Helicity Distributions and Orbital Angular Momentum at Small x

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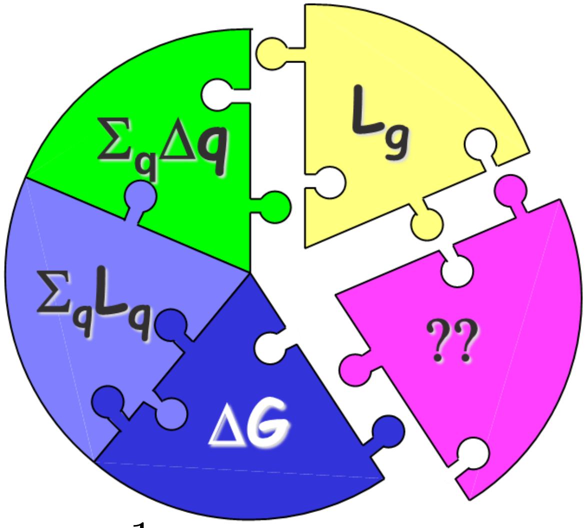
Based on work done with Dan Pitonyak and Matt Sievert (2015-2018) and with Florian Cougoulic (2019).

Outline

- Goal: understanding the proton spin coming from helicities and OAM of small-x quarks and gluons.
- Quark Helicity ("simplify-evolve-solve" prescription):
    - Quark helicity distribution at small x
    - Small-x evolution equations for quark helicity
    - Small-x asymptotics of quark helicity
- Gluon Helicity:
    - Gluon helicity distribution at small x
    - Small-x evolution equations for gluon helicity
    - Small-x asymptotics of gluon helicity TMDs
- Quark and Gluon OAM at small x: results.
- Valence quark transversity at small x.
- Conclusions

Introduction and goals

Proton Spin Puzzle



- Helicity sum rule (Jaffe-Manohar form):

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

with the net quark and gluon spin

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x,Q^2) \qquad S_g(Q^2) = \int_0^1 dx\, \Delta G(x,Q^2)$$

- The helicity parton distributions are

$$\Delta f(x,Q^2) \equiv f^+(x,Q^2) - f^-(x,Q^2)$$

with the net quark helicity distribution

$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

- L_q and L_g are the quark and gluon orbital angular momenta

Our goal

- The goal is to constrain theoretically the amount of proton spin and OAM coming from small x.

- Any existing and future experiment probes the helicity distributions and OAM down to some x_{min} .

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx \, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx \, \Delta G(x, Q^2)$$

$$L_{q+\bar{q}}(Q^2) = \int_0^1 dx \, L_{q+\bar{q}}(x, Q^2) \qquad L_G(Q^2) = \int_0^1 dx \, L_G(x, Q^2)$$

- At very small x (for the proton), saturation sets in: that region likely carries a negligible amount of proton spin. But what happens at larger (but still small) x?

Our goal

- Ultimately the aim is to make predictions for helicity distributions at EIC.
- If the predictions are not too far off, one could extrapolate the theory curves down to $x=0$, getting a (hopefully) good estimate for quark and gluon spin coming from small x.
- For OAM the story is more complicated, but perhaps we may be able to constrain small-x OAM this way too.

Quark Helicity at Small x
(flavor-singlet case)

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph],
arXiv:1610.06197 [hep-ph], arXiv:1610.06188 [hep-ph],
arXiv:1703.05809 [hep-ph], arXiv:1808.09010 [hep-ph]

Quark Helicity TMD

- We start with the definition of the quark helicity TMD with a future-pointing Wilson line staple.

$$g_{1L}^q(x, k_T^2) = \frac{1}{(2\pi)^3} \frac{1}{2} \sum_{S_L} S_L \int d^2 r \, dr^- \, e^{ik\cdot r} \langle p, S_L | \bar{\psi}(0) \, \mathcal{U}[0, r] \, \frac{\gamma^+ \gamma^5}{2} \, \psi(r) | p, S_L \rangle_{r^+=0}$$

- At small-x, in anticipation of the shock-wave formalism, we rewrite the quark helicity TMD as (in A^-=0 gauge for the + moving proton)

$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta-\xi)} \left(\frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-, \infty] \;\; V_{\underline{\zeta}}[\infty, \zeta^-] \, \psi_\beta(\zeta) \right\rangle$$

where the fundamental light-cone Wilson line is

$$V_{\underline{x}}[b^-, a^-] = \mathrm{P} \exp \left\{ ig \int\limits_{a^-}^{b^-} dx^- \, A^+(x^-, \underline{x}) \right\}$$

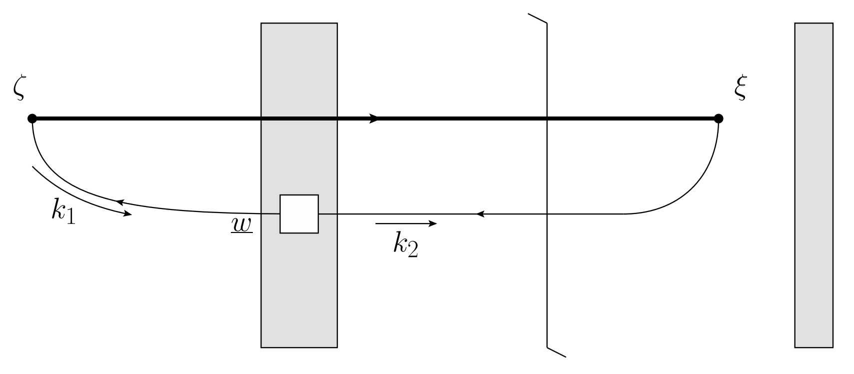
Quark Helicity TMD at Small x

- Only one diagram contributes, giving

$$g_{1L}^{q}(x, k_T^2) = \frac{4N_c}{(2\pi)^6} \int d^2\zeta \, d^2w \, d^2y \, e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \, \frac{\underline{\zeta}-\underline{w}}{|\underline{\zeta}-\underline{w}|^2} \cdot \frac{\underline{y}-\underline{w}}{|\underline{y}-\underline{w}|^2} \, G_{\underline{w},\underline{\zeta}}(zs)$$

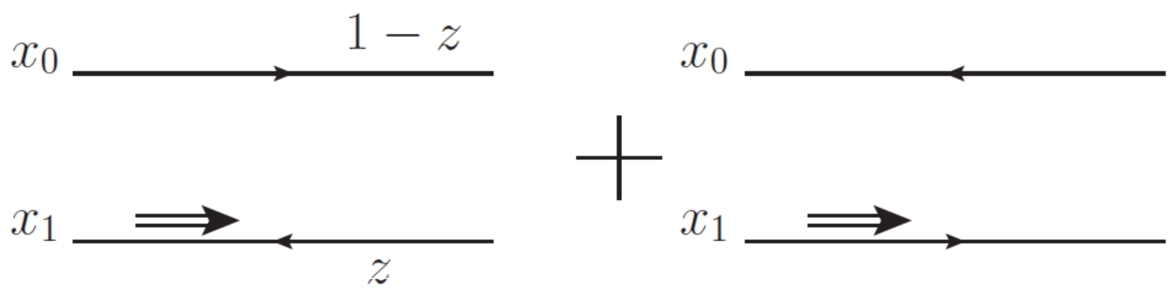
  where $G_{w\zeta}$ is the polarized dipole amplitude (defined on the next slide).

- Here s is the cms energy squared, Λ is some IR cutoff, underlining denotes transverse vectors, z = smallest longitudinal momentum fraction of the dipole momentum out of those carried by the quark and the antiquark



Polarized Dipole

- All flavor-singlet small-x helicity observables depend on one object, "polarized dipole amplitude":



$$G_{10}(z) \equiv \frac{1}{2N_c} \, \text{Re} \left\langle\!\!\left\langle \text{T tr} \left[V_{\underline{0}} V_{\underline{1}}^{pol \, \dagger} \right] + \text{T tr} \left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle (z)$$

unpolarized quark

polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

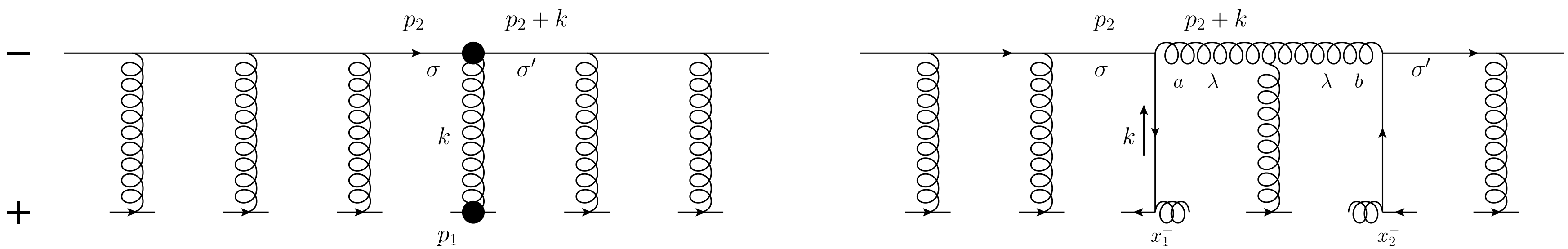
$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle (z) \equiv zs \left\langle \mathcal{O} \right\rangle (z)$$

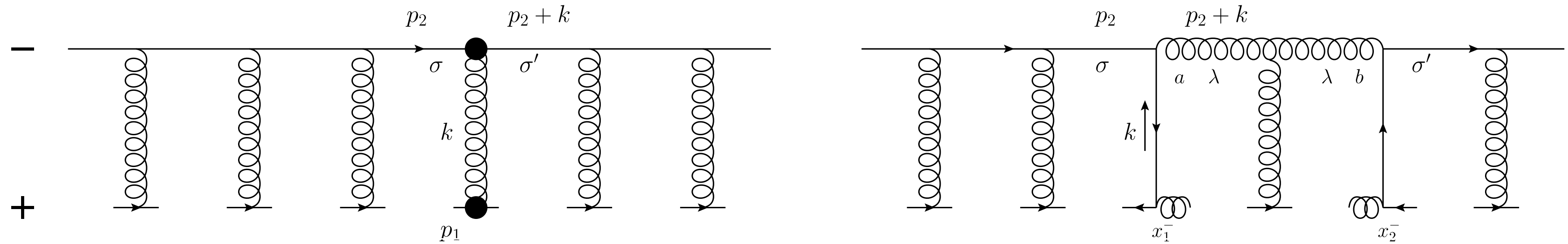
Polarized fundamental “Wilson line”

- To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized “Wilson line” V^{pol}, which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.



- At the leading order we can either exchange one non-eikonal t-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two t-channel quarks, as shown above.

Polarized fundamental "Wilson line"



- In the end one arrives at (KPS '17; YK, Sievert, '18; cf. Chirilli '18)

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- \, V_{\underline{x}}[+\infty, x^-] \, F^{12}(x^-, \underline{x}) \, V_{\underline{x}}[x^-, -\infty]$$

$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[+\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \, \gamma^+ \, \gamma^5 \right]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty].$$

- We have employed an adjoint light-cone Wilson line

$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ig \int\limits_{a^-}^{b^-} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$

- Note the simple physical meaning of the first term:

$$-\vec{\mu} \cdot \vec{B} = -\mu_z \, B_z = \mu_z \, F^{12}$$

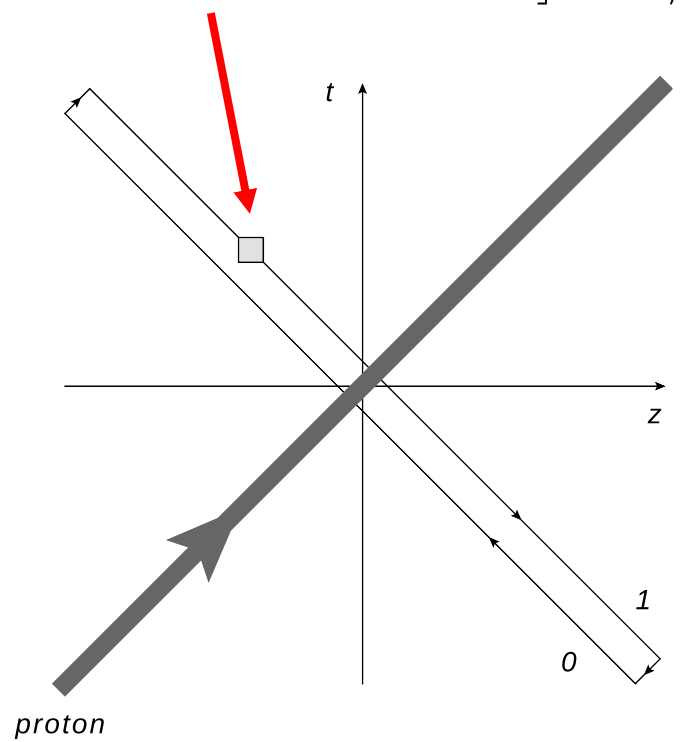
Polarized Dipole Amplitude

- The polarized dipole amplitude is then defined by

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

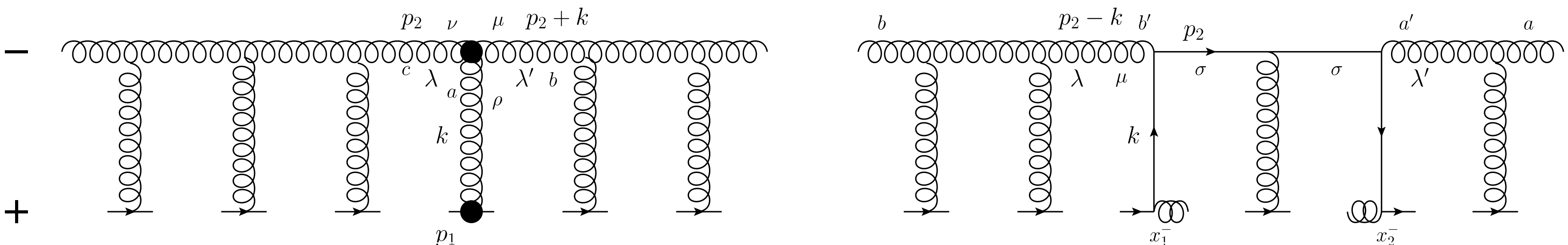
with the standard light-cone Wilson line

$$V_{\underline{x}}[b^-, a^-] = \text{P} \exp \left\{ ig \int\limits_{a^-}^{b^-} dx^- \, A^+(x^-, \underline{x}) \right\}$$



Polarized adjoint “Wilson line”

- Quarks mix with gluons. Therefore, we need to construct the adjoint polarized Wilson line --- the leading helicity-dependent part of the gluon scattering amplitude on the longitudinally polarized target.



