A momentum representation for loop computations in gravity

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Aim: Gravity for particle phenomenologists

How is gravity different?

Here we will approach gravity with the tools and mindset of

a (H) effective field theorist

Indeed gravity is a non-renormalizable, non-linear theory,

& in most applications in particle physics an EFT with a very good convergence

$$\sqrt{G_N}E = \frac{E}{M_{\rm pl}}$$

In fact too good. There are however 'zero background' processes e.g.

Global symmetry breaking

Gravity from local symmetry

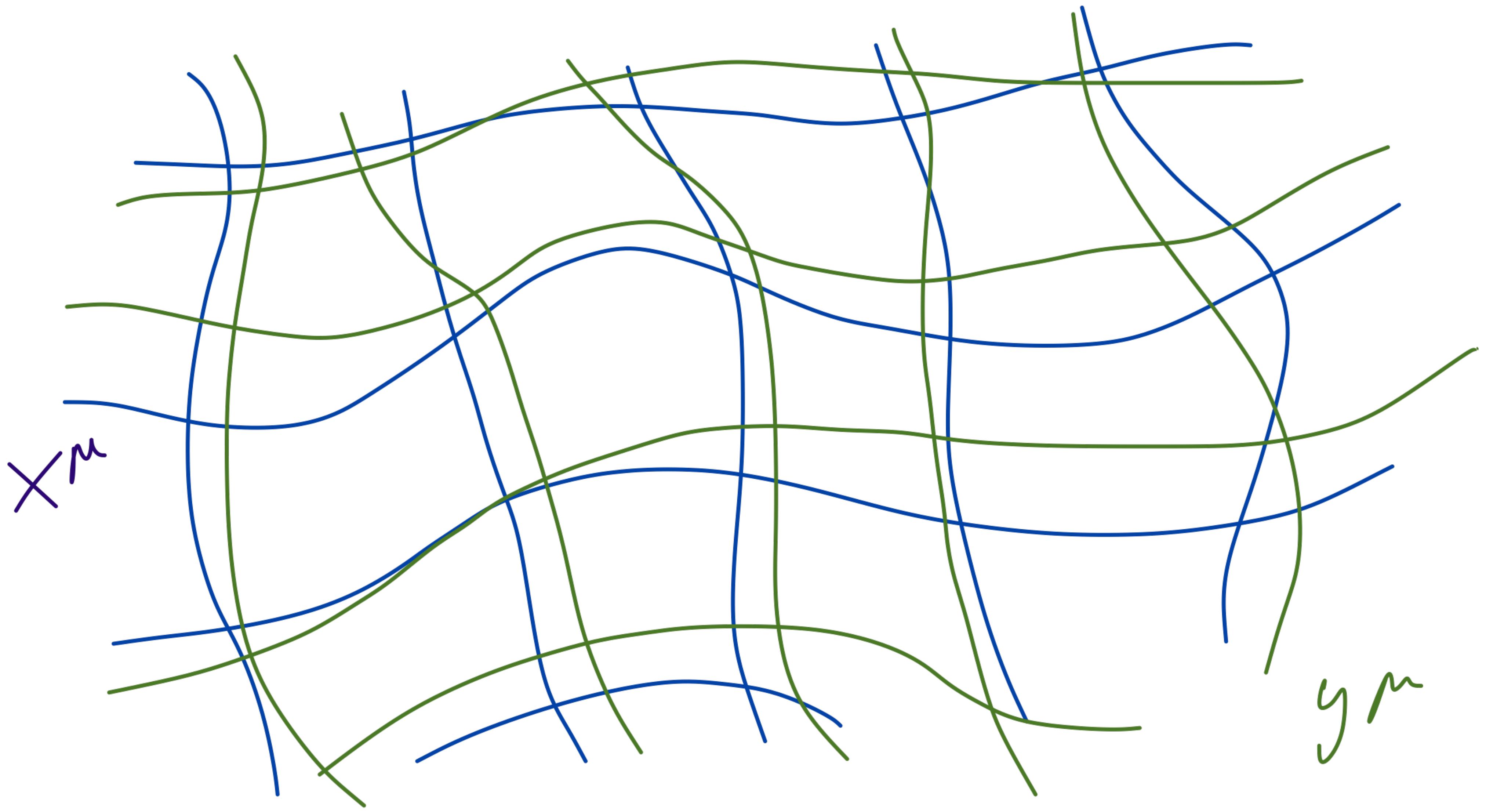
The physics (the action) is the same for all observers

