The static self-energy (and the plaquette) at large orders in perturbation theory

Based on
Bali, Bauer, Pineda: Phys. Rev. Lett. 108 (2012) 242002; PoS LATTICE2013 (2014) 371
Bali, Bauer, Pineda, Torrero: Phys. Rev. D87 (2013) 094517
Bali, Bauer, Pineda: Phys.Rev.D 89 (2014) 054505; Phys.Rev.Lett. 113 (2014) 092001

See also hep-ph/0208031, hep-ph/0310130, hep-lat/0509022

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Motivation

m_Q. Fundamental parameter of the Standard Model.

Usual definitions:

- $m_{\overline{\rm MS}}$ $\rightarrow$ short distance mass.
- $m_{\rm OS}$ $\rightarrow$ natural definition for heavy quark physics.

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1} \,,$$

Renormalon (OPE) analysis predict $r_n \sim n!$.

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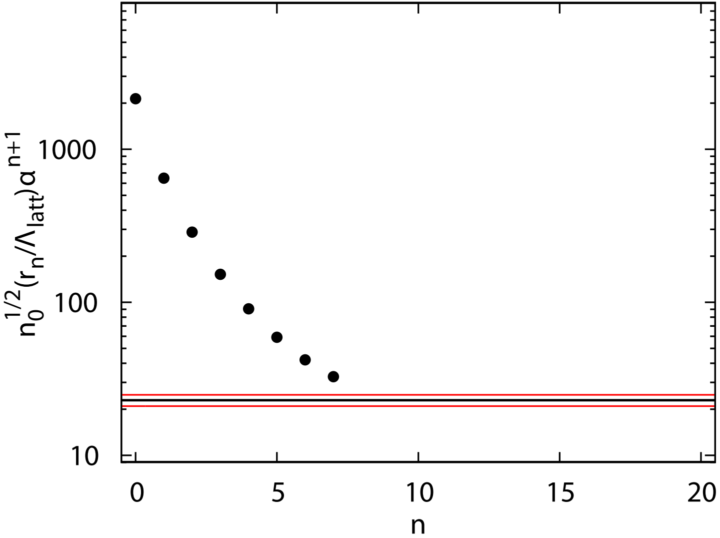
$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS})\,, \quad m_{\tilde{G}} = m_{\tilde{g},\rm OS} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},\rm OS}\right)$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

$$m_{\rm OS} = m_{\overline{\rm MS}} + r_0\alpha_s + r_1\alpha_s^2 + \cdots$$

with $r_n \sim n!$. In other words

$$\delta_{np}^{(\rm pert.)} m_{\rm OS} = \delta_{np}^{(\rm pert.)}(m_{\overline{\rm MS}} + r_0\alpha_s + r_1\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$



$n_0^{1/2}(r_n/\Lambda_{latt})\alpha^{n+1}$
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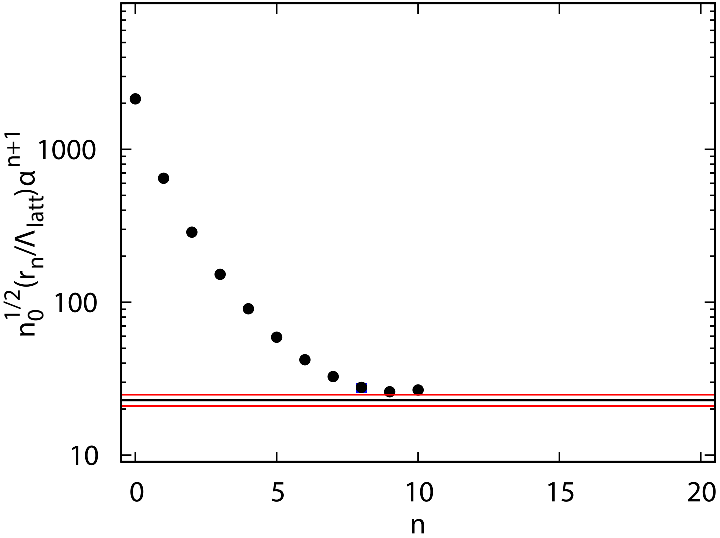