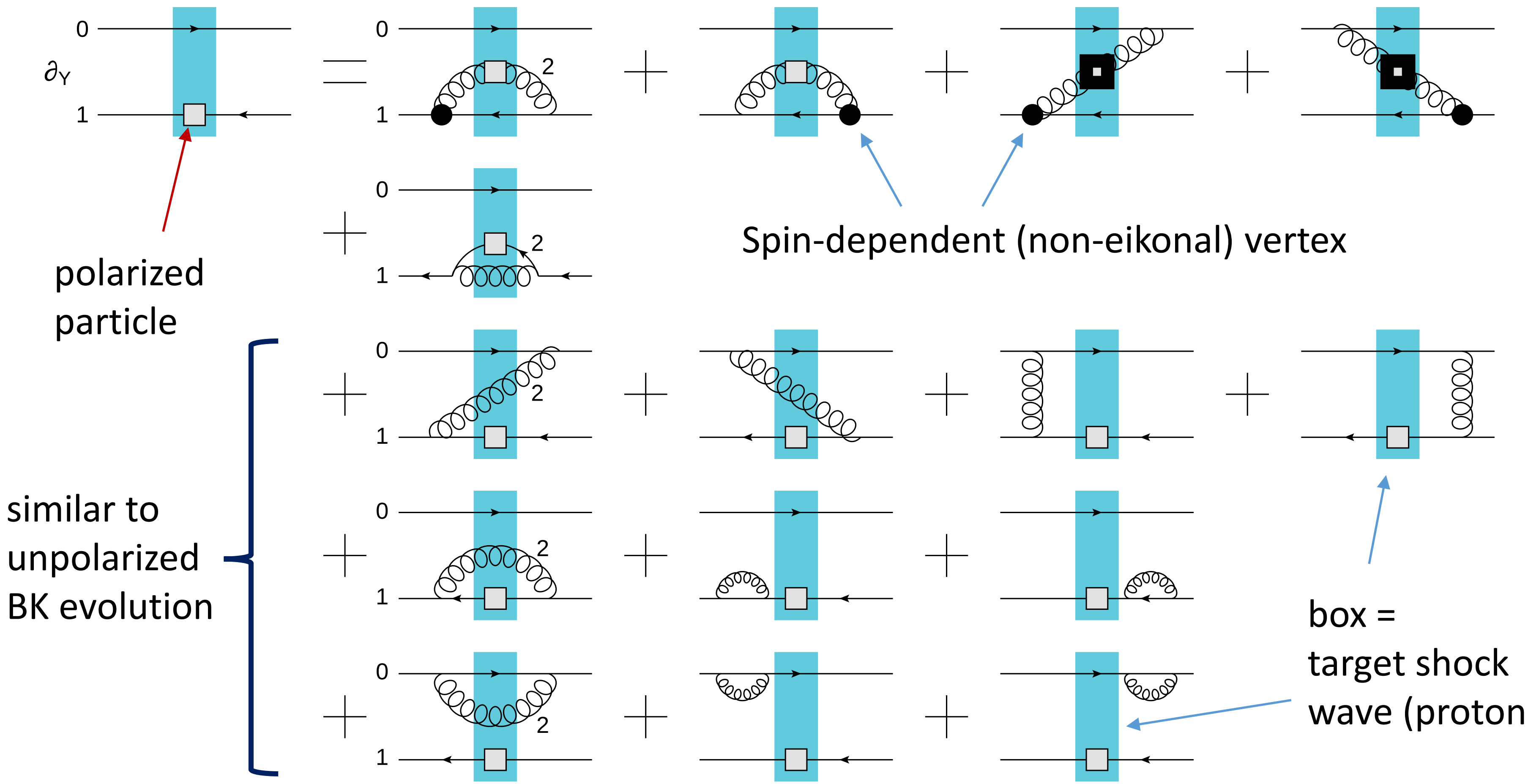
- The calculation is similar to the quark scattering case. It yields (cf. Chirilli ‘18)

$$(U_{\underline{x}}^{pol})^{ab} = \frac{2i\,g\,p_1^+}{s}\int\limits_{-\infty}^{+\infty} dx^- \ \left(U_{\underline{x}}[+\infty, x^-]\,\mathcal{F}^{12}(x^+=0, x^-, \underline{x})\,U_{\underline{x}}[x^-, -\infty]\right)^{ab}$$

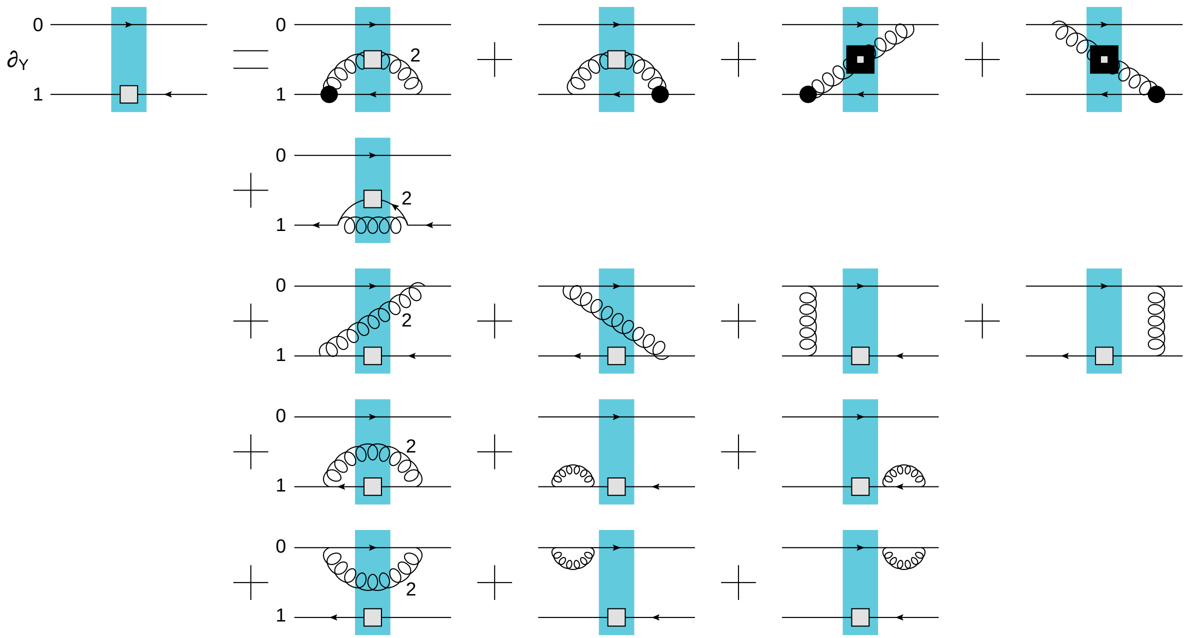
$$-\frac{g^2\,p_1^+}{s}\int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \ U_{\underline{x}}^{aa'}[+\infty, x_2^-]\,\bar{\psi}(x_2^-, \underline{x})\,t^{a'}\,V_{\underline{x}}[x_2^-, x_1^-]\,\frac{1}{2}\,\gamma^+\gamma_5\,t^{b'}\,\psi(x_1^-, \underline{x})\,U_{\underline{x}}^{b'b}[x_1^-, -\infty] - \text{c.c.}$$

Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\, s} \langle \ldots \rangle$$

$$\rho'^{\,2} = \frac{1}{z'\, s}$$

$$\frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle (z) = \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0 (z) + \frac{\alpha_s}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2}$$

$$\times \Bigg\{ \theta(x_{10} - x_{21})\, \frac{2}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_1^{unp\,\dagger}\right]\, U_2^{pol\,ba} \right\rangle\!\!\right\rangle (z')$$

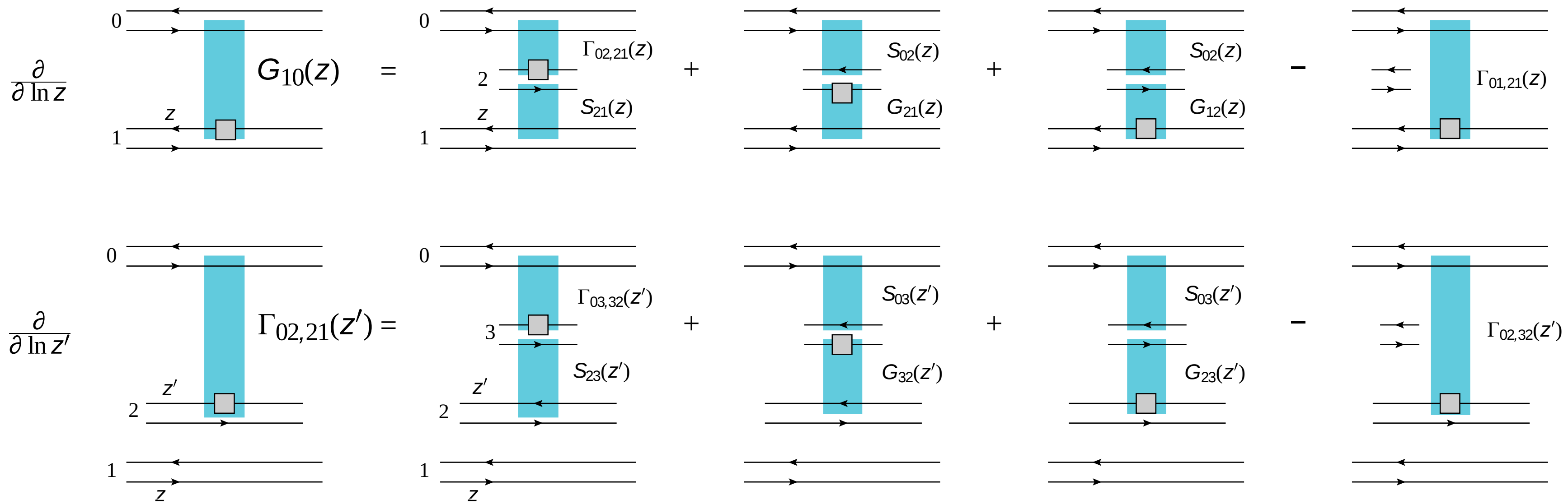
$$+ \theta(x_{10}^2 z - x_{21}^2 z')\, \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_2^{pol\,\dagger}\right]\, U_1^{unp\,ba} \right\rangle\!\!\right\rangle (z')$$

$$+ \theta(x_{10} - x_{21})\, \frac{1}{N_c} \left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_2^{unp\,\dagger}\right]\, \mathrm{tr}\left[V_2^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle (z') - N_c \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle (z') \right] \Bigg\}$$

Equation does not close!

Polarized Dipole Evolution in the Large-N_c Limit

In the large-N_c limit the equations close, leading to a system of 2 equations:



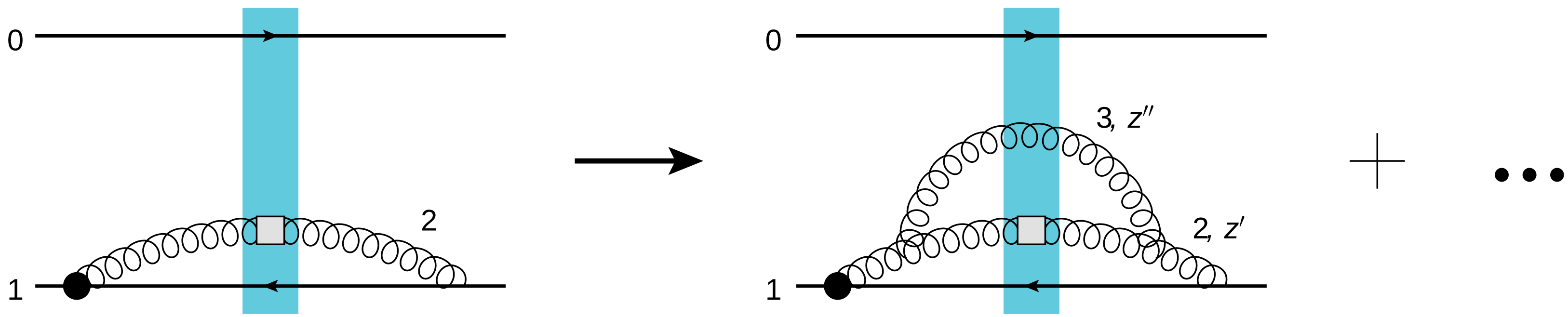
$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\,\Gamma_{02,\,21}(z')\, S_{21}(z') + 2\, G_{21}(z')\, S_{02}(z') + G_{12}(z')\, S_{02}(z') - \Gamma_{01,\,21}(z')\right]$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\,\Gamma_{03,\,32}(z'')\, S_{23}(z'') + 2\, G_{32}(z'')\, S_{03}(z'') + G_{23}(z'')\, S_{03}(z'') - \Gamma_{02,\,32}(z'')\right]$$

S = found from BK/JIMWLK, it is LLA

“Neighbor” dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may `know’ about another dipole:



$$x_{21}^2 \, z' \gg x_{32}^2 \, z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ \ln(1/x)$$

- Helicity evolution resummation parameter is double-logarithmic (DLA):

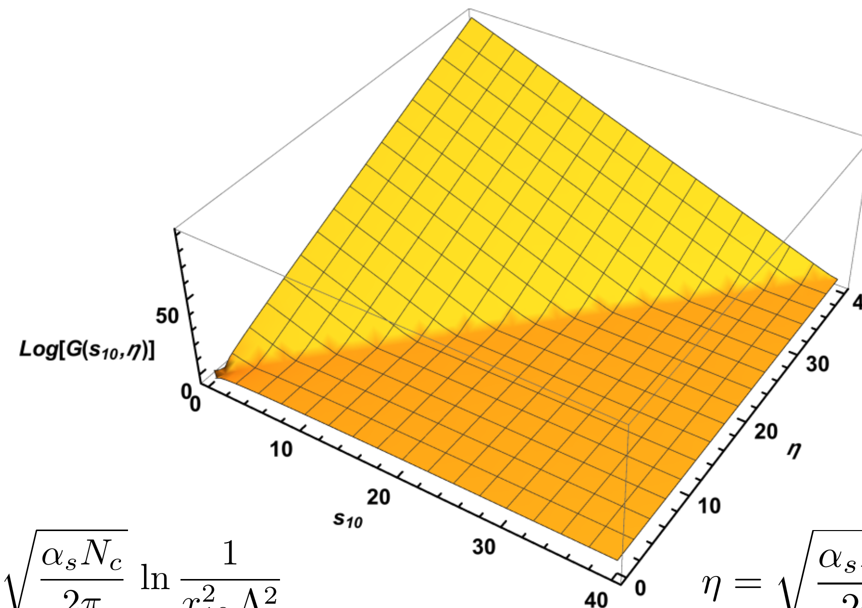
$$\alpha_s \ \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in the UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Quark Helicity at Small x

- These equations can be solved both numerically and analytically. (KPS '16-'17)



