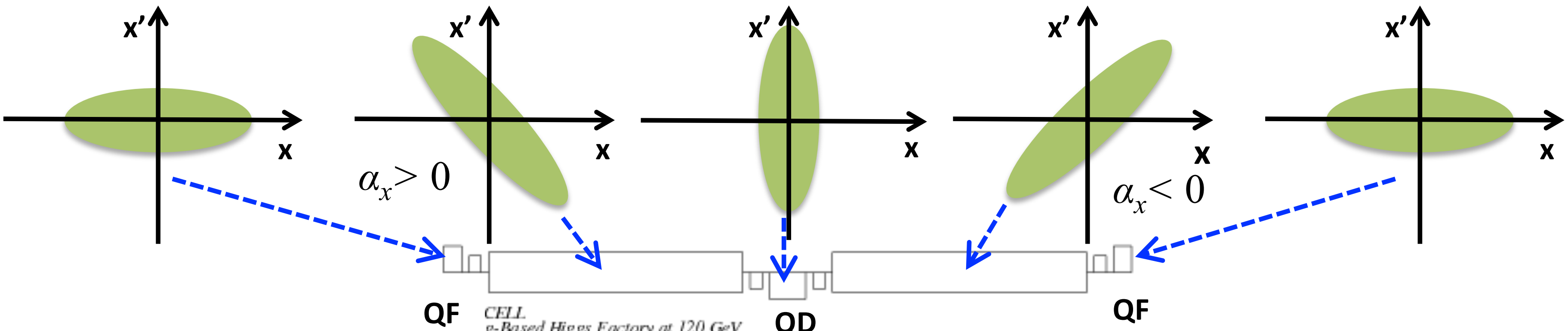
$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

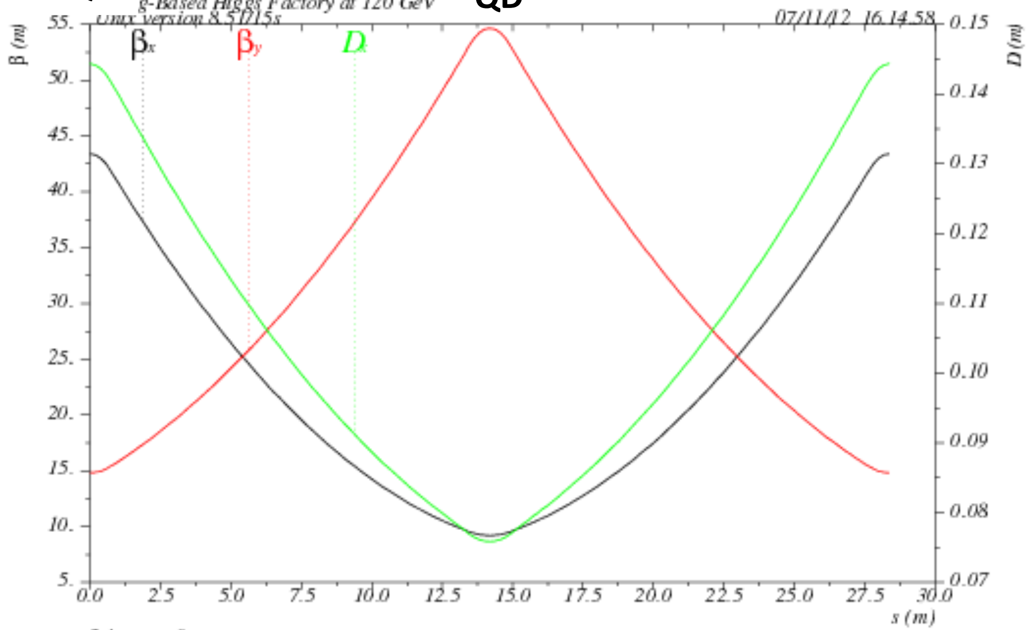
$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ -\frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}}$$

Evolution of phase space

- A large β-function corresponds to a large beam size and a small beam divergence