Under coordinate transformations

$$x^\mu = x^\mu(y) \simeq y^\mu + \varepsilon^\mu(y)$$

Tensors transform linearly, e.g. the metric g

$$\tilde{g}_{\alpha\beta} = \left(\frac{\partial x}{\partial y} g(y) \frac{\partial x}{\partial y}\right)_{\alpha\beta} \simeq \left(\frac{\partial \varepsilon}{\partial y} g + g \frac{\partial \varepsilon}{\partial y}\right)_{\alpha\beta}$$



Perturbative dynamics of gravity follow from splitting the metric $g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \sqrt{G_N}\, h_{\alpha\beta}$

such that h transforms as

$$\sqrt{G_N}\, \tilde{h}_{\alpha\beta} = \partial_\alpha \varepsilon_\beta + \partial_\beta \varepsilon_\alpha + \cdots$$

the dynamical metric picks a shift

which forbids mass and removes longitudinal d.o.f.

Gravity from local symmetry

A covariant index transforms as
$$V^\mu \to \widetilde{V}^\mu(y) = \frac{\partial y^\mu}{\partial x^\nu} V^\nu \simeq -\frac{\partial \varepsilon^\mu}{\partial x^\nu} V^\nu \equiv -\Theta^\mu_\nu V^\nu$$

The derivative of a covariant index
$$\partial_\mu V^\nu \to \widetilde{\partial_\mu V}^{\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} V^\beta$$
$$\simeq -\underbrace{\frac{\partial^2 \varepsilon^\nu}{\partial x^\mu \partial x^\beta}}_{\partial_\mu \Theta} V^\beta + (\text{linear on } {}^{\nu}_{\mu})$$

$\neq 0$ for x-dependent $\Theta(x)$

We introduce a covariant derivative
$$\tilde{\Gamma}^\nu_{\mu\beta} = \frac{\partial^2 \varepsilon^\nu}{\partial x^\mu \partial x^\beta} + (\text{linear on } {}^{\nu}_{\mu\beta}) = \frac{\partial \Theta^\nu_\beta}{\partial x^\mu} + (\text{linear on } {}^{\nu}_{\mu\beta})$$

same

left out

In analogy with internal gauge symmetries
$$\tilde{A}_\mu \cdot T^i{}_j = \frac{\partial(\theta \cdot T)^i{}_j}{\partial x^\mu} + [A_\mu \cdot T, \theta \cdot T]^i{}_j = \frac{\partial(\theta \cdot T)^i{}_j}{\partial x^\mu} + \mu(\text{linear on } {}^i{}_j)$$

The elementary field is the metric though!