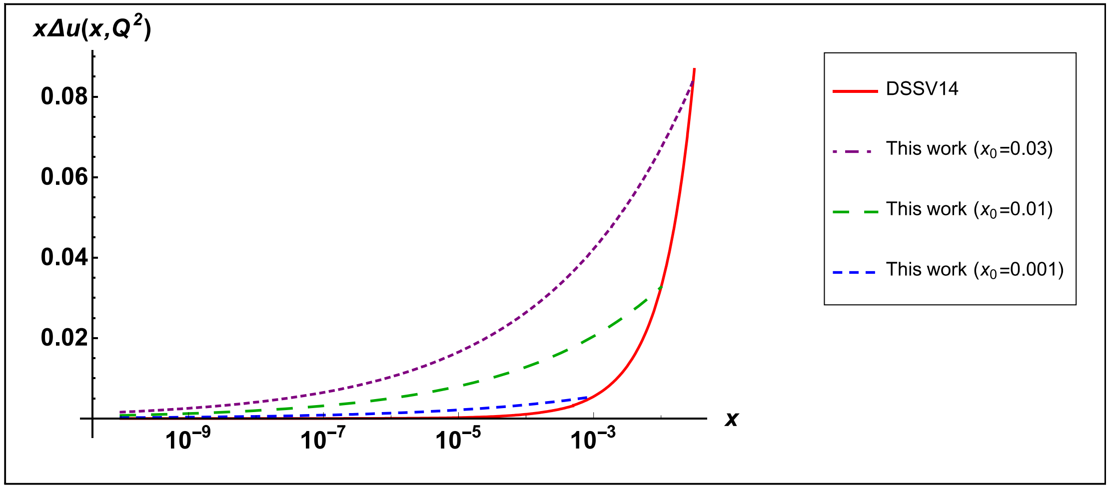
$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2} \qquad \eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{z s}{\Lambda^2}$$

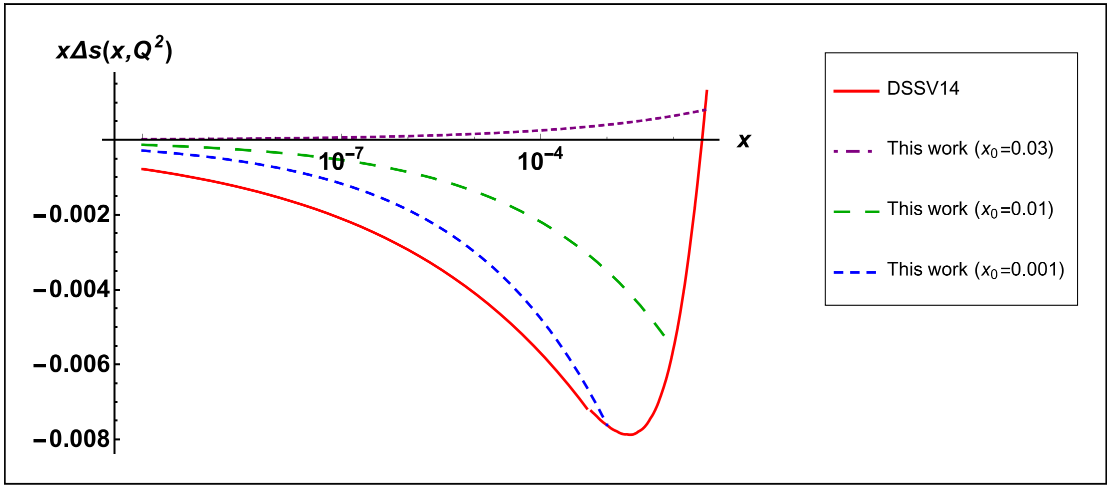
- The small-x asymptotics of quark helicity is (at large N_c)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Impact of our $\Delta\Sigma$ on the proton spin

- We have attached a $\Delta\tilde{\Sigma}(x, Q^2) = N\, x^{-\alpha_h}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :

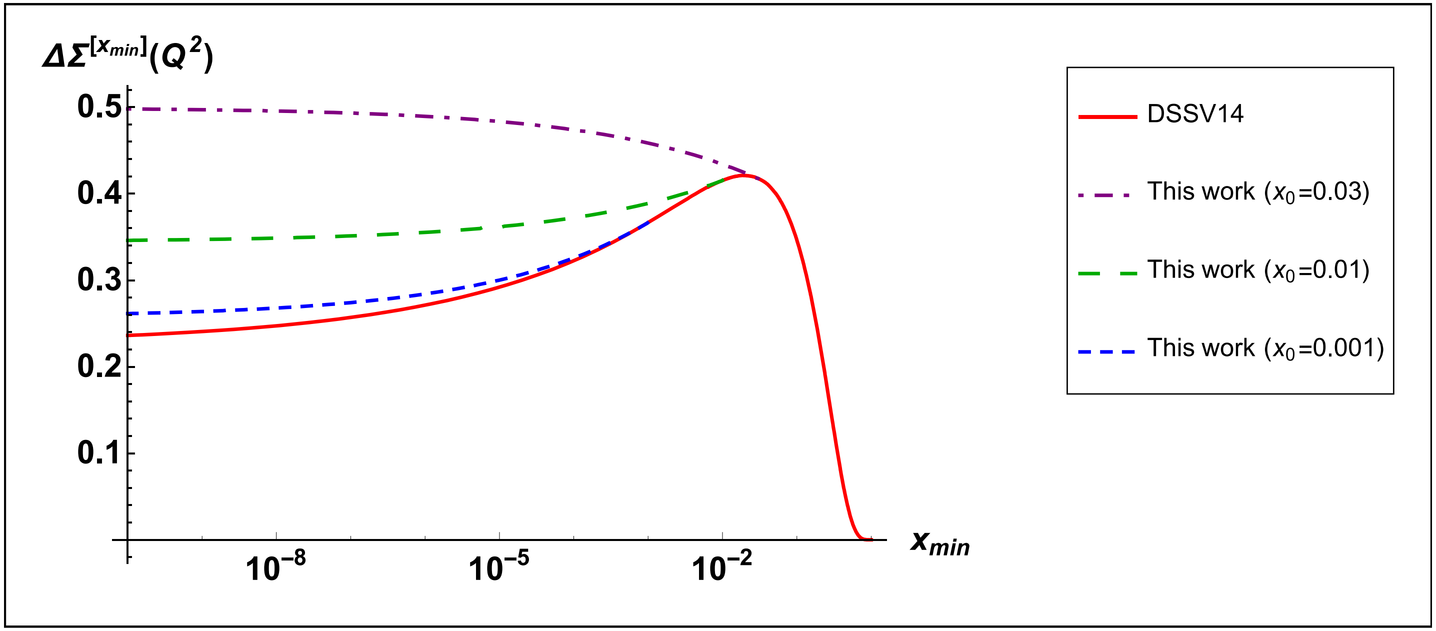




"ballpark" phenomenology

Impact of our $\Delta\Sigma$ on the proton spin

- Defining $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta\Sigma(x, Q^2)$ we plot it for x_0=0.03, 0.01, 0.001:



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

Gluon Helicity at Small x

Yu.K., D. Pitonyak, M. Sievert, arXiv:1706.04236 [nucl-th]

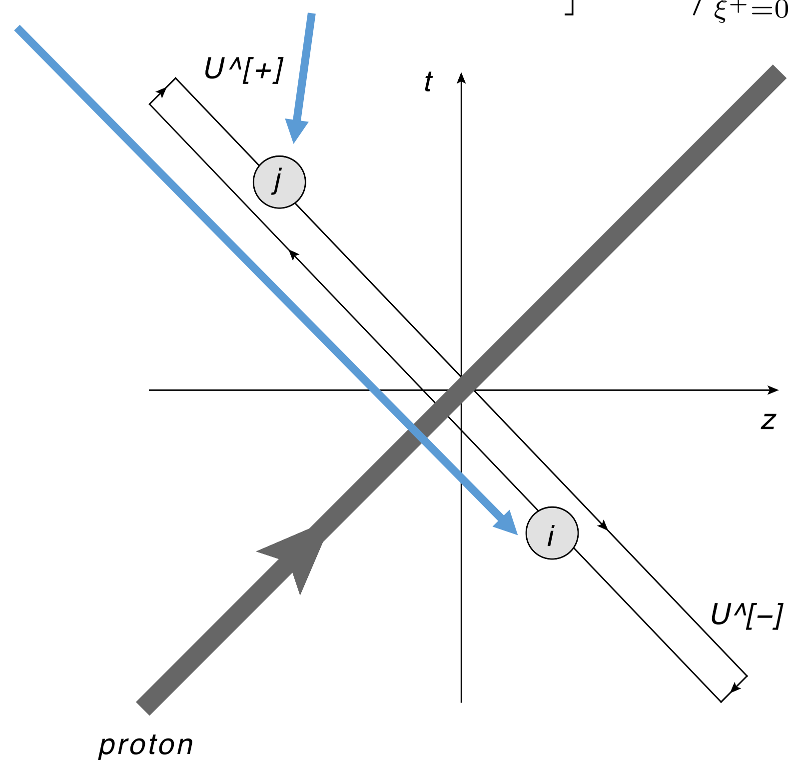
Dipole Gluon Helicity TMD

- Now let us repeat the calculation for gluon helicity TMDs.

- We start with the definition of the gluon dipole helicity TMD:

$$g_1^G(x, k_T^2) = \frac{-2i\, S_L}{x\, P^+} \int \frac{d\xi^-\, d^2\xi}{(2\pi)^3}\, e^{ixP^+\,\xi^- - i\underline{k}\cdot\underline{\xi}} \left\langle P, S_L | \epsilon_T^{ij}\, \mathrm{tr}\left[F^{+i}(0)\, \mathcal{U}^{[+]\dagger}[0,\xi]\, F^{+j}(\xi)\, \mathcal{U}^{[-]}[\xi,0] \right] | P, S_L \right\rangle_{\xi^+=0}$$

- Here $U^{[+]}$ and $U^{[-]}$ are future and past Wilson line staples (hence the name `dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Dipole Gluon Helicity TMD

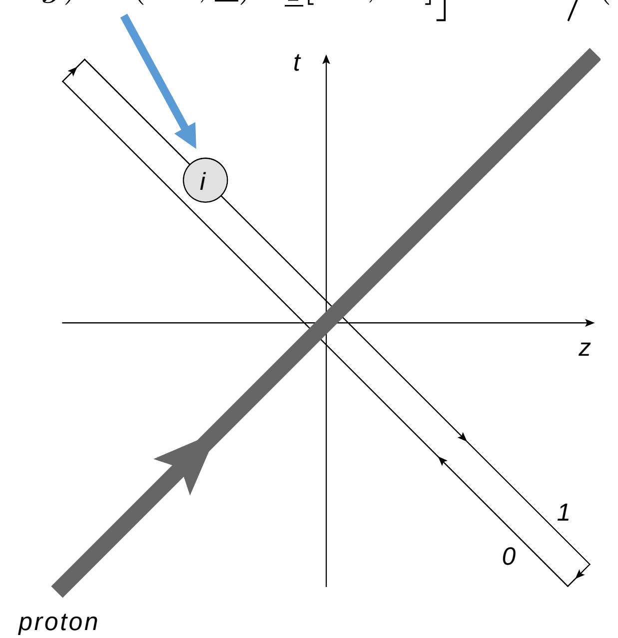
- At small x, the definition of dipole gluon helicity TMD can be massaged into

$$g_1^{G\,dip}(x, k_T^2) = \frac{8i\,N_c\,S_L}{g^2(2\pi)^3} \int d^2x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}}\; k_\perp^i\, \epsilon_T^{ij} \left[\int d^2b_{10}\, G_{10}^j(zs = \frac{Q^2}{x}) \right]$$

- Here we obtain a new operator, which is a transverse vector (written here in A^-=0 gauge):

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig)\, \tilde{A}^i(x^-, \underline{x})\, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

- Note that $k_\perp^i\, \epsilon_T^{ij}$ can be thought of as a transverse curl acting on $G_{10}^i(z)$ and not just on $\tilde{A}^i(x^-, \underline{x})$ -- different from the polarized dipole amplitude!



Dipole TMD vs dipole amplitude

- Note that the operator for the <u>dipole</u> gluon helicity TMD

$$G^i_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \tilde{A}^i(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

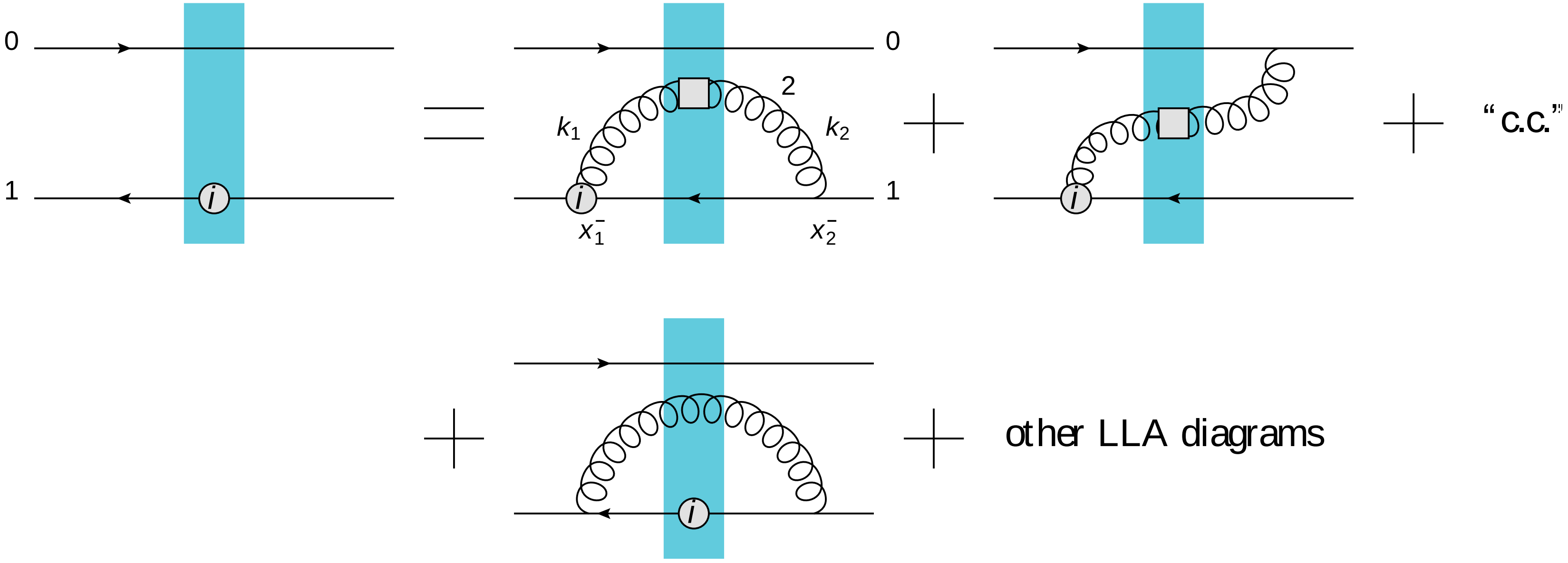
  is different from the polarized <u>dipole</u> amplitude

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

- We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the 'dipole' name may not even be valid for such TMDs.)
- This is different from the unpolarized gluon TMD case.