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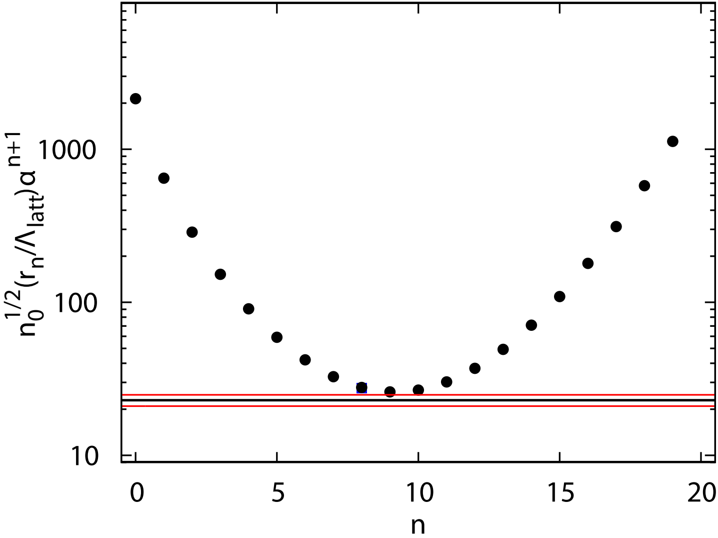
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$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}\,,$$

$$m_{\rm OS} = m_{\overline{\rm MS}} + \int_0^{\infty} {\rm d}t\, e^{-t/\alpha_s} B[m_{\rm OS}](t)\,, \qquad B[m_{\rm OS}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}.$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\rm OS}](t) = N_m \nu \frac{1}{(1-2u)^{1+b}}\left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right) + (\text{analytic term}),$$

Next renormalon at $u = 1$.

$$r_n \stackrel{n\rightarrow\infty}{=} N_m\, \nu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right).$$

$$b = \frac{\beta_1}{2\beta_0^2}\,, \qquad c_1 = \frac{1}{4\, b \beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right)\,, \qquad \cdots$$

Beneke; Pineda

Over the years a lot of evidence in favour of the existence of the renormalon. Particularly important for heavy quark physics.

Two that I specially like:

- Static potential: $2m + V_s$ is renormalon free
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$$r_n \overset{n\to\infty}{\sim} m_{\overline{\rm MS}} \left(\frac{\beta_0}{2\pi}\right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s[\nu/m_{\overline{\rm MS}}]}{s!} \sim \nu$$

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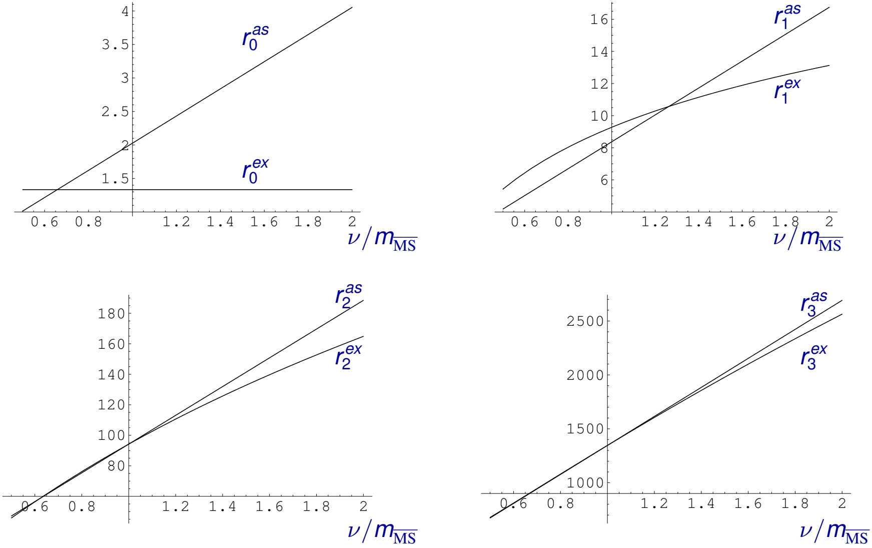

Figure: Plots of the exact (r_n^{ex}) and asymptotic (r_n^{as}) value of $r_n(\nu)$ at different orders in perturbation theory as a function of $\nu/m_{\overline{\rm MS}}$. From hep-lat/0509022.

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon dominance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD (and in heavy quarkonium physics).