Gravity as a non-renormalizable QFT

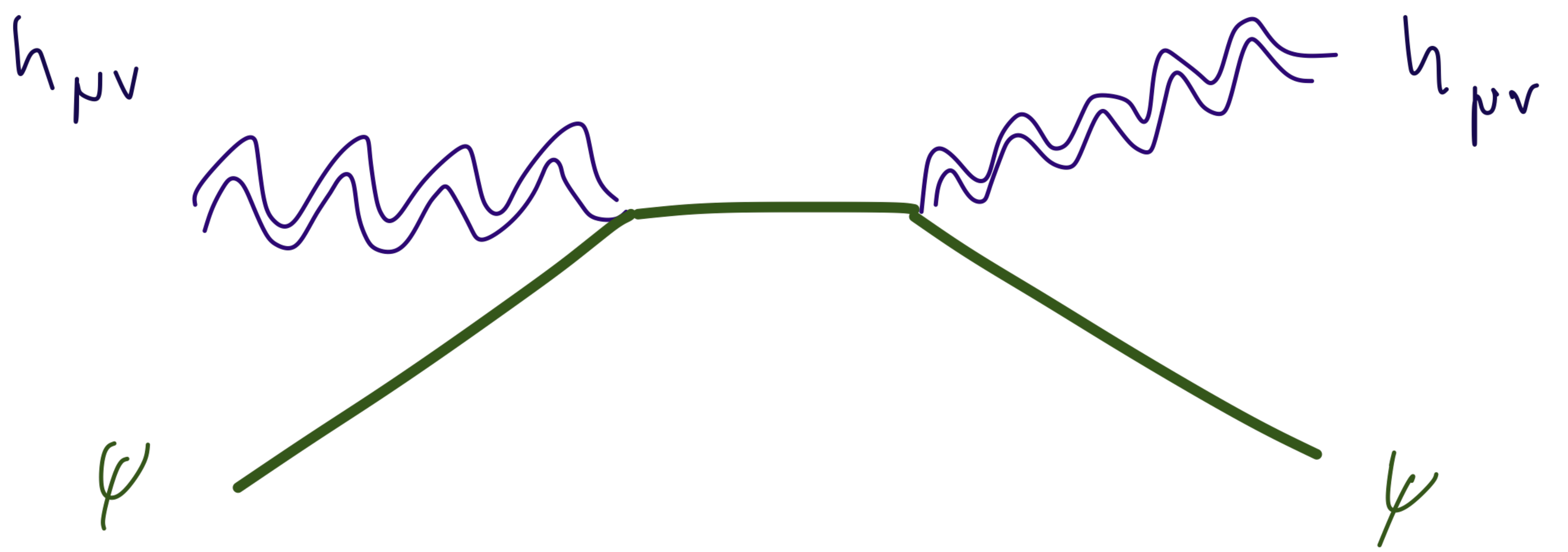
One can now expand on the metric around a vacuum solution and do QFT as usual

$$S = \int d^d x \sqrt{g} \left(\mathcal{L}_{\text{matter}} - \frac{R - 2\Lambda}{16\pi G_N} \right) \quad \text{with}$$

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \sqrt{G_N}\, h_{\alpha\beta}$$

$$\begin{aligned} g^{\alpha\beta} &= \left(\bar{g} + \sqrt{G_N} h \right)^{-1} \\ &= \bar{g}^{\alpha\beta} - \sqrt{G_N} (\bar{g} h \bar{g})^{\alpha\beta} + G_N (\bar{g} h \bar{g} h \bar{g})^{\alpha\beta} + \dots \end{aligned}$$

It is a non-linear and non-renormalizable theory and computable in the weak field or perturbative regime



One loop computations

… and in particular loop computations

Functional methods are specially useful in this scenario

to avoid cumbersome Feynman rules, summing over diagrams

& to obtain the effective action

$$e^{i\Gamma_{\rm eff}} = \int D\delta\Phi e^{iS[\Phi]+i\delta S\delta\Phi+i\delta^2 S\delta\Phi^2+\cdots} = \left(\det\left(-\frac{\delta^2 S}{\delta^2\Phi}\right)\right)^{-\frac{1}{2}} e^{iS[\Phi]}$$

Computing a functional determinant

One has therefore, formally

$$\det(\mathcal{O}) = \exp\left(\mathrm{tr}(\log(\mathcal{O}))\right) \ .$$

the evaluation requires specification of this functional trace determinant

whereas the operators of interest will have the form

$$\mathcal{O} = \sqrt{g}\left(\nabla^{\mu}\nabla_{\mu} + U(\Phi(x))\right) \ ,$$

There's a number of ways to this, for one using momentum or coordinate representations