Evolution Equation

- To construct evolution equation for the operator G^i governing the gluon helicity TMD we resum similar (to the quark case) diagrams:



Large-N_c Evolution: Equations

- This results in the following evolution equations:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{21})^j_\perp}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{20})^j_\perp}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's)\right]$$

$$\Gamma^i_{10\,21}(z's) = G^{i\,(0)}_{10}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{31})^j_\perp}{x_{31}^2} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + G_{31}(z''s)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{30})^j_\perp}{x_{30}^2} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + \Gamma^{gen}_{31\,,\,30}(z''s)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left[x_{10}^2\,,\,x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{31}^2}{x_{31}^2} \left[G^i_{13}(z''s) - \Gamma^i_{10\,,\,31}(z''s)\right].$$

Large-N_c Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region yielding the small-x gluon helicity intercept

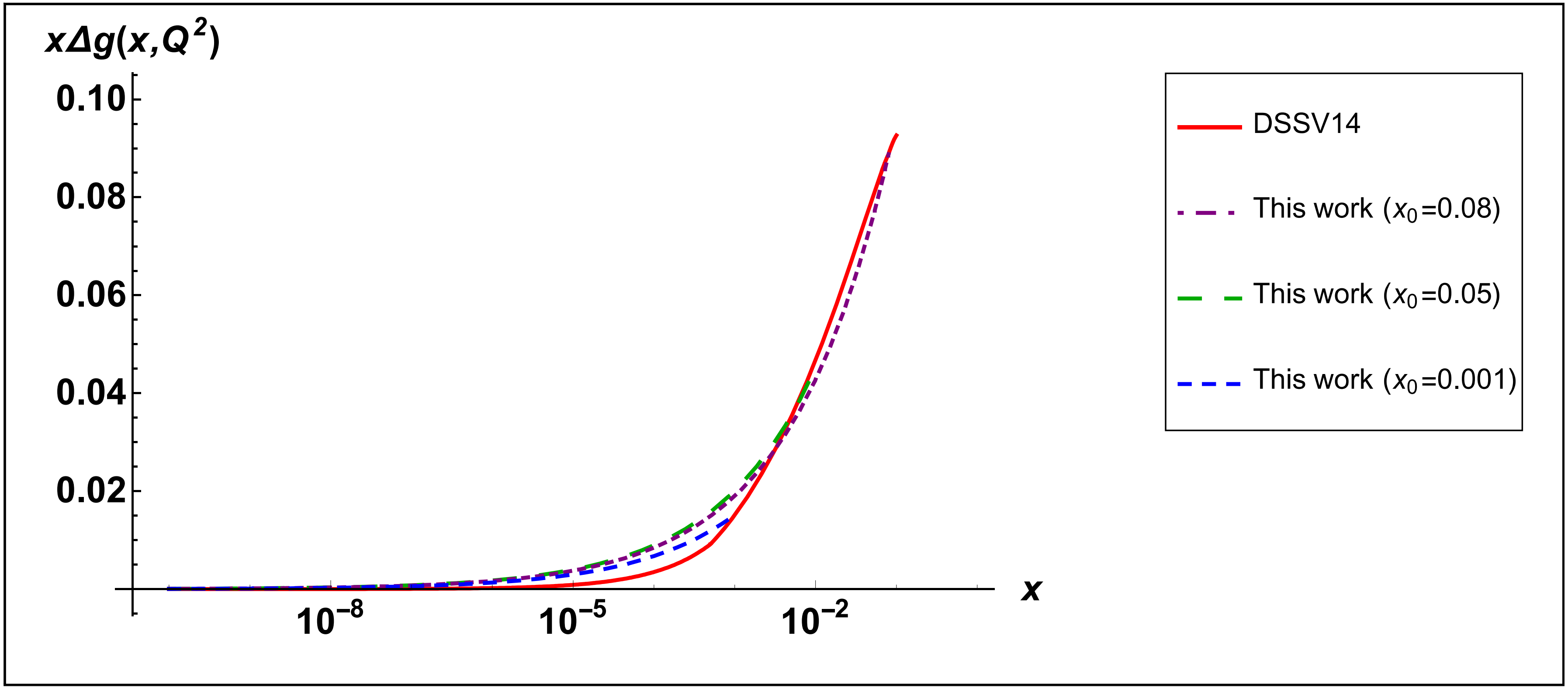
$$\alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- We obtain the small-x asymptotics of the gluon helicity distributions:

$$\Delta G(x, Q^2) \sim g_{1L}^{G\,dip}(x, k_T^2) \sim \left(\frac{1}{x} \right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}}$$

Impact of our ΔG on the proton spin

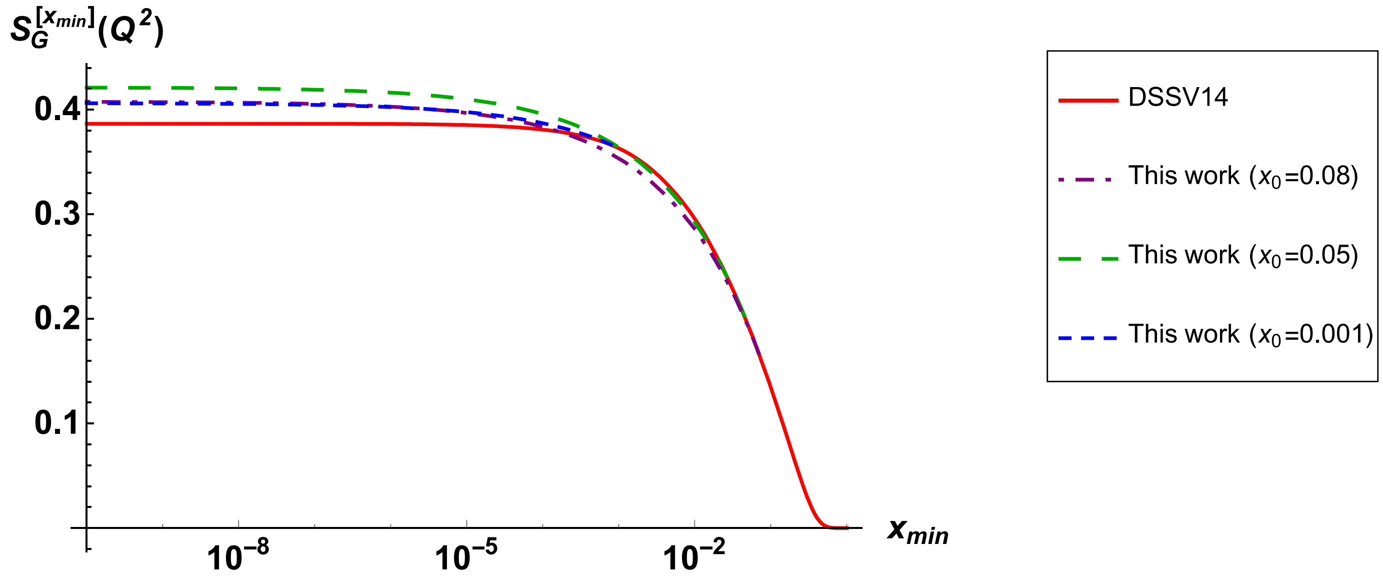
- We have attached a $\Delta\tilde{G}(x,Q^2) = N\,x^{-\alpha_h^G}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :



"ballpark"
phenomenology

Impact of our ΔG on the proton spin

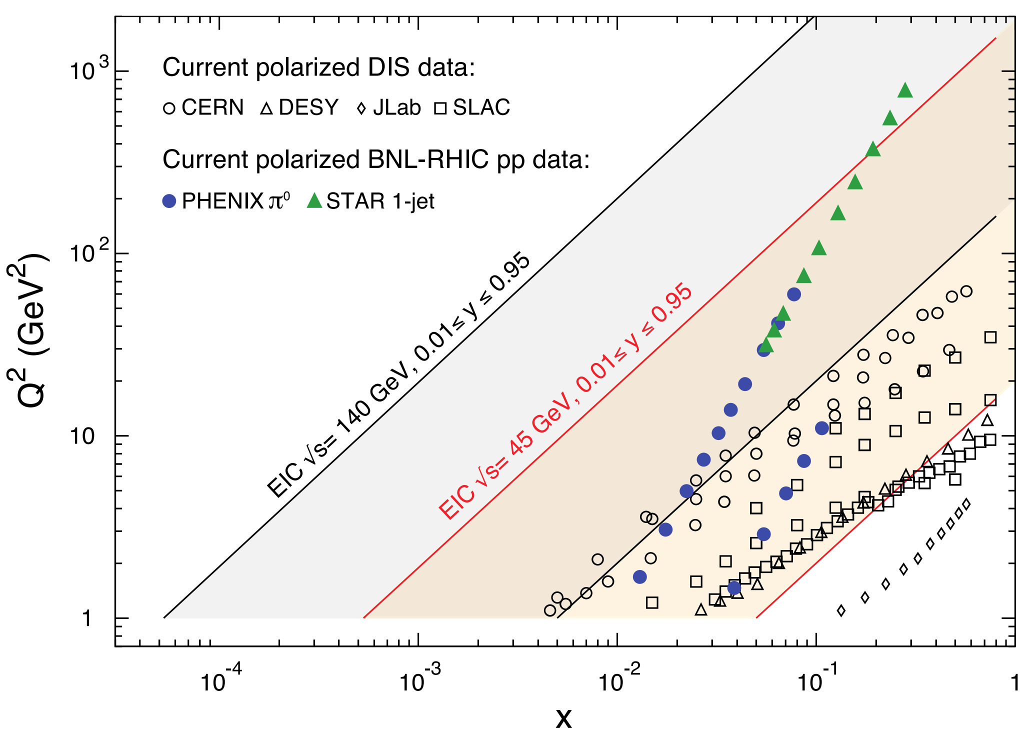
- Defining $S_G^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta G(x, Q^2)$ we plot it for x_0=0.08, 0.05, 0.001:

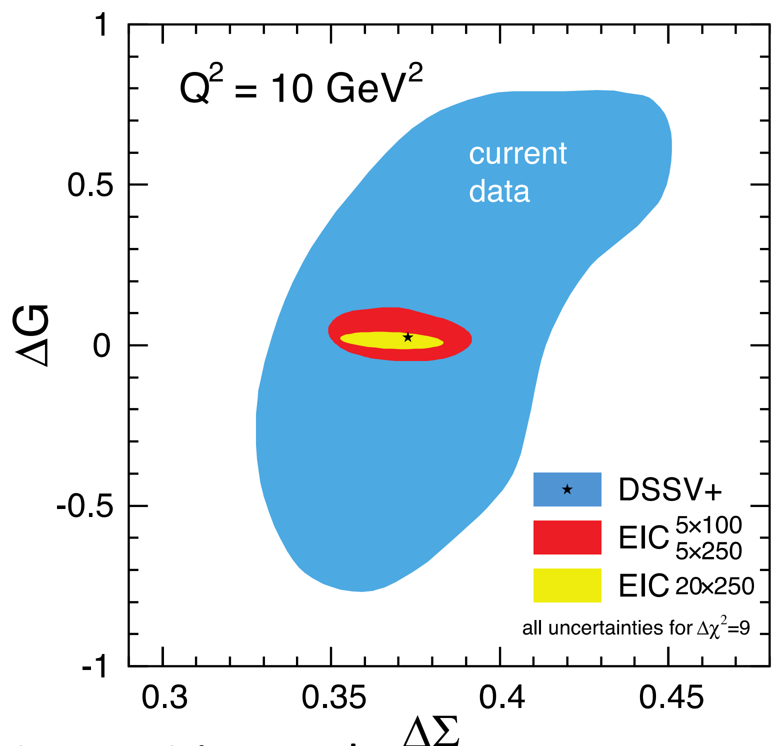


- We observe a moderate enhancement of gluon spin.
- More detailed phenomenology is needed in the future.

EIC & Spin Puzzle

- Parton helicity distributions are sensitive to low-x physics.
- EIC would have an unprecedented low-x reach for a polarized DIS experiment, allowing to pinpoint the values of quark and gluon contributions to proton's spin:





- ΔG and $\Delta\Sigma$ are integrated over x in the $0.001 < x < 1$ interval.

Outlook

- To go beyond the large-N_c and large-N_c&N_f limits need to write a helicity analogue of JIMWLK evolution.
- This has been done recently (F. Cougoulic, YK, arXiv:1910.04268 [hep-ph]):

$$W_\tau[\alpha, \beta, \psi, \bar{\psi}] = W_\tau^{(0)}[\alpha, \beta, \psi, \bar{\psi}] + \int d^3\tau' \ \mathcal{K}_h[\tau, \tau'] \cdot W_{\tau'}[\alpha, \beta, \psi, \bar{\psi}]$$

with the kernel

$$\mathcal{K}_h[\tau, \tau'] = \frac{\alpha_s}{\pi^2} \int d^2 w_\perp \frac{\underline{X}' \cdot \underline{Y}'}{X'^2 \, Y'^2} \ \theta^{(3)}(\tau - \tau') \theta\left(z' - \frac{\Lambda^2}{s}\right) \theta\left(X'^2 - \frac{1}{z' \, s}\right) \theta\left(Y'^2 - \frac{1}{z' \, s}\right)$$
$$\times \left\{ U_w^{ba} D^+_{x,a,<} \, D^+_{y,b,>} - \frac{1}{2}\left(D^+_{x,a,<} D^+_{y,a,<} + D^+_{x,a,>} \, D^+_{y,a,>}\right) \right.$$
$$+ \frac{1}{2} U_w^{pol,ba} (D^+_{x,a,<} D^\perp_{y,b,>} + D^\perp_{x,a,<} D^+_{y,b,>})$$
$$\left. + \left(\frac{1}{2}\gamma^5 \gamma^-\right)_{\beta\alpha} \frac{1}{2}\left((V_w^{pol})_{ij} D^{\bar{\psi}}_{x,j,\alpha,<} D^{\psi}_{y,i,\beta,>} + (V_w^{pol \, \dagger})_{ij} D^{\bar{\psi}}_{x,j,\alpha,>} D^{\psi}_{y,i,\beta,<}\right) \right\}$$

Quark and Gluon OAM at Small x

Quark OAM: Definition

- We begin by writing the (Jaffe-Manohar) quark OAM in terms of the Wigner distribution as

$$L_z = \int \frac{d^2 b_\perp db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k, b)$$

with the quark SIDIS Wigner distribution

$$W^{q,SIDIS}(k,b) = 2 \sum_X \int d^2 r \, dr^- \; e^{ik \cdot r} \left\langle \bar{\psi}_\alpha \left(b - \tfrac{1}{2} r\right) \, V_{\underline{b} - \frac{1}{2} \underline{r}} \left[b^- - \tfrac{1}{2} r^-, \infty\right] |X\rangle \left(\tfrac{1}{2} \gamma^+\right)_{\alpha\beta} \right.$$
$$\left. \times \; \langle X | \, V_{\underline{b} + \frac{1}{2} \underline{r}} \left[\infty, b^- + \tfrac{1}{2} r^-\right] \; \psi_\beta \left(b + \tfrac{1}{2} r\right) \right\rangle$$

- Here, and above, the angle brackets denote ”CGC averaging” in the (polarized) proton target:

$$\left\langle \hat{\mathcal{O}}(b, r) \right\rangle = \frac{1}{2P^+} \int \frac{d^2 \Delta \, d\Delta^+}{(2\pi)^3} \, e^{ib \cdot \Delta} \left\langle P + \frac{\Delta}{2} \right| \hat{\mathcal{O}}(0, r) \left| P - \frac{\Delta}{2} \right\rangle$$

