A Gauge Principle in Financial Markets

In honour of Stan Brodsky’s 80th Birthday

(CERN Feb 2020)

- Covariant derivative in QCD

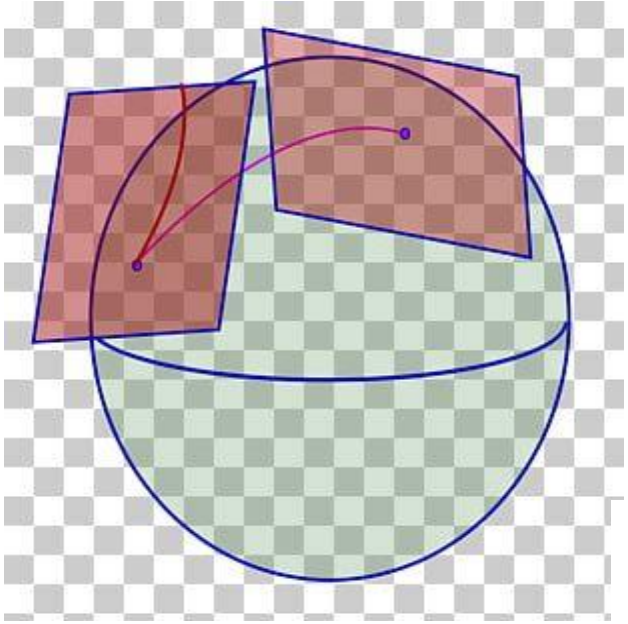
$$\partial_\mu \to D_\mu = \partial_\mu + ig\,\mathscr{A}_\mu$$

$$\downarrow$$

$$\mathscr{A}_\mu = A_\mu T^a$$

Cov-derivatives is an important concept in differential geometry, not just in QED,QCD,…

- Example 1:



Component wise differentiation of a vector

$$\frac{\partial Y^{\mu}}{\partial X^{\nu}} =: Y^{\mu}{}_{,\nu}$$

cannot be full story since basis itself changes along the sphere. Better candidate for differentiation:

Christoffel-symbol

$$\underset{\substack{\uparrow\\ \text{tensor}}}{Y^{\mu}{}_{;\nu}} =: \underset{\substack{\uparrow\\ \text{no tensor}}}{Y^{\mu}{}_{,\nu}} + Y^{\lambda}\,\underset{\substack{\uparrow\\ \text{no tensor}}}{\Gamma^{\mu}_{\lambda\,\nu}} \leftarrow$$

Levy-Civitae connection

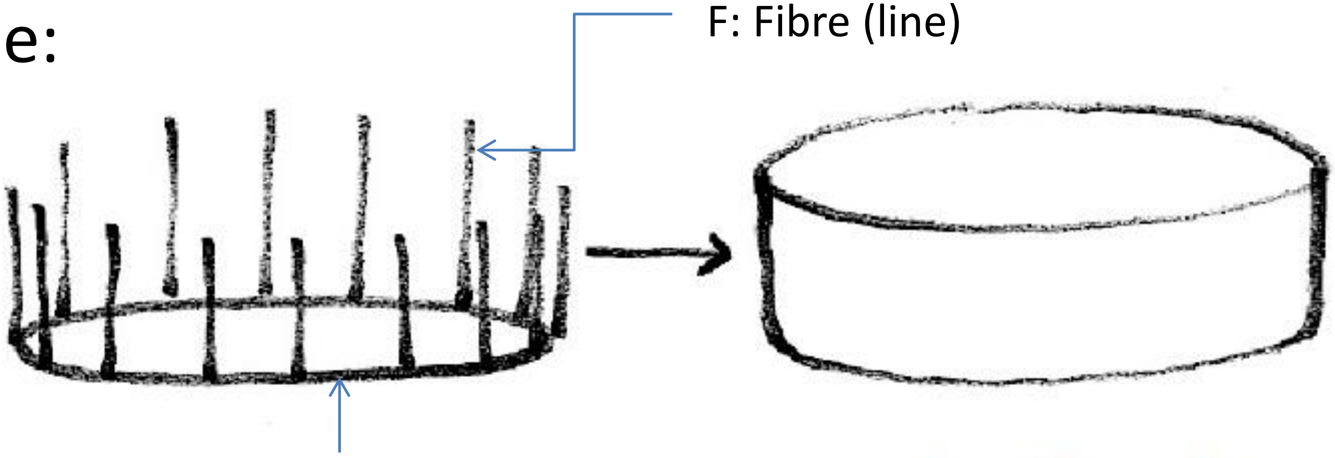
Cov -derivative:

$$D_{\nu} := \partial_{\nu} + \Gamma^{\cdot}{}_{\cdot,\nu}$$

Γ is a covariant derivative just like $\mathscr{A}_{\mu}$!

Need for cov-derivatives arise whenever fibre-bundle structure applies and one wants to move from one point of a fibre to a neighbouring fibre point. Extra structure required to do that is provided by a connection

Simplest fibre bundle:



F: Fibre (line)

M: base manifold (circle)

$$E = M \times F$$

Total space

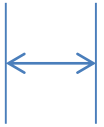
Base mfd

Fibre

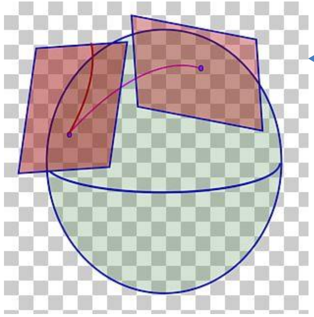
Projection $\pi: E \rightarrow M$
$p \rightarrow \pi(p)$

fibre at p: $\pi^{-1}(p)$

This is a trivial bundle. It admits a natural flat connection:



Recall: **Example 1**; this is a fibre bundle with

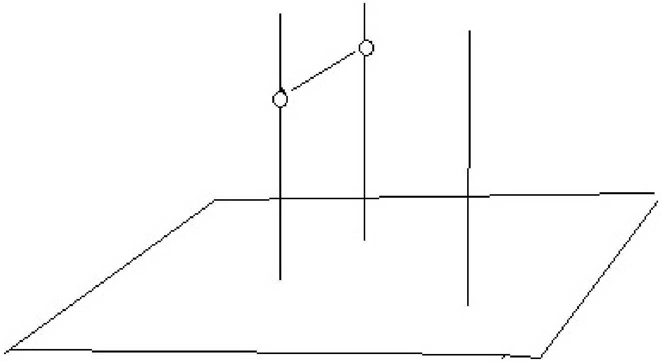


Base mfd: $M = S^2$ ← Tangent bundle

Fibre: $F = \mathbb{R}^2$

Attach a vector space at each point of the base mfd (vector bundle)

Example 2: QCD ; M: Minkovski space $F = \mathbb{C}$



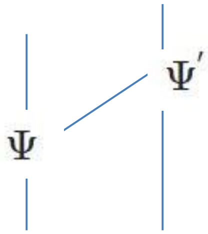
QCD wavefunctions are sections of a line bundle with base manifold=Minkovki space

Def: the map $\sigma: M \rightarrow E\ (total\ space)$

is called a section if $\pi \circ \sigma = id_M$

Identify “similar” neighbouring pts via cov-deriv

$$D\,\Psi = 0$$



Ψ agrees with Ψ' modulo a gauge transformation

$$D\big(g(x)\Psi\big) = g(x)D\,(\Psi) = 0$$

Note: Not all fibre-bundles are of the form

$$E = M \times F$$

In general one observes $E = M \times F$ only locally for open subsets $U_i \subset M$

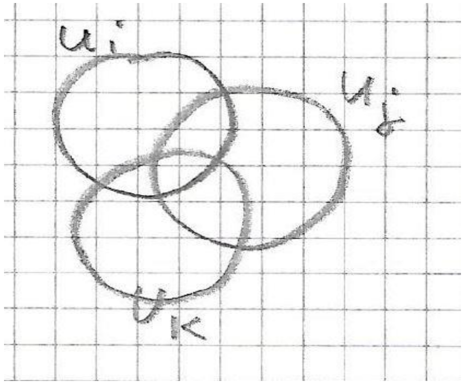
Let $\{U_i : U_i \subset M;\ i \in N\}$ be an open cover of M $\cup_i U_i = M$