Position representation: Heat-Kernel

In this technique one uses a differential equation formalism; the kernel

$$\mathrm{Ker}(x,y;\tau) \equiv e^{i\tau\mathcal{O}}$$

which obeys a partial differential equation resembling the Heat equation

$$\frac{\partial}{i\partial\tau}\mathrm{Ker}(x,y;\tau) = \mathcal{O}\,\mathrm{Ker}(x,y;\tau) = (\Box + U(\Phi))\,\mathrm{Ker}(x,y;\tau)$$

with the decomposition in De-Witt coefficients 'a'

$$\mathrm{Ker}(x,y;\tau) = \frac{i\sqrt{\mathcal{D}(x,y)}}{(4\pi i\tau)^{d/2}}\exp\left(i\frac{\sigma(x,y)}{2\tau}\right)\Omega(x,y;\tau) \qquad \Omega(x,y;\tau) = \sum (i\tau)^n a_n(x,y)$$

which turn the PDE into an iterative process and in terms of which

$$\mathrm{tr}(\log(\mathcal{O})) = -\frac{i}{(4\pi)^{d/2}}\int\sqrt{g}d^dx\int\frac{d\tau}{\tau}(i\tau)^{n-d/2}a_n(x,x)$$

See e.g.
[Barvinsky & Vilkovisky QGPR119 1985]

Momentum representation: Gauge theory

On the other hand one can directly evaluate the trace into a double integral

$$\mathrm{tr}\,(\log\mathcal{O}) = \int \frac{d^d q d^d x}{(2\pi)^4} e^{-iqx} \log(\mathcal{O}) e^{iqx}$$

Where we have taken the field we integrate over to momentum representation

$$D_\mu \Phi(x) \to i(q_\mu + A_\mu(x))\Phi(q)$$

however the problem with this naive transformation is that it

does not maintain a covariant description throughout

Covariant Derivative Expansion

Take a covariant derivative in general between operators that depend on x,q

$$\mathcal{O}(q,x)\ (iq_\mu + \partial_\mu + iA_\mu)\ \mathcal{P}(q,x)$$

One can restore a covariant description with a transformation

cancel out

$$e^{iD\partial_q}(iq_\mu + D_\mu)e^{-iD\partial_q} = iq_\mu - [D\partial_q, q_\mu] + D_\mu$$
$$+ [iD\partial_q, D_\mu] + \frac{1}{2}[iD\partial_q, [iD\partial_q, iq_\mu]] + \dots$$
$$= iq_\mu + \frac{i}{2}\partial_q^\nu [D_\nu, D_\mu] + \dots$$

[M.K. Gaillard NPB268]
[Henning, Lu & Murayama, 1412.1837]
[Drodz, Ellis, Quevillon & You, 1512.03003]
....

With the Baker-Campbell-Hausdorff

we work this out to all orders in q

$$= iq_\mu + \sum_{n=1}^{\infty} \frac{n}{(n+1)!} \left([i\partial_q D\ ,\ )^n\ , D_\mu(]\right)^n$$

Covariant Derivative Expansion for Gravity

How about gravity where we just change the gauge field for the connection?

$$\nabla_\mu = \partial_\mu + \Gamma^{\cdot}_{\mu\cdot}$$

applying the same procedure now

$$\begin{aligned} e^{i\nabla\partial_q}(iq_\mu + \nabla_\mu)e^{-i\nabla\partial_q} &= iq_\mu - [\nabla\partial_q, q_\mu] + \nabla_\mu + \dots \\ &= iq_\mu - [\nabla, q_\mu]\,\partial_q - \nabla\,[\partial_q, q_\mu] + \nabla_\mu + \dots \end{aligned}$$