Quark OAM: small-x simplifications

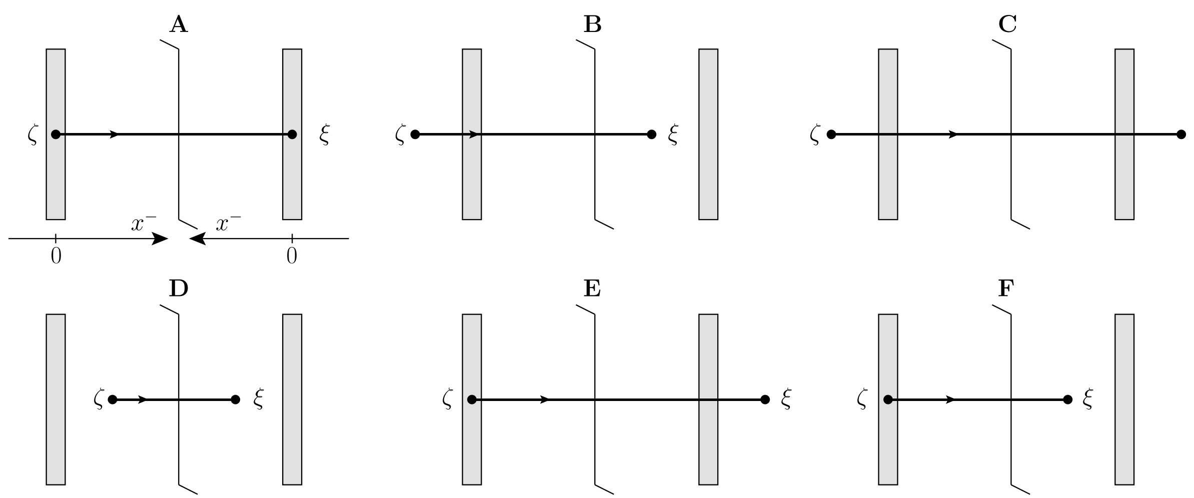
- The resulting quark OAM "PDF" is

$$L_q(x,Q^2) = \frac{2P^+}{(2\pi)^3} \sum_X \int d^2k_\perp \, d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta-\xi)} \left(\frac{\underline{\zeta}+\underline{\xi}}{2} \times \underline{k} \right) \left\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-,\infty] \, |X\rangle \, \left(\tfrac{1}{2}\gamma^+\right)_{\alpha\beta} \right.$$
$$\left. \times \, \langle X| \, V_{\underline{\zeta}}[\infty,\zeta^-] \, \psi_\beta(\zeta) \right\rangle$$

- This can be compared to quark helicity,

$$g_{1L}^q(x,k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta-\xi)} \left(\frac{1}{2}\gamma^+\gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-,\infty] \;\; V_{\underline{\zeta}}[\infty,\zeta^-] \, \psi_\beta(\zeta) \right\rangle$$

- The operators are different, but the structure is similar. The quark OAM can be evaluated in the same way as the quark helicity operator: only diagram B survives.



Quark OAM: small-x expression

- After some algebra we arrive at the following small-x expression for quark OAM:

$$L_{q+\bar{q}}(x,Q^2) = \frac{8N_c}{(2\pi)^5}\int d^2k_\perp\, d^2x_{10}\, d^2x_1\, e^{i\underline{k}\cdot\underline{x}_{10}}\, \frac{\underline{x}_{10}}{x_{10}^2}\times\frac{\underline{k}}{\underline{k}^2}\; \underline{x}_1\times\underline{k}\int\limits_{\Lambda^2/s}^{1}\frac{dz}{z}\, G_{10}(zs) - \sum_f [\Delta q^f(x,Q^2)+\Delta\bar{q}^f(x,Q^2)]$$

- The result is written in terms of the polarized dipole amplitude G_{10} (z). It seems we are done, right?
- This is almost correct. The remaining minor technicality is that the above quark OAM depends on the "first moment" of the polarized dipole amplitude

$$I^k(\underline{x}_{10}, zs) = \int d^2x_1\, x_1^k\, G_{10}(zs)$$

while all our earlier results for the quark helicity were derived for the "zeroth moment", the impact-parameter integrated polarized dipole amplitude

$$G(x_{10}^2, zs) = \int d^2x_1\, G_{10}(zs)$$

Quark OAM: small-x asymptotics

- It turns out that the "first moment" of the polarized amplitude is subleading. It grows with energy as a smaller power of energy

$$I^k(\underline{x}_{10}, zs) \sim \left(zs x_{10}^2\right)^{2\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

  than the flavor-singlet quark helicity distribution

$$\Delta\Sigma(x, Q^2) = \sum_f [\Delta q^f(x, Q^2) + \Delta \bar{q}^f(x, Q^2)] \sim \left(\frac{1}{x}\right)^{\alpha_h^q} = \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \approx \left(\frac{1}{x}\right)^{2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Since 2.31 > 2, we get (cf. Y. Hatta & D.-J. Yang, 2018)

$$L_{q+\bar{q}}(x, Q^2) = -\Delta\Sigma(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Note that this is not a complete cancellation, the contribution to the proton spin is

$$\tfrac{1}{2}\,\Delta\Sigma(x, Q^2) + L_{q+\bar{q}}(x, Q^2) = -\tfrac{1}{2}\,\Delta\Sigma(x, Q^2)$$

Gluon OAM: definition

- The gluon OAM story is similar. We start with the Wigner distribution definition

$$L_z = \int \frac{d^2 b_\perp \, db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k, b)$$

with the dipole Wigner distribution for gluons

$$W^{G\, dip}(k,b) = \frac{4}{xP^+} \int d\xi^- \, d^2\xi_\perp \;\; e^{ixP^+ \xi^- - i\underline{k}\cdot\underline{\xi}} \times \left\langle \text{tr} \left[F^{+i}(b - \tfrac{1}{2}\xi) \, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \, F^{+i}(b + \tfrac{1}{2}\xi) \, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

- We obtain the following expression for the gluon OAM "PDF" (cf. Hatta et al, 2016)

$$L_G(x, Q^2) = \frac{4}{(2\pi)^3 \, x} \int d^2 b_\perp db^- \, d^2 k_\perp \, d\xi^- \, d^2 \xi_\perp \;\; (\underline{b} \times \underline{k}) \;\; e^{ixP^+ \xi^- - i\underline{k}\cdot\underline{\xi}} \times \left\langle \text{tr} \left[F^{+i}(b - \tfrac{1}{2}\xi) \, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \, F^{+i}(b + \tfrac{1}{2}\xi) \, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

Gluon OAM: small-x expression

- Gluon OAM at small x can (similarly to the quark OAM) be rewritten in terms of the "moment" of the polarized dipole amplitude G_{10}^{i} for the gluon helicity TMD. This object is different from the polarized amplitude for the quark.
- We get

$$L_G(x, Q^2) = - \frac{8 i N_c}{g^2 \, (2\pi)^3} \int d^2 x_{10} \, d^2 k_\perp \, e^{i \underline{k} \cdot \underline{x}_{10}} \, (\underline{k} \cdot \underline{x}_{10}) \, G_5 \left(x_{10}^2, zs = \frac{Q^2}{x} \right)$$

where

$$\int d^2 x_1 \; x_1^j \, \nabla_{10}^i \, G_{10}^i (zs) = x_{10}^j \, G_4 (x_{10}^2, zs) + \epsilon^{jk} x_{10}^k \, G_5 (x_{10}^2, zs)$$

- We write down and solve the equations for G_5.

Gluon OAM: small-x asymptotics

- We arrive at the following relation

$$L_G(x, Q^2) = \left(\frac{\alpha_h^q}{4} \, \ln \frac{Q^2}{\Lambda^2} \right) \Delta G(x, Q^2)$$

where

$$\alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- We conclude that

$$L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^G} \sim \left(\frac{1}{x} \right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}} \sim \left(\frac{1}{x} \right)^{1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}}$$

- Note that with the DLA accuracy we could also simply conclude that

$$|L_G| \ll |\Delta G|$$

Valence Quark Transversity at Small x

Yu.K., M. Sievert, arXiv:1808.10354 **[hep-ph]**

Small-x Asymptotics of Quark Transversity

- Solution of the transversity evolution equation is straightforward.
- The resulting small-x asymptotics is (cf. Kirschner et al, 1996)

$$h_{1T}^{NS}(x, k_T^2) \sim h_{1T}^{\perp\, NS}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2\sqrt{\frac{\alpha_s\, C_F}{\pi}}$$

- Note the suppression by x^2 compared to the unpolarized quark TMDs.
- For α_s = 0.3 we get

$$h_{1T}^{NS}(x, k_T^2) \sim h_{1T}^{\perp\, NS}(x, k_T^2) \sim x^{0.243}$$

- This certainly satisfies the Soffer bound, but is not likely to produce much tensor charge from small x.

$$\delta q(Q^2) = \int_0^1 dx\, h_1(x, Q^2)$$

Conclusions

- At large N_c we have obtained the following small-x asymptotics:

$$\Delta q(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$\Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

$$L_{q+\bar{q}}(x,Q^2) = -\Delta\Sigma(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}}},$$

$$L_G(x,Q^2) \sim \Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}}}$$

- Future helicity and OAM work will involve solving the large-N_C&N_f equations + including running coupling corrections + LLA corrections + phenomenology to constrain the spin+OAM coming from small-x quarks and gluons.
- EIC should be able to measure helicity TMDs with high precision and down to fairly small x. We may also be able to learn something about OAM.

$$h_{1T}^{NS}(x,k_T^2) \sim h_{1T}^{\perp\, NS}(x,k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2\sqrt{\frac{\alpha_s\, C_F}{\pi}}$$

Stan Brodsky's impact on my work

Happy Birthday, Stan!

- Light-front perturbation theory: I have had the appendix of the original Brodsky-Lepage paper with the LFPT rules since I started graduate research in 1995. I still have that copy and consult it regularly.
- Al Mueller's dipole model and my derivation of the BK equation were constructed using LFPT rules.
- First time I understood what nuclear shadowing was: walking with Stan from ECT* in Trento back to town for dinner (1998).
- Brodsky-Lepage-Mackenzie (BLM) scale fixing for the strong coupling: we employed it to construct rcBK and rcJIMWLK.
- Transverse single spin asymmetry: groundbreaking work by Stan with Hwang and Schmidt. Clarified single spin asymmetry to all of us, myself including.

Backup Slides

Proton Spin Puzzle

$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx\, \Delta\Sigma(x, Q^2)$$

- The spin puzzle began when the EMC collaboration measured the proton g_1 structure function ca 1988. Their data resulted in

$$\Delta\Sigma \approx 0.1 \div 0.2$$

- It appeared quarks do not carry all of the proton spin (which would have corresponded to $\Delta\Sigma = 1$).

- Missing spin can be

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

  - Carried by gluons
  - In the orbital angular momenta of quarks and gluons
  - At small x (both helicity and OAM):

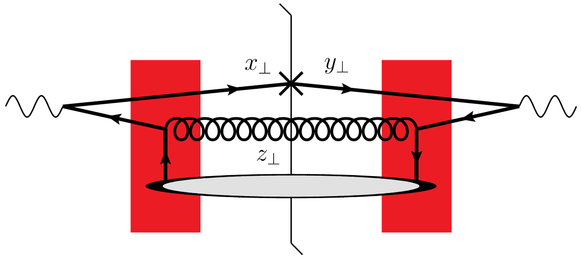
$$S_q(Q^2) = \frac{1}{2} \int_0^1 dx\, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$

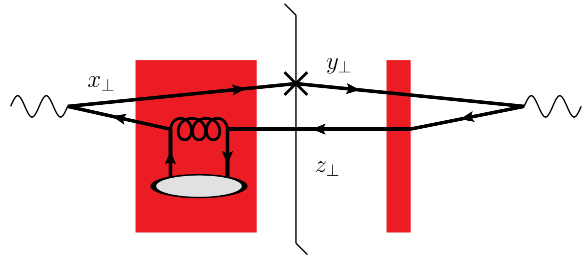
Can't integrate down to zero, use x_{min} instead!

  - Or all of the above!

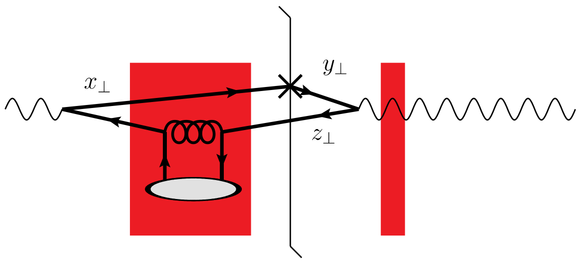
Quark Helicity TMD at Small x

- Dominance of diagram B can also be obtained by applying crossing symmetry to the SIDIS process (KS ‘15):



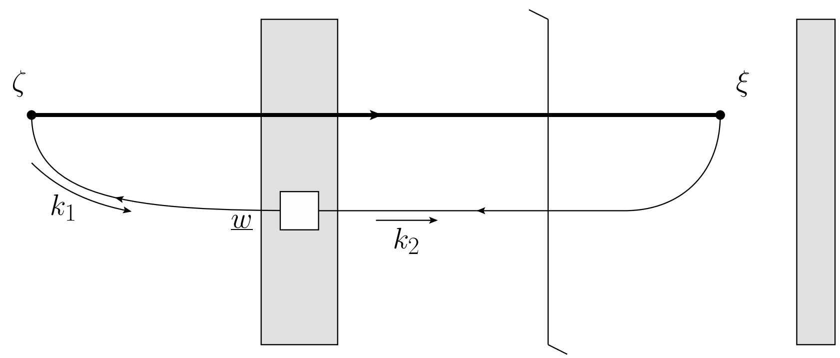
+


+ c.c. =0

+


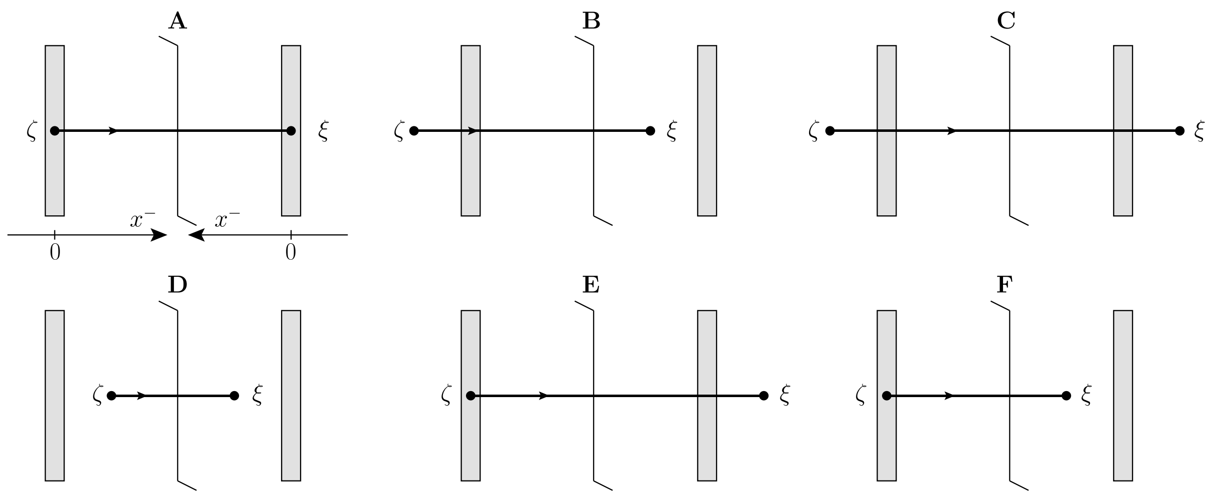
+ c.c. = the answer

- Compare the last line to the diagram B: reflecting the cc amplitude into the amplitude reduces the above diagram to the one on the right.