Locally there is a diffeomorphism:

Local trivialization: $\Phi_i \ : U_i \times F \to \pi^{-1}(U_i)$

$$e \in E : \Phi_i^{-1}(e) = (p, f) \, ; \, p \in M, f \in F \qquad \Phi_i = \Phi(p, f) =: \Phi_p(f)$$

The whole bundle is reconstructed by patching together the local pieces U_i subject to gluing conditions



The transition maps

$$t_{ij}(p) = \Phi_{i,p}^{-1} \circ \Phi_{j,p}$$

$$F \to F$$

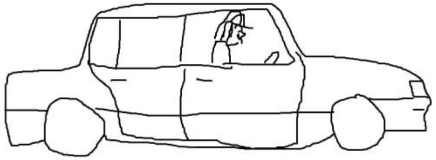
induce a group structure

$$t_{ii}(p) \ : \ identity \ map$$

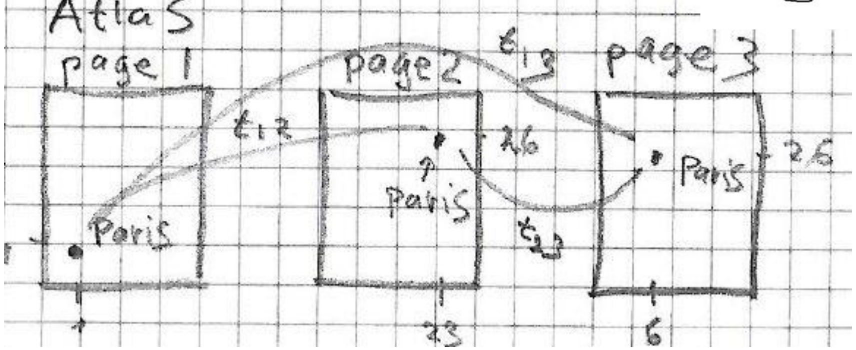
$$t_{ij}^{-1} = t_{ji}$$

$$t_{ik} = t_{ij} t_{jk}$$

Example: Holiday (in the old days)



Navigate via map: Atlas

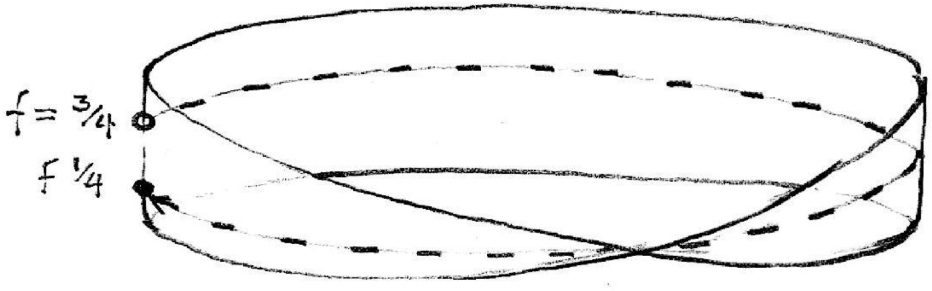


Aliens could deduce that earth is a sphere by studying group structure of transition maps of the atlas

The group structure of transition maps encodes aspects of the geometry in an intrinsic way.