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POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory (Di Renzo et al.)

$$L^{(R)}(N_S,N_T)=\frac{1}{N_S^3}\sum_{\mathbf{n}}\frac{1}{d_R}\mathrm{tr}\left[\prod_{n_4=0}^{N_T-1}U_4^R(n)\right]=e^{-aN_TP^{(R,\rho)}(N_S,N_T)}\qquad U_\mu^R(n)\approx e^{iA_\mu^R(n+1/2)a}$$

We implement triplet and octet representations R ($d_R=3,8$).

$$P^{(R,\rho)}(N_S,N_T)=\sum_{n=0}^{\infty}c_n^{(R,\rho)}(N_S,N_T)\alpha^{n+1}\,,$$

$$\delta m=\lim_{N_S,N_T\to\infty}P^{(3,\rho)}(N_S,N_T)\,,\qquad \delta m_{\tilde g}=\lim_{N_S,N_T\to\infty}P^{(8,\rho)}(N_S,N_T)\,,$$

$$c_n^{(R,\rho)}=\lim_{N_S,N_T\to\infty}c_n^{(R,\rho)}(N_S,N_T)\,.$$

$$\delta m=\frac{1}{a}\sum_{n=0}^{\infty}c_n^{(3,\rho)}\alpha^{n+1}(1/a)\,(\text{fundamental}),\quad \delta m_{\tilde g}=\frac{1}{a}\sum_{n=0}^{\infty}c_n^{(8,\rho)}\alpha^{n+1}(1/a)\,(\text{adjoint})$$

$$\lim_{n\to\infty}c_n^{(3,\rho)}=r_n(\nu)/\nu$$

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	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^{20})$	$\mathcal{O}(\alpha^{32})$
$N_S(N_T)$	4(4)	8(8, 10, 12, 14)	4(8)

Table: *The first arrow states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ have been computed for each specific lattice volume for PBC.*

$\mathcal{O}(\alpha^3)$	$N_S(N_T)$	5(5, 6, 7, 8, 10)			
$\mathcal{O}(\alpha^4)$	$N_S(N_T)$	4(5, 6, 7, 8, 10, 12, 16, 20, 24)	12(16, 20)		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	6(6, 8, 10, 12, 16)	8(12, 16)		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	10(8, 12, 16, 20)	16(12, 16, 20)		
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	7(7, 8)	8(8, 10)	9(12)	10(10)
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	11(16)	12(12)	14(14)	

Table: *The first column states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ and the associated ratios have been computed for each specific lattice volume for TBC.*

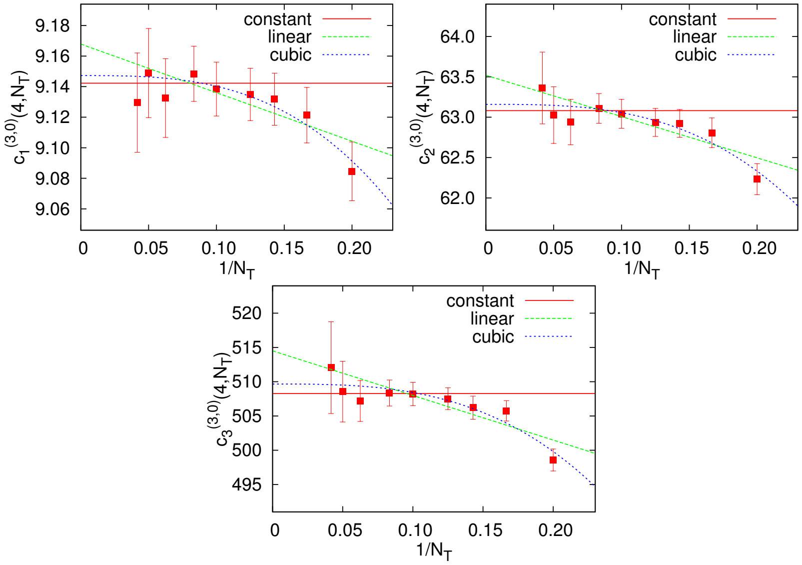

Figure: $c^{(3,0)}_{1,2,3}(4, N_T)$ *as a function of* $1/N_T$, *in comparison to a constant plus linear fit, a constant plus cubic fit, and a constant fitted only to the* $N_T > 10$ *points.*