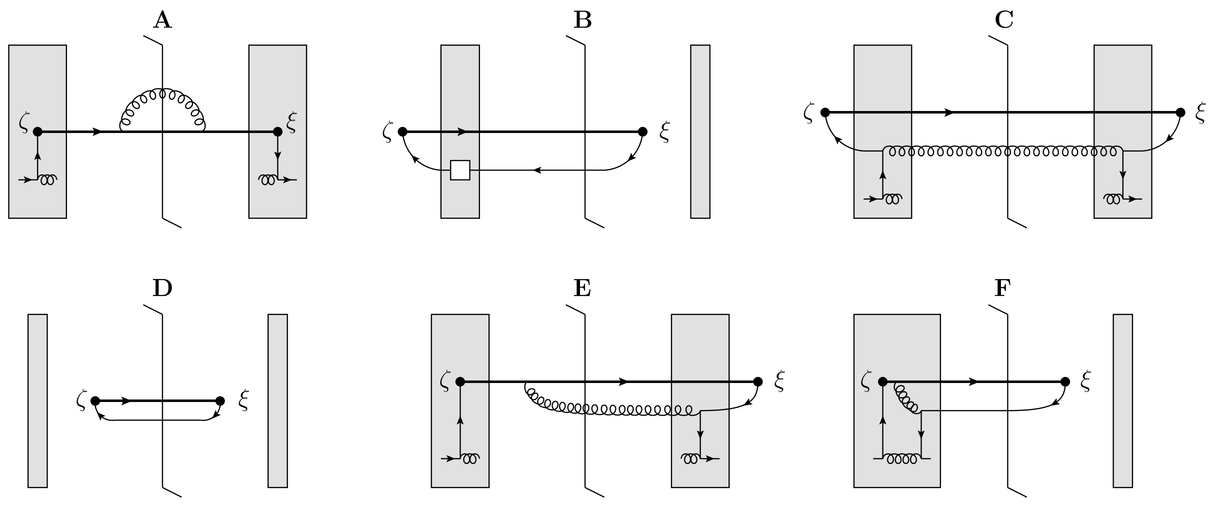
Quark Helicity TMD at Small x

- At high energy/small-x the proton is a shock wave, and we have the following contributions to the SIDIS quark helicity TMD:



$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \sum_X \int d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta-\xi)} \left(\tfrac{1}{2}\gamma^+\gamma^5\right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-, \infty] \, |X\rangle \, \langle X| \, V_{\underline{\zeta}}[\infty, \zeta^-] \, \psi_\beta(\zeta) \right\rangle$$

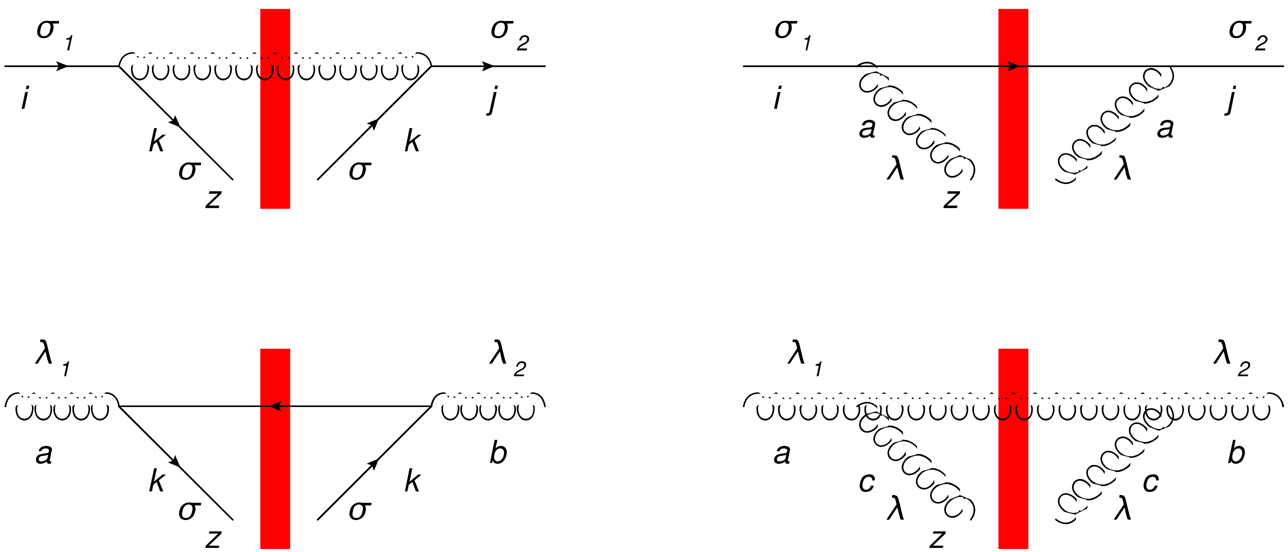
Quark Helicity TMD at Small x



- Diagram D does not transfer spin information from the target. Diagram C is canceled as we move t-channel quarks across the cut.
- Diagram F is energy-suppressed, since the gluon should have no time to be emitted and absorbed inside the shock wave.
- Diagrams of the types A and E++ can be shown to cancel each other at the leading (DLA) order (Ward identity).
- We are left with the diagram B.

Helicity Evolution Ingredients

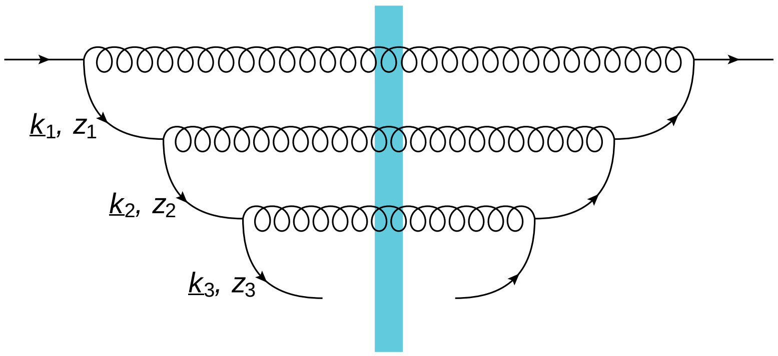
- Unlike the unpolarized evolution, in one step of helicity evolution we may emit a soft gluon or a soft quark (all in A^+=0 LC gauge of the projectile):

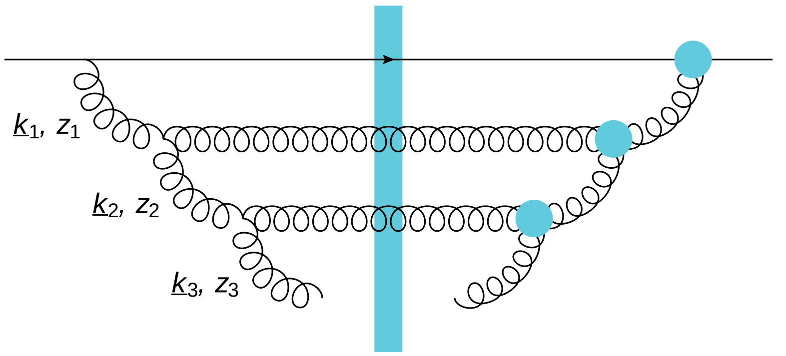


- When emitting gluons, one gluon is eikonal, while another one is soft, but non-eikonal, as is needed to transfer polarization down the cascade/ladder.

Helicity Evolution: Ladders

- To get an idea of how the helicity evolution works let us try iterating the splitting kernels by considering ladder diagrams (circles denote non-eikonal gluon vertices):



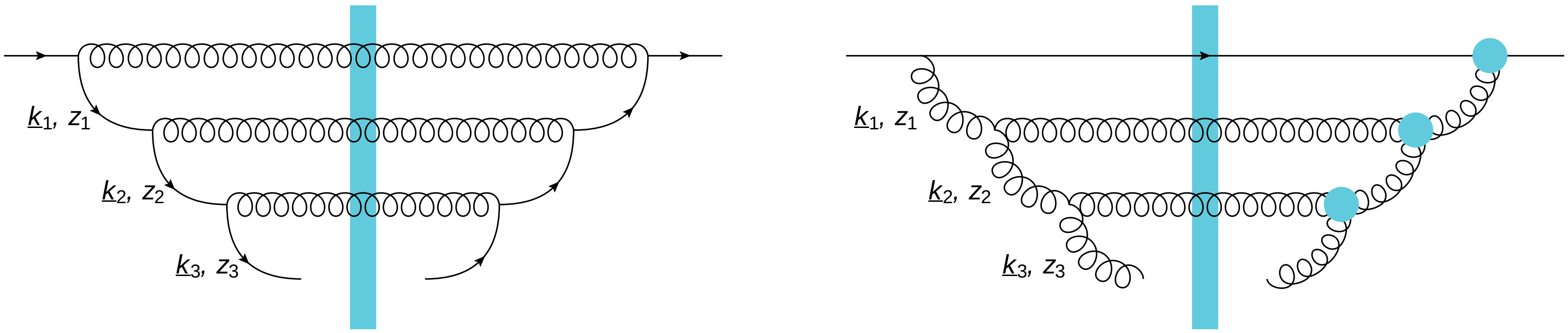


- To get the leading-energy asymptotics we need to order the longitudinal momentum fractions of the quarks and gluons (just like in the unpolarized evolution case) $1 \gg z_1 \gg z_2 \gg z_3 \gg \ldots$

obtaining a nested integral

$$\alpha_s^3 \int\limits_{z_i}^{1} \frac{dz_1}{z_1} \int\limits_{z_i}^{z_1} \frac{dz_2}{z_2} \int\limits_{z_i}^{z_2} \frac{dz_3}{z_3} \, z_3 \otimes \frac{1}{z_3 \, s} \sim \frac{1}{s} \, \alpha_s^3 \, \ln^3 s$$

Helicity Evolution: Ladders



- However, these are not all the logs of energy one can get here. Transverse momentum (or distance) integrals have UV and IR divergences, which lead to logs of energy as well.
- If we order transverse momenta / distances as (Sudakov-β ordering)

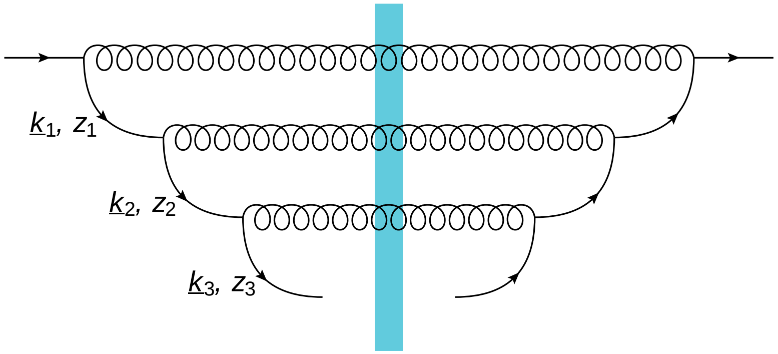
$$\frac{\underline{k}_1^2}{z_1} \ll \frac{\underline{k}_2^2}{z_2} \ll \frac{\underline{k}_3^2}{z_3} \ll \ldots \qquad z_1\, \underline{x}_1^2 \gg z_2\, \underline{x}_2^2 \gg z_3\, \underline{x}_3^2 \gg \ldots$$

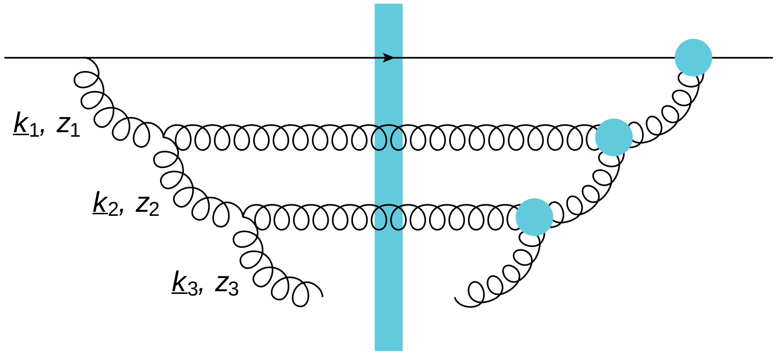
we would get integrals like

$$\int\limits_{1/(z_n\, s)}^{x^2_{n-1,\perp}\, z_{n-1}/z_n} \frac{dx^2_{n,\perp}}{x^2_{n,\perp}}$$

also generating logs of energy.

Helicity Evolution: Ladders





- To summarize, the above ladder diagrams are parametrically of the order

$$\frac{1}{s}\,\alpha_s^3\,\ln^6 s$$

- Note two features:
  - 1/s suppression due to non-eikonal exchange
  - two logs of energy per each power of the coupling!

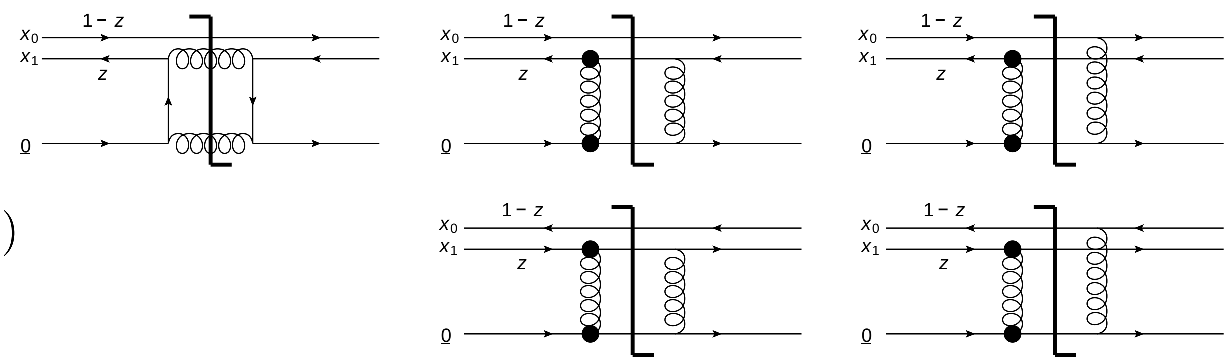
Large-N_c Evolution

- In the strict DLA limit (S=1) and at large N_c we get (here Γ is an auxiliary function we call the 'neighbor dipole amplitude') (KPS '15)

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z') + 3\, G(x_{21}^2, z')\right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\right\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z'') + 3\, G(x_{32}^2, z'')\right]$$