Non-trivial bundles carry their own “gauge theory” that is inherent In their geometry

Trivial bundle



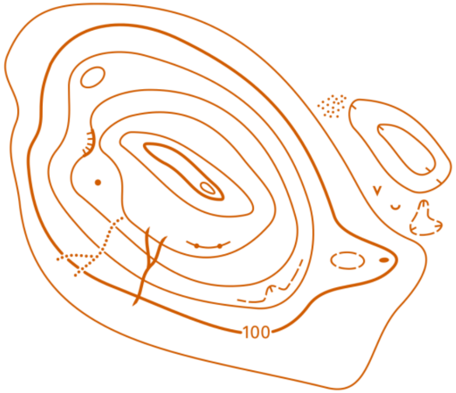
Mobius band:non-trivial;goup: $\mathbb{Z}_2$

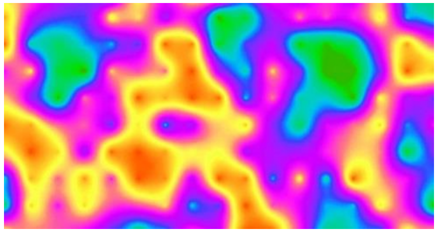
Note: in trivial bundles transition maps can be chosen as identity map and flat-connections are possible.

However even for trivial bundles non-trivial connections are possible. Example: Standard Model

M:Minkovski-space is contractible to a point. Hence bundle is trivial, yet SU(3) as gauge symmetry

Think of DX=0 as level lines or heat map





Many connections ∇ are plausible as long as the adhere to a certain mathematical structure:

$$\nabla : \Gamma(E) \to \Gamma(E \oplus T^*(M))$$

where ∇ obeys the Leibniz rule:

$$\begin{cases} \nabla(\sigma f) = (\nabla\sigma)f + \sigma \oplus df \\ \sigma \in \Gamma(E), f \in C^\infty(M) \end{cases}$$

Contracting with a tangent vector gives cov-deriv:

$$\begin{cases} \nabla_X : \Gamma(E) \to \Gamma(E) \\ \nabla_X(\sigma) = (\nabla\sigma)(X) \end{cases}$$

On trivial (vector) bundles, every connection can be written as

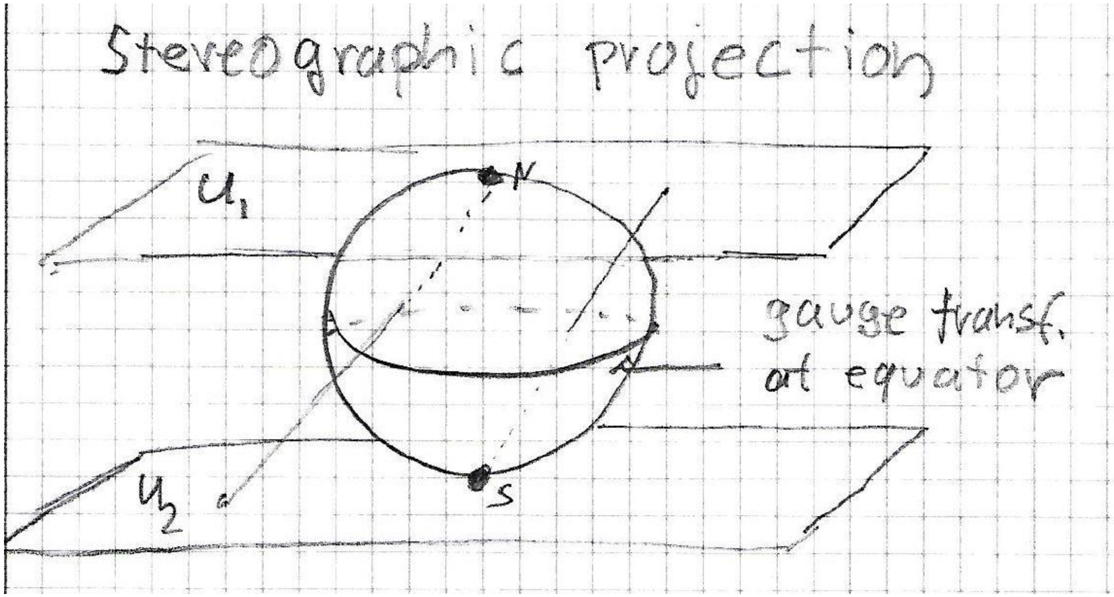
$\nabla e^i = \omega^i_j \, e^j$ or $\nabla_X e^i = \omega^i_j(X) \, e^j$ where ω^i_j is a matrix of 1-forms