Perturbative OPE at finite volume

$$\delta m(N_S) = \lim_{N_T\to\infty} P(N_S,N_T) \qquad \text{and} \qquad c_n(N_S) = \lim_{N_T\to\infty} c_n(N_S,N_T)\,.$$

For large N_S, we write (OPE: $\frac{1}{a} \gg \frac{1}{N_S a} \gg \Lambda_{\rm QCD}$)

$$\delta m(N_S) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Taylor expansion of $\alpha\left((aN_S)^{-1}\right)$ in powers of $\alpha(a^{-1})$:

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right); \qquad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S)\,,$$

$f_n^{(0)} = f_n$ and the coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m with $m < n$ and β_j with $j \le n-1$.

$$f_1(N_S) = f_1 + f_0\frac{\beta_0}{2\pi}\ln(N_S)\,,$$

$$f_2(N_S) = f_2 + \left[2f_1\frac{\beta_0}{2\pi} + f_0\frac{\beta_1}{8\pi^2}\right]\ln(N_S) + f_0\left(\frac{\beta_0}{2\pi}\right)^2 \ln^2(N_S)\,,$$

and so on.

"Physical interpretation"

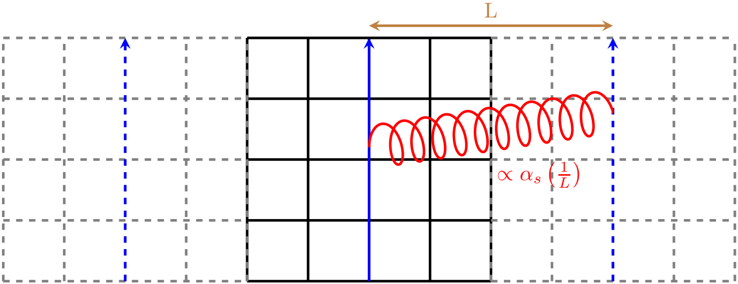

Figure: *Self-interactions with replicas producing $1/L = 1/(aN_S)$ Coulomb terms.*

$$P \propto \int_{1/(aN_S)}^{1/a} dk\, \alpha(k) \sim \frac{1}{a}\sum_n c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_n c_n \alpha^{n+1}\left((aN_S)^{-1}\right),$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n!\,, \qquad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!}\,.$$

	$c_n^{(3,0)}$	$c_n^{(3,1/6)}$	$c_n^{(8,0)} C_F/C_A$	$c_n^{(8,1/6)} C_F/C_A$
c_0	2.117274357	0.72181(99)	2.117274357	0.72181(99)
c_1	11.136(11)	6.385(10)	11.140(12)	6.387(10)
$c_2/10$	8.610(13)	8.124(12)	8.587(14)	8.129(12)
$c_3/10^2$	7.945(16)	7.670(13)	7.917(20)	7.682(15)
$c_4/10^3$	8.215(34)	8.017(33)	8.197(42)	8.017(36)
$c_5/10^4$	9.322(59)	9.160(59)	9.295(76)	9.139(64)
$c_6/10^6$	1.153(11)	1.138(11)	1.144(13)	1.134(12)
$c_7/10^7$	1.558(21)	1.541(22)	1.533(25)	1.535(22)
$c_8/10^8$	2.304(43)	2.284(45)	2.254(51)	2.275(45)
$c_9/10^9$	3.747(95)	3.717(97)	3.64(11)	3.703(98)
$c_{10}/10^{10}$	6.70(22)	6.65(22)	6.49(25)	6.63(22)
$c_{11}/10^{12}$	1.316(52)	1.306(53)	1.269(59)	1.303(53)
$c_{12}/10^{13}$	2.81(13)	2.79(13)	2.71(14)	2.78(13)
$c_{13}/10^{14}$	6.51(35)	6.46(35)	6.29(37)	6.45(35)
$c_{14}/10^{16}$	1.628(96)	1.613(97)	1.57(10)	1.614(97)
$c_{15}/10^{17}$	4.36(28)	4.32(28)	4.22(29)	4.33(28)
$c_{16}/10^{19}$	1.247(86)	1.235(86)	1.206(89)	1.236(86)
$c_{17}/10^{20}$	3.78(28)	3.75(28)	3.66(28)	3.75(28)
$c_{18}/10^{22}$	1.215(93)	1.204(94)	1.176(95)	1.205(94)
$c_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