- The initial conditions are given by the Born-level graphs

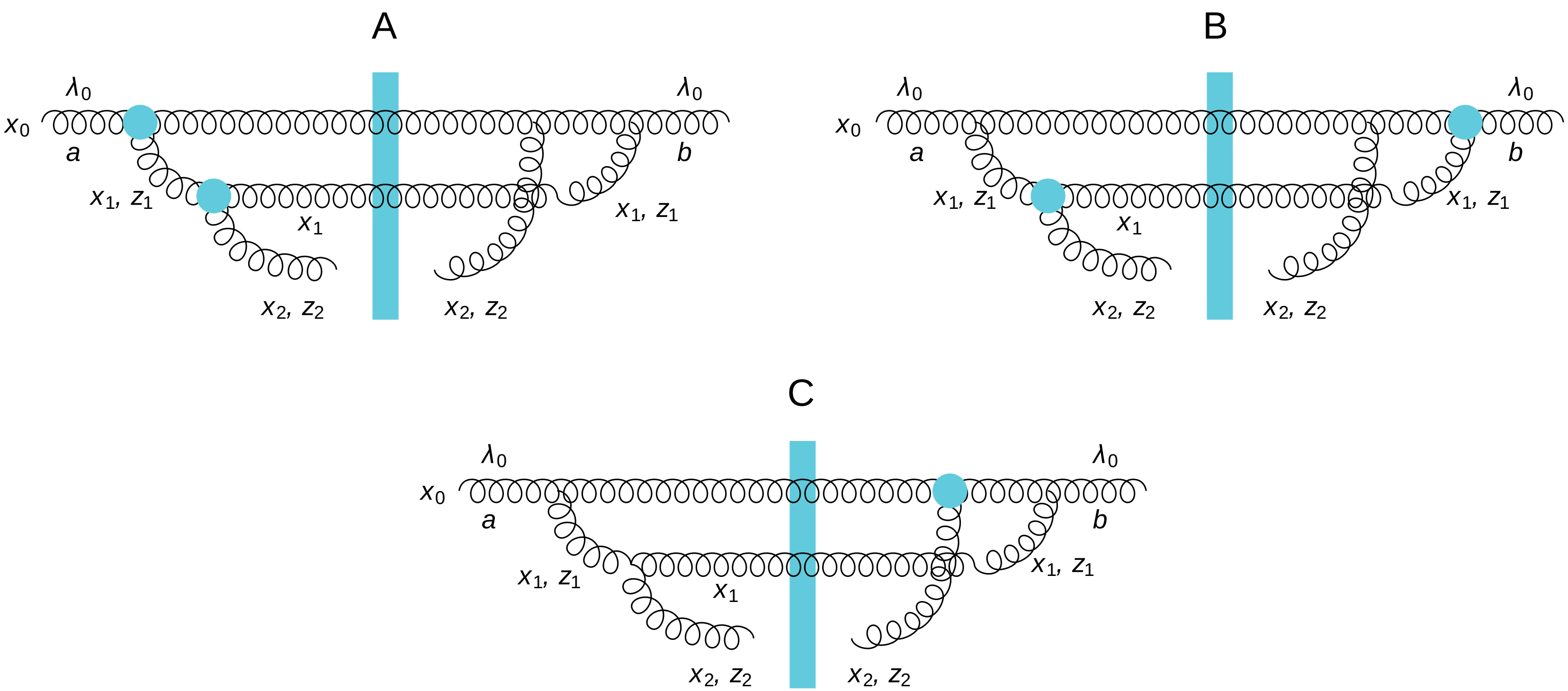


$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2\, \ln(zs\, x_{10}^2)\right]$$

Non-Ladder Diagrams

- Ladder diagrams are not the whole story. The non-ladder diagrams below are also leading-order (that is, DLA).



- Non-ladder soft quark emissions cancel for flavor-singlet observables we are primarily interested in. Non-ladder soft gluons do not cancel.

Large-N_c Evolution: Equations

- Here

$$\Gamma^{gen}_{20,21}(z's) = \theta(x_{20} - x_{21})\,\Gamma_{20,21}(z's) + \theta(x_{21} - x_{20})\,G_{20}(z's)$$

  is an object which we know from the quark helicity evolution, as the latter gives us G and Γ.

- Note that our evolution equations mix the gluon (G^i) and quark (G) small-x helicity evolution operators:

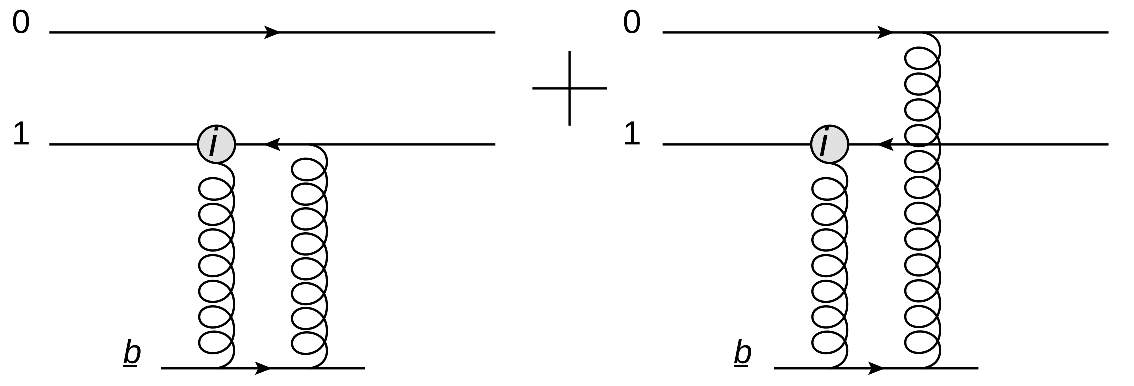
$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\,(x_{21})^j_\perp}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\,(x_{20})^j_\perp}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's)\right]$$

Initial Conditions

- Initial conditions for this evolution are given by the lowest order t-channel gluon exchanges:



$$\int d^2 b_{10}\, G_{10}^{i\,(0)}(zs) = \int d^2 b_{10}\, \Gamma_{10,21}^{i\,(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\, \pi\, \epsilon^{ij}\, x_{10}^j\, \ln\frac{1}{x_{10}\,\Lambda}$$

- Note that these initial conditions have no ln s, unlike the initial conditions for the quark evolution:

$$\int d^2 b_{10}\, G_{10}^{(0)}(zs) = \int d^2 b_{10}\, \Gamma_{10,21}^{(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\pi\, \ln(zs\, x_{10}^2)$$

Small-x Evolution at large N_c

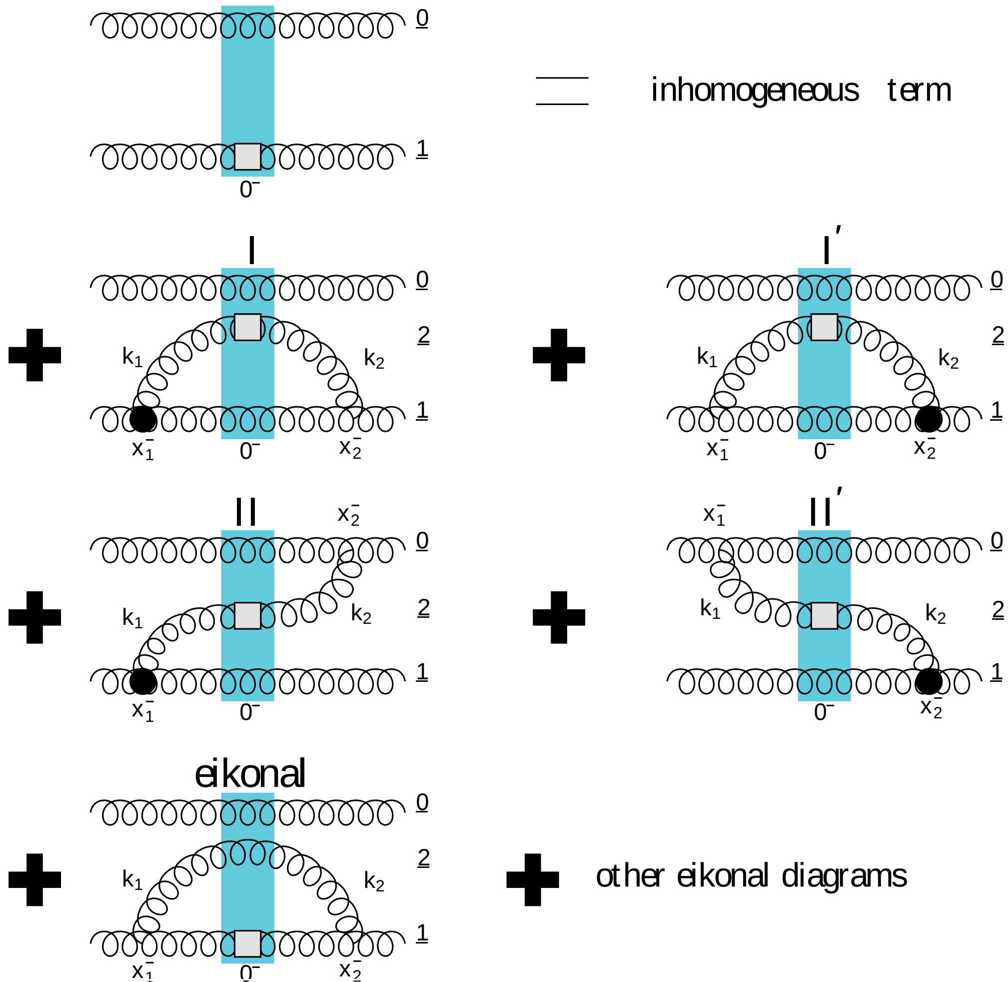
- At large N_c the evolution is gluon-driven. We will evolve a gluon dipole, remembering that at large N_c the relation between the adjoint and fundamental longitudinally-polarized gluon dipoles is

$$G_{10}^{adj}(z) = 4\, G_{10}(z)$$

(Note that the factor is 4, not 2 like in the unpolarized dipole case.)

Small-x Evolution at large N_c

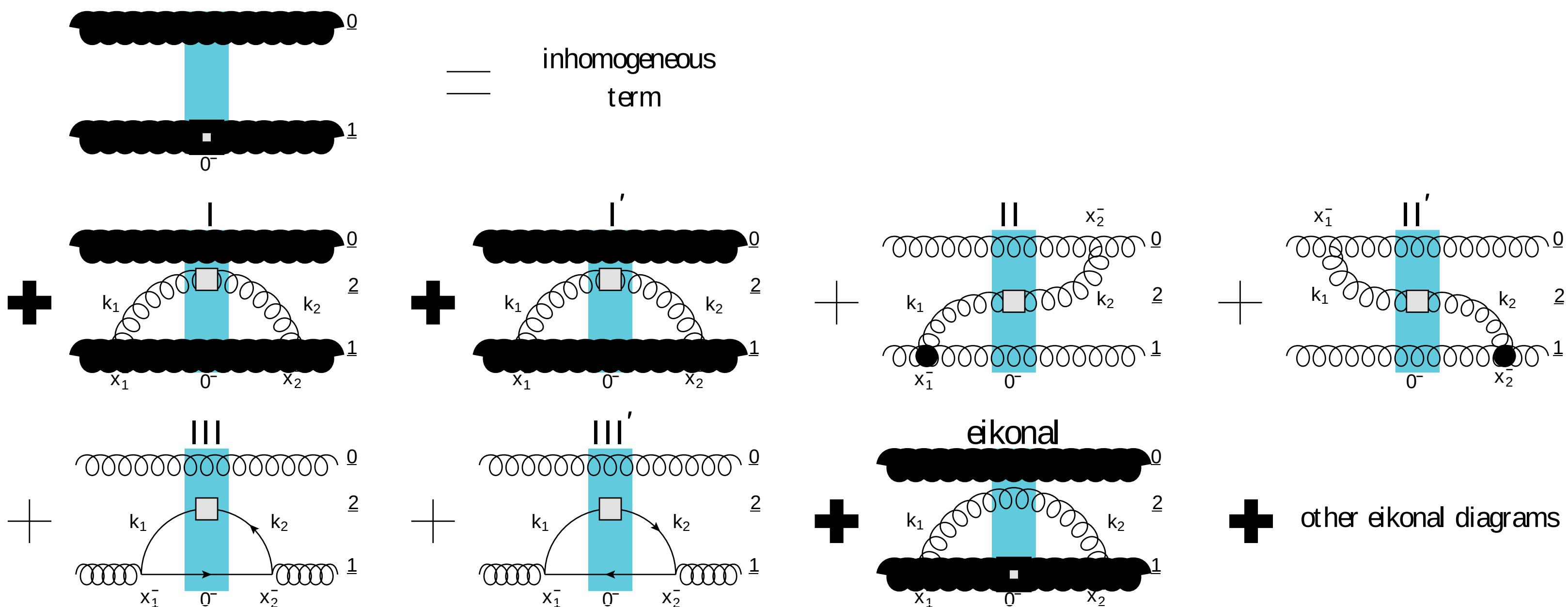
- We need to sum the following diagrams (box denotes the polarized "Wilson lines"):



Small-x Evolution at large N_c&N_f

- At large N_c&N_f there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.

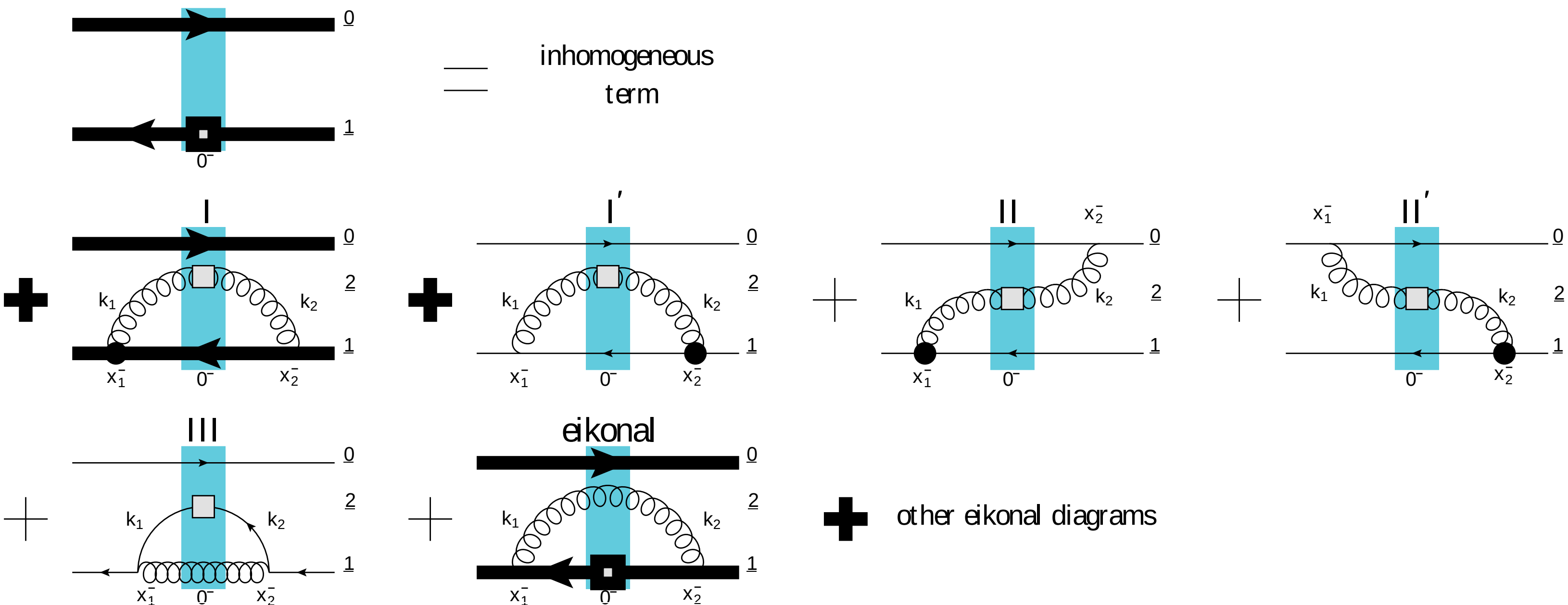
- Here's the adjoint dipole evolution:



Small-x Evolution at large N_c&N_f

- At large N_c&N_f there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.