This is still true for non-trivial bundles but only locally. Going from one patch with basi $\{e\}_i$ with $\nabla e^i = \Theta^i_k e^k$ to a new patch with basis $\{\tilde{e}_i\}$ with $\nabla \tilde{e}^i = \tilde{\Theta}^i_k e^k$

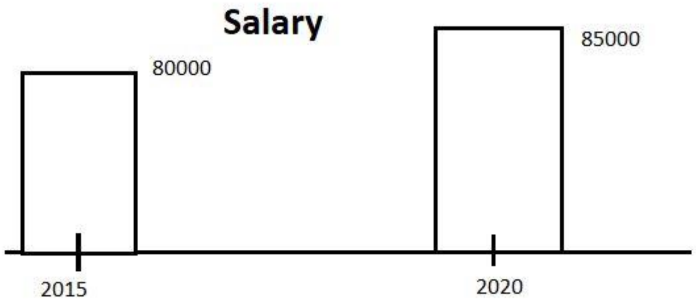
It is easy to show $e =: g \, \tilde{e}$ $g \in GL(n, \mathbb{R})$

$$\tilde{\Theta} = g^{-1} \Theta \; g - g^{-1} \, dg$$

∇ transforms between patches via gauge transformation

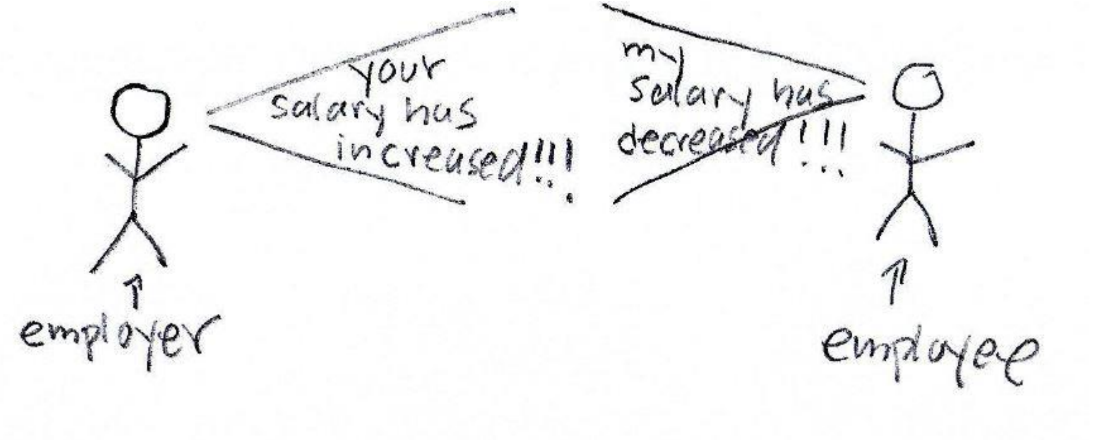


Connections in Finance



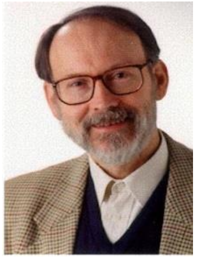
Question:
Who is right?

Answer: Both! But they are using a different (covariant) derivative!



$$D^{Employer} = \frac{d}{dt}$$

$$D^{Employee} = \frac{d}{dt} - r_{inflation}$$



Connections in Finance

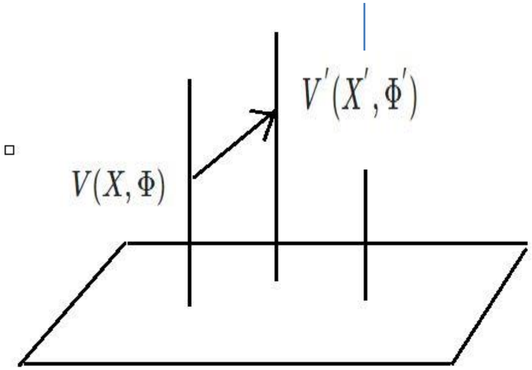
Let $X_\mu(t)\ :\ \mu = 1,\ldots,n$ prices of financial instruments (stocks,bonds,..)