	$f_n^{(3,0)}$	$f_n^{(3,1/6)}$	$f_n^{(8,0)} C_F/C_A$	$f_n^{(8,1/6)} C_F/C_A$
f_0	0.7696256328	0.7810(59)	0.7696256328	0.7810(69)
f_1	6.075(78)	6.046(58)	6.124(87)	6.063(68)
$f_2/10$	5.628(91)	5.644(62)	5.60(11)	5.691(78)
$f_3/10^2$	5.87(11)	5.858(76)	6.00(18)	5.946(91)
$f_4/10^3$	6.33(22)	6.29(17)	6.57(40)	6.26(23)
$f_5/10^4$	7.73(35)	7.71(26)	7.67(66)	7.78(42)
$f_6/10^5$	9.86(53)	9.80(42)	9.68(99)	9.79(69)
$f_7/10^7$	1.388(81)	1.378(71)	1.35(15)	1.38(11)
$f_8/10^8$	2.12(12)	2.11(12)	2.06(22)	2.10(17)
$f_9/10^9$	3.54(20)	3.52(20)	3.40(37)	3.51(27)
$f_{10}/10^{10}$	6.49(33)	6.44(34)	6.23(67)	6.44(43)
$f_{11}/10^{12}$	1.296(64)	1.286(66)	1.24(13)	1.286(74)
$f_{12}/10^{13}$	2.68(19)	2.64(18)	2.65(33)	2.65(21)
$f_{13}/10^{14}$	6.70(54)	6.68(52)	6.36(90)	6.66(57)
$f_{14}/10^{16}$	1.58(14)	1.56(14)	1.55(22)	1.57(15)
$f_{15}/10^{17}$	4.41(34)	4.37(33)	4.24(47)	4.37(35)
$f_{16}/10^{19}$	1.241(92)	1.230(91)	1.20(11)	1.231(94)
$f_{17}/10^{20}$	3.79(28)	3.75(28)	3.67(30)	3.76(28)
$f_{18}/10^{22}$	1.215(94)	1.204(94)	1.176(97)	1.205(94)
$f_{19}/10^{23}$	4.12(33)	4.08(33)	3.99(34)	4.08(33)

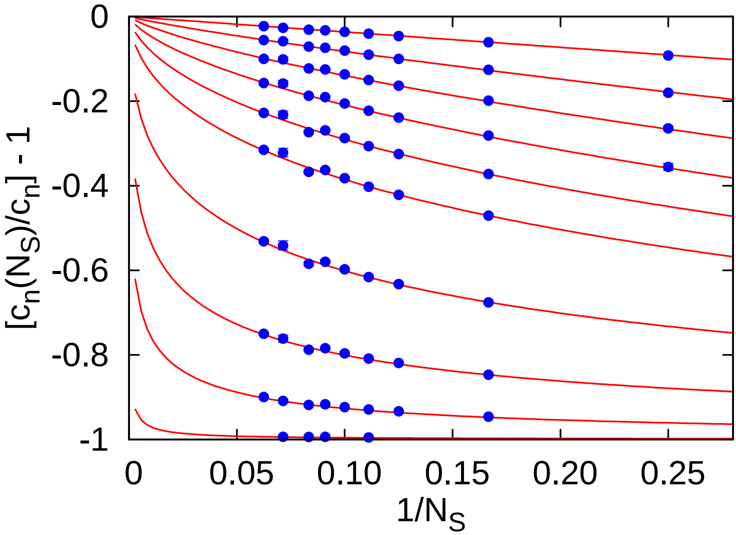

Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ *for* $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ *(top to bottom). For each value of* N_S *we have plotted the data point with the maximum value of* N_T*. The curves represent the global fit.* $-(1/N_S) f_{0,\mathrm{DLPT}}^{(3,0)}/c_{0,\mathrm{DLPT}}^{(3,0)}$ *is shown for* $n = 0$*.*