- Here's the fundamental dipole evolution:



Small-x Evolution at large N_c&N_f

- The resulting equations are

$$Q_{10}(zs) = Q_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{1/(z's)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left\{ \frac{1}{2}\,\Gamma_{02,21}^{adj}(z') + \frac{1}{2}\,G_{21}^{adj}(z') + Q_{12}(z') - \bar{\Gamma}_{01,21}(z') \right\}$$

$$+ \frac{\alpha_s\, N_c}{4\,\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/(z's)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2}\, Q_{21}(z'),$$

$$G_{10}^{adj}(z) = G_{10}^{adj\,(0)}(z) + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z} \frac{dz'}{z'} \int\limits_{1/(z'\,s)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma_{10,21}^{adj}(z') + 3\, G_{21}^{adj}(z') \right]$$

$$- \frac{\alpha_s\, N_f}{2\,\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/(z'\,s)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2}\, \bar{\Gamma}_{02;21}(z'),$$

These are yet to be solved.

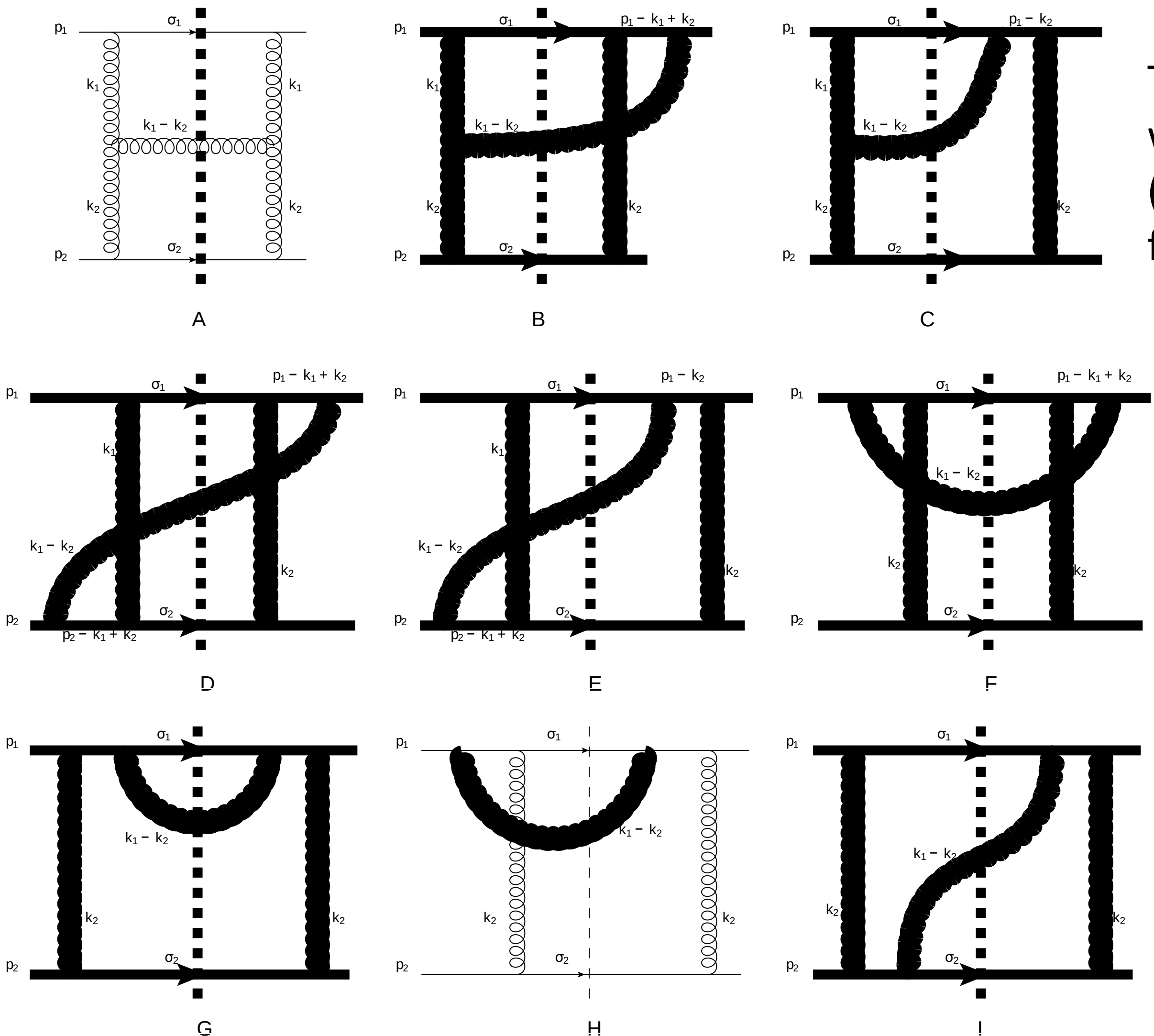
$$\Gamma_{10,21}^{adj}(z') = \Gamma_{10,21}^{adj\,(0)}(z') + \frac{\alpha_s\, N_c}{2\pi} \int\limits_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''\,s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma_{10,32}^{adj}(z'') + 3\, G_{32}^{adj}(z'') \right]$$

$$- \frac{\alpha_s\, N_f}{2\,\pi} \int\limits_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''\,s)}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2}\, \bar{\Gamma}_{03;32}(z''),$$

$$\bar{\Gamma}_{10,21}(z') = \bar{\Gamma}_{10,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left\{ \frac{1}{2}\,\Gamma_{03,32}^{adj}(z'') + \frac{1}{2}\,G_{32}^{adj}(z'') + Q_{32}(z'') - \bar{\Gamma}_{01,32}(z \right.$$

$$+ \frac{\alpha_s\, N_c}{4\,\pi} \int\limits_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int\limits_{1/(z''s)}^{x_{21}^2 z/z'} \frac{dx_{32}^2}{x_{32}^2}\, Q_{32}(z'').$$

Comparison with BER

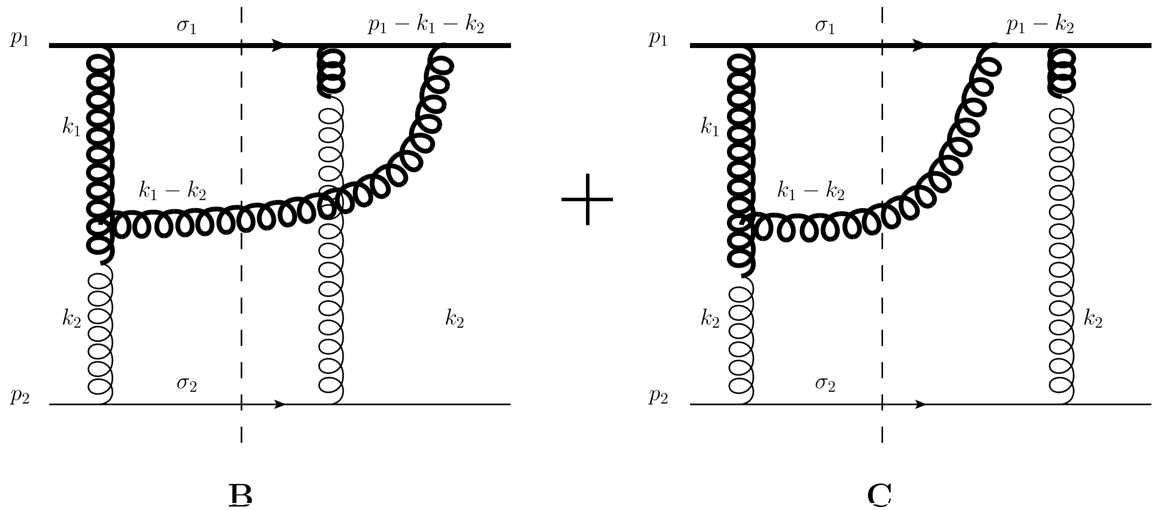


To better understand BER work, we (KPS) tried calculating one (real) step of DLA helicity evolution for the qq->qq scattering (ca 2016).

It appears that we have identified the k_2>> k_1 and k_1>> k_2 regimes in which diagrams A, B, C, D, E, I are DLA, which were possibly not considered by BER for B, C, … I. (Clarification to follow.)

Diagrams B and C: neither ladders nor bremsstrahlung gluons

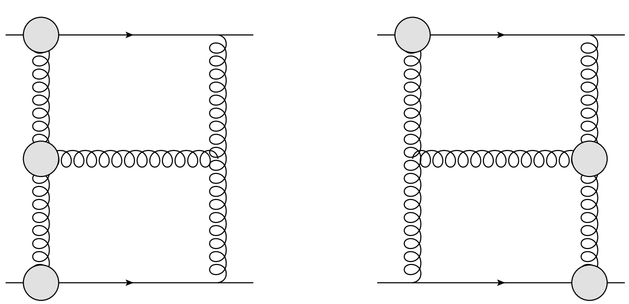
- Consider now diagrams B and C in the opposite kinematics, k2 << k1.

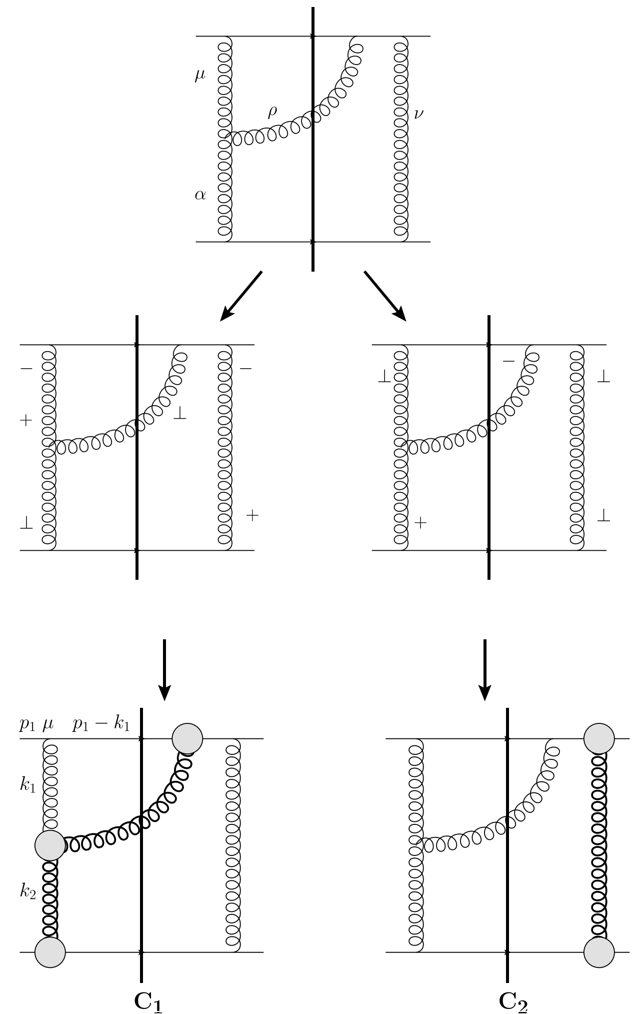


- There are 2 softest gluons k2. But it does not look like a ladder. If we apply IREE prescription, we split the diagram in two, but the upper part does not appear to look like a ladder.

Diagram C

- At this order it is OK. One can redefine how the ladder goes (now thick lines describe helicity flow, circles = non-eikonal helicity-dependent vertices).
- Also, C1 = C2.
- This is based on the observation by Boussarie, Hatta and Yuan ‘19.
- This is all one needs for NNLO anomalous dimension.
- `Regular’ ladders look like this:

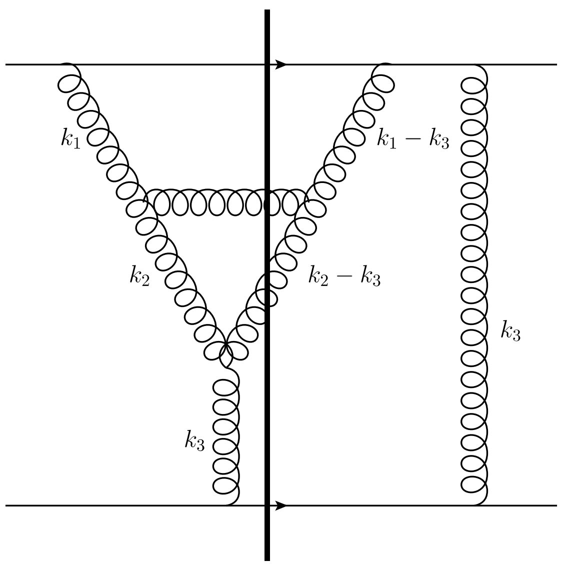




Higher orders?

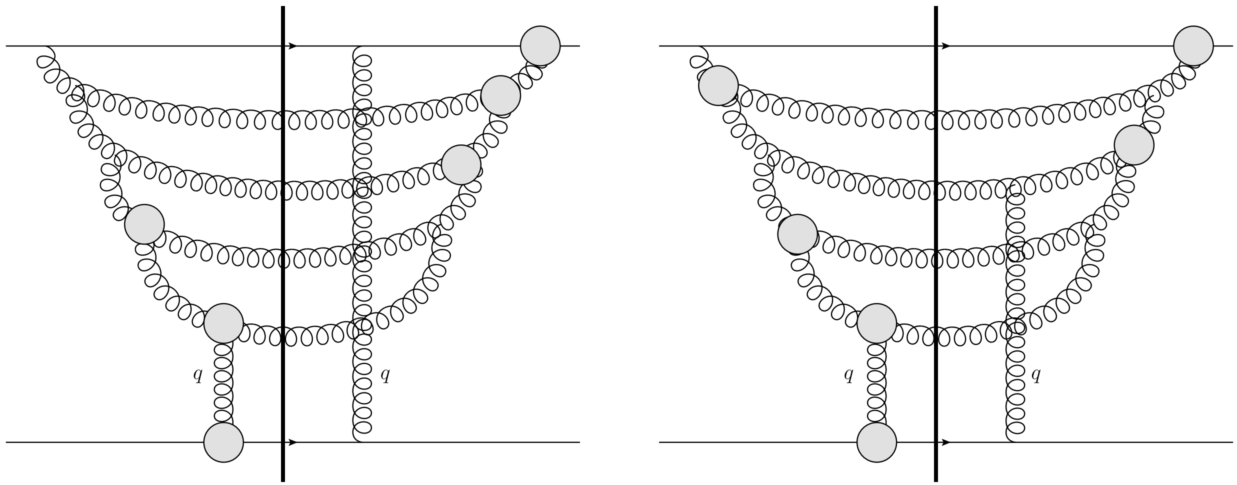
- I do not see how this iterates to higher orders.

- How do we define the ladder in diagrams like this one? (if this is DLA)



Higher orders

- Iterating further, one arrives at diagrams like the ones in our shock wave approach:



- If q are the softest 2 gluons, how do these graphs arise from IREE? I know for sure these are DLA in our shock wave calculation.