Let $\Phi_\mu(t)$ holdings at time t of instrument µ. A portfolio V with allocations $\Phi_\mu(t)$ is worth

$$V = \sum_\mu \Phi_\mu X_\mu$$

This defines a fibre-bundle with base space $M = (t, X, \Phi)$

$X = (X_1, \ldots, X_n), \Phi = (\Phi_1, \ldots, \Phi_n)$ with fibres consisting of all possible portfolios



In order to measure the quality of a portfolio manager (who controls $\Phi_\mu(t)$) do not allow injection or withdrawal of money

Connections in Finance

$$dV = \sum_\mu \Phi_\mu dX_\mu + \underbrace{\sum_\mu X_\mu d\Phi_\mu}_{=0}$$

$$\boxed{\sum_\mu X_\mu d\Phi_\mu = 0}$$

⇕

Self-financing condition

Define covariant derivative:

$$D_t = d - \frac{\sum_\mu X_\mu dX_\mu}{\sum_\mu X_\mu \phi_\mu}$$

What are the constant level lines?

$$0 = D_t V = \Phi_\mu dX_\mu + X_\mu d\Phi_\mu - \Phi_\mu dX_\mu = X_\mu d\Phi_\mu$$

Hence

$$D_t V = 0 \quad \Leftrightarrow \quad \textit{self financing investment}$$

This motivates the gauge potential:

$$\boxed{A := \frac{\sum_\mu \Phi_\mu dX_\mu}{\sum_\mu \Phi_m u X_\mu}}$$

Connections in Finance

What is the corresponding gauge group?

Try local gauge transformation:

$$\left\{ \begin{aligned} & X_\mu \to X_\mu^{'} = \Lambda(x) X_\mu \\ & A_\mu \to A^{'} = A + \frac{d\Lambda(X)}{\Lambda(X)} \end{aligned} \right.$$

$$\left\{ \begin{aligned} & V = \Phi_\mu X_\mu \\ & V \to V^{'} = \Lambda \, \Phi_\mu X_\mu \\ & D \to D^{'} = D - \frac{d\Lambda}{\Lambda} \end{aligned} \right.$$

$$D^{'} V^{'} = \left(d - \frac{\Phi_\mu dX_\mu}{\Phi_\nu X_\nu} - \frac{d\Lambda}{\Lambda} \right) \left(\Lambda \Phi_\lambda X_\lambda \right) = \Lambda DV + \Phi_\lambda X_\lambda d\Lambda - \Phi_\lambda X_\lambda d\Lambda$$

$$= \Lambda \, DV$$

D is indeed a cov-derivative. The symmetry group describes dilations

Curvature and arbitrage

There is no absolute scale in finance: Scaling all assets by an common factor does not change the economics. Any (non-zero) assets can be used equally as a reference.

Now that A is identified, apply machinery of differential geometry, calculate curvature (field strength)

$$R = dA + [A \wedge A]$$

It turns out

$$R = \frac{1}{V^2}\sum_{\mu,\nu}\Phi_\mu X_\nu X_\mu\left(\frac{dX_\mu}{X_\mu} - \frac{dX_\nu}{X_\nu}\right) \underset{\substack{\downarrow \\ \text{wedge}}}{\wedge} d\Phi_\nu$$

Curvature and arbitrage

What is arbitrage?

Riskless way to make profits: enter 1 Euro into slot machine retrieve 2 Euros

It can be shown:

$$R = 0 \leftrightarrow \textit{no arbitrage opportunities}$$

Differential geometry is a natural picture for finance

Geometrically, curvature describes net effect of parallel transport of a vector around a closed loop. $R \neq 0$ would allow making systematic money by shifting assets around after ending up at the original allocation.