$\neq 0!$

cancel out

the covariant derivative index does itself transform under coordinate changes

Covariant Derivative Expansion for Gravity

there's then the need for a more general transformation which we write as

$$e^{iT}; \qquad T = \sum_n T_{(n)}\,, \qquad T_{(n)}(\lambda q) = \lambda^{-n} T_{(n)}(q)\,,$$

where we solve iteratively in terms of inverse powers of momenta to obtain

Extra terms in Gravity

$$\begin{aligned} e^{iT}(iq_\mu + \nabla_\mu)e^{-iT} =& iq_\mu + \frac{i}{4}\{\partial_q^\nu, [\nabla_\nu, \nabla_\mu]\} + \frac{i}{12} R^\nu{}_{..\mu}\{\partial_q^{.2}, q_\nu\} \\ & - \frac{1}{6}\{[\partial_q\nabla, [\nabla_\nu, \nabla_\mu]]\,, \partial_q^\nu\} - \frac{1}{24}[\nabla_., R^\nu{}_{..\mu}]\{\partial_q^{.3}, q_\nu\} \\ & - \frac{i}{16}\{[\partial_q\nabla, [\partial_q\nabla, [\nabla_\nu, \nabla_\mu]]]\,, \partial_q^\nu\} - \frac{i}{80}[\nabla_., [\nabla_., R^\nu{}_{..\mu}]]\{\partial_q^{.4}, q_\nu\} \\ & + \frac{i}{48}\{R^\nu{}_{..\mu}\partial_q^{.3}, [\nabla_., \nabla_\nu]\} + \frac{7i}{720} R^\nu{}_{..\rho} R^\rho{}_{..\mu}\{\partial_q^{.4}, q_\nu\} + \mathcal{O}(q^{-4}) \\ \equiv & \, i(q_\mu + \mathcal{K}_\mu) \end{aligned}$$

same as gauge th

$\frac{1}{2}\frac{n}{n+1!} = \frac{1}{2}\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \dots\right)$

[R.A. 1912.09671]

Covariant Derivative Expansion for Gravity

From where however one can proceed as usual in this method

$$\begin{aligned}\operatorname{tr}\log(\mathcal{O}) &= \int \frac{d^4x d^4q}{(2\pi)^4} \int dm^2 \frac{1}{\mathcal{O}(i(q+\mathcal{K}), \tilde{\Phi}) + m^2} \\ &= -\int \frac{d^dx d^dq}{(2\pi)^d} \int dm^2 \frac{1}{q^2 - m^2 + \{q, \mathcal{K}\} + \mathcal{K}^2 - \mathcal{U}} \\ &= -\int \frac{d^4x d^4q}{(2\pi)^4} \int dm^2 \sum_n \left[\frac{1}{q^2 - m^2}\left(\mathcal{U} - \{q, \mathcal{K}\} - \mathcal{K}^2\right)\right]^n \frac{1}{q^2 - m^2}\end{aligned}$$

remembering however there is a background metric with which we're contracting covariant indexes

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^\mu g_{\mu\nu} q^\nu - m^2)^2} = \frac{1}{16\pi^2(d-4)}\sqrt{g} + \text{finite}$$

Ultraviolet divergences

"NEW" TERMS IN GRAVITY

SAME AS GAUGE TH

"NEW" TERMS IN GRAVITY

$$\frac{1}{16\pi^2(4-d)}\left[\left(\frac{R^2_{....}}{180}-\frac{R^2_{..}}{180}\right)\mathrm{tr}(\mathbb{I})+\frac{1}{12}\mathrm{tr}\left(\mathscr{R}_{\mu\nu}\mathscr{R}^{\mu\nu}\right)+\frac{1}{2}\mathrm{tr}\left(\mathcal{U}_{(0)}+\frac{R}{6}\right)^2\right]$$