In the middle of quad. QF , β_x is maximum (β_y minimum), and $\alpha_{x,y}=0$



In the middle of quad. QD, β_y is maximum (β_x minimum), and $\alpha_{x,y}=0$

Nonlinearities

In reality, some lattices have significant higher order terms

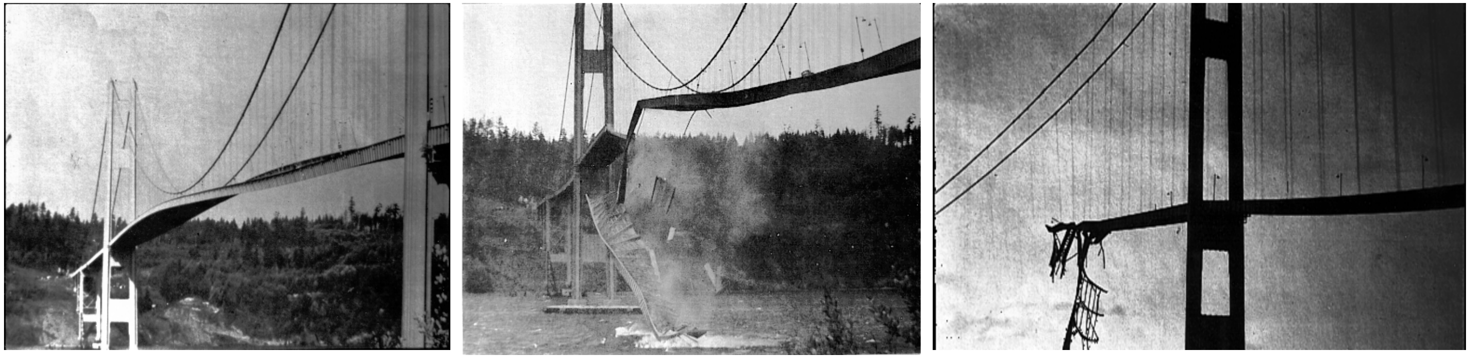
$$x'' + K(s)x = O(x^2) + \ldots$$

and magnetic imperfections, e.g. dipole errors

$$x'' + K(s)x = \delta(s - s_0)\theta_{\mathrm{error}}$$

which can drive **resonances**

Tacoma Narrow bridge 1940



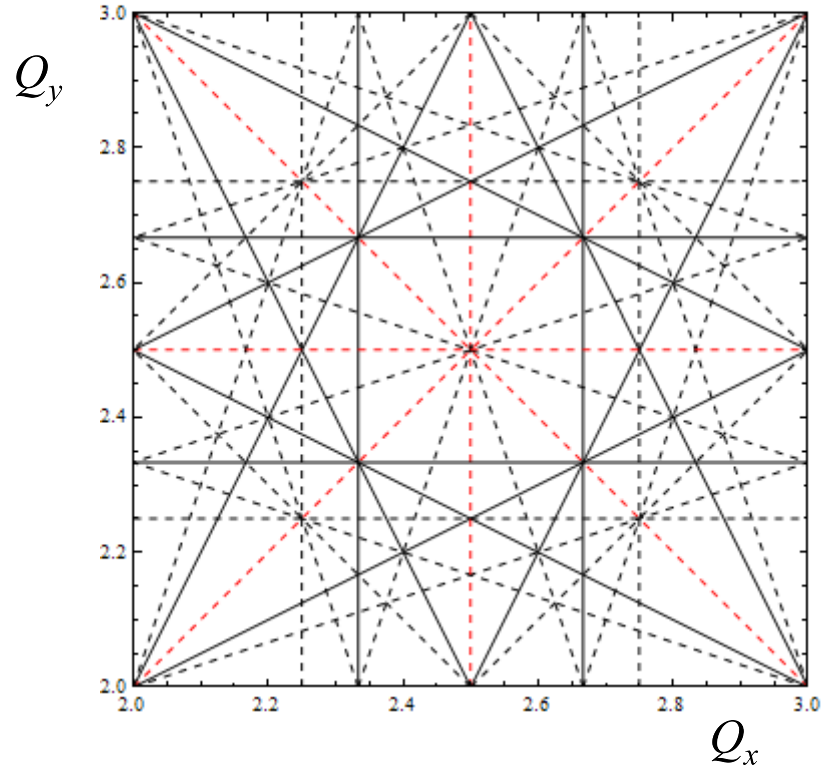
(Excitation by strong wind on the eigenfrequencies)

Resonances

Hill's equation is quasiharmonic, and whenever we have a harmonic system, the danger of exciting a resonance exists. Multiple sources of resonant driving terms exist in accelerators:

Tune diagram

- Linear magnet imperfections
- Time varying fields
- Nonlinear magnets
- Collective effects
- etc., etc.



$$lQ_x + mQ_y = r$$

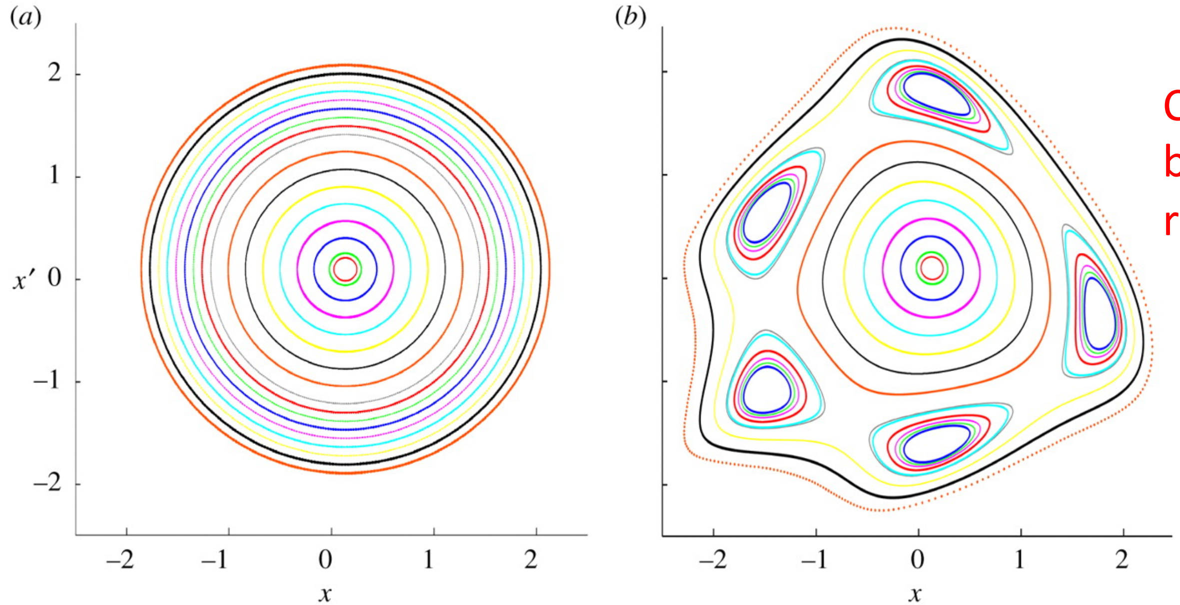
where (l, m, r) are integers

Nonlinearities

- Example:

Sextupoles are the most common magnet nonlinearities in accelerators

Phase space plot of particle motion close to a fifth-order resonance



Chaotic regimen between the resonant islands!

Linear machine **With a sextupole element**

(Courtesy of W. Herr and E. Forest)