$$\mathscr{R}_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$$

Standard Model + Gravity Renormalization

in the gravitational coupling $\kappa^2 = 8\pi G_N$

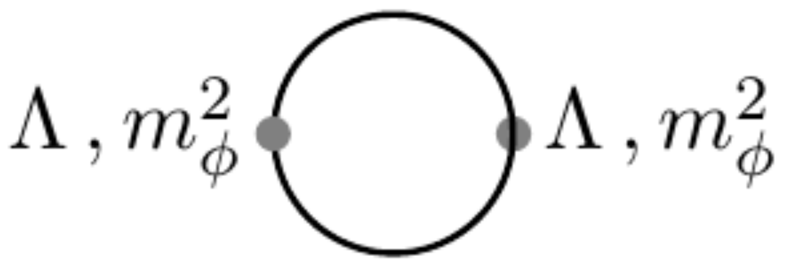
We put this to use to compute the UV divergences coming from gravitational interactions

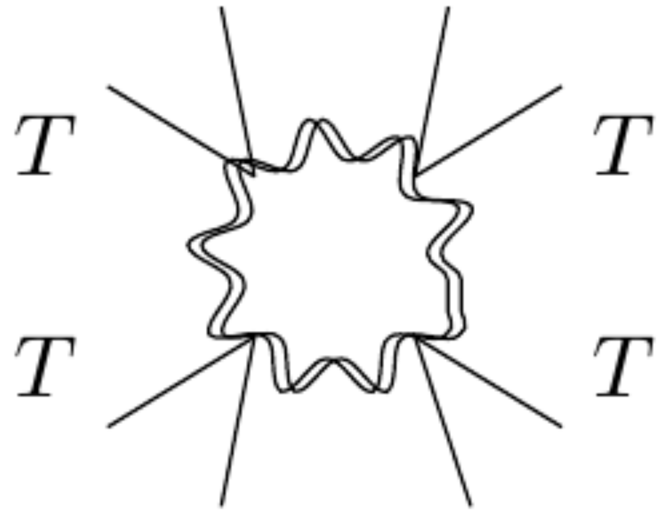
$$\frac{1}{16\pi^2(4-d)}\left(\frac{1}{2}m_\phi^4\,\mathrm{tr}(\mathbb{I}_\phi) + 5\Lambda^2 + 8\Lambda m_\phi^2\kappa^2\phi^2 + \cdots + \ldots\frac{45\kappa^8}{2048}\left(\psi^\dagger\sigma\psi\right)^4\right)$$

$\kappa^2 = 8\pi G_N = \frac{8\pi}{M_{Pl}^2}$

[R.A. 1912.09671]

contributions from dimension 0 (CC) ..... to dimension 12 fermion x 8





Standard Model + Gravity Renormalization

in the gravitational coupling $\kappa^2 = 8\pi G_N$

We put this to use to compute the UV divergences coming from gravitational interactions

$$\frac{1}{16\pi^2(4-d)}\left(2m_H^4 + 5\Lambda^2 + 16\Lambda\kappa^2 m_H^2 H^\dagger H + \cdots + \cdots + \frac{45\kappa^8}{2048}\left(\psi^\dagger\sigma\psi\right)^4\right)$$

contributions from dimension 0 (CC) ..... to dimension 12 fermion x 8

In addition there is the mixed contributions, also given in

[R.A. 1912.09671] .

Checked against
[t'Hooft & Veltman, 1974]
[Desser & Nieuwenhuizen PRD10, 401]
[Desser & Nieuwenhuizen PRD10, 411]
[Duff hep-th/9308075]

Summary

How is Gravity different?

Gravity at low energies /weak field is a model example of EFT

This means computationally there is no unsurmountable obstacle

and furthermore has some similarities with ordinary gauge theories

In particular here for one loop computations with a covariant momentum representation we found that

what is a known series in the gauge theories we solve iteratively in gravity

We worked out this series to obtain the UV divergences for in $\left(\kappa = \frac{\sqrt{8\pi}}{M_{Pl}}\right)^n$

the Standard Model and General Relativity with Cosmological Constant