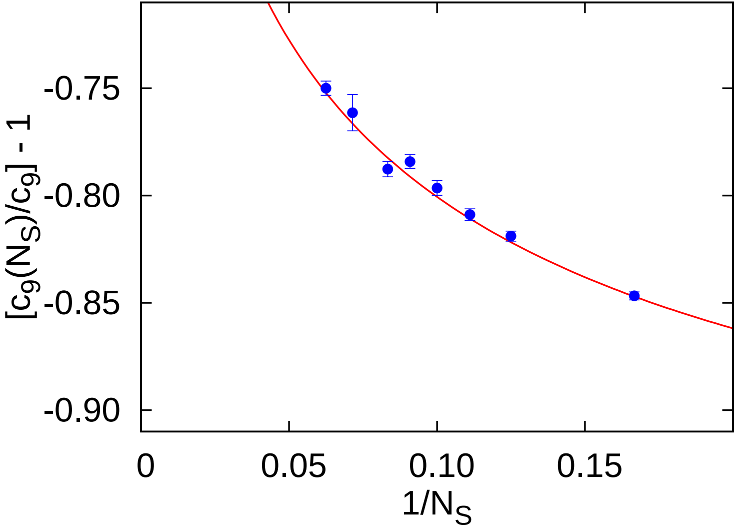

Figure: $c_9^{(3,0)}(N_S)/c_9^{(3,0)} - 1$. *For each value of N_S we have plotted the data point with the maximum value of N_T. The curve represents the global fit.*

Ratios

$$\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}}\frac{1}{n}=\frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}}\frac{1}{n}$$

$$=\frac{\beta_0}{2\pi}\left\{1+\frac{b}{n}-\frac{bs_1}{n^2}+\frac{1}{n^3}\left[b^2s_1^2+b(b-1)(s_1-2s_2)\right]+\mathcal{O}\left(\frac{1}{n^4}\right)\right\} \, .$$

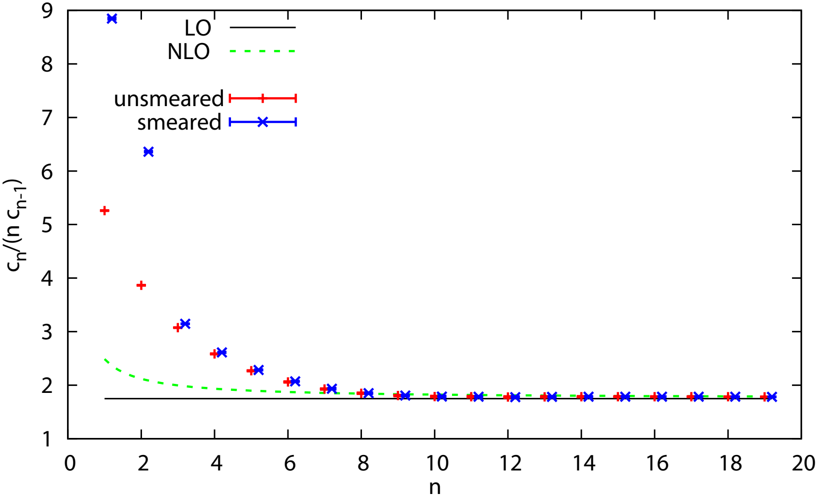

Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients c_n in comparison to the theoretical prediction at different orders in the $1/n$ expansion.

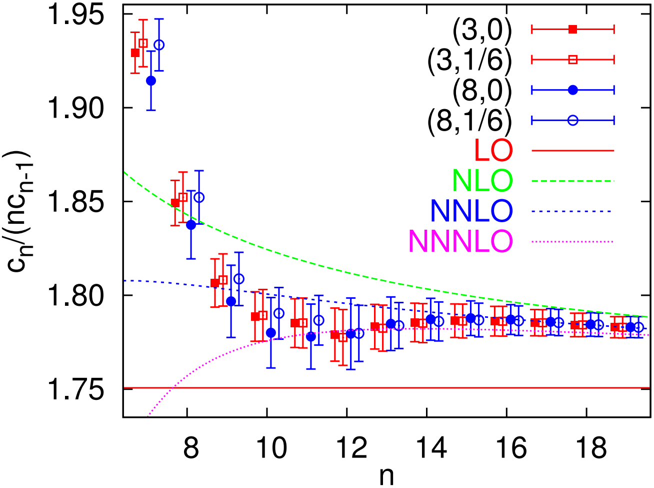

Figure: *The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet fundamental static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.*

N_m

$$c_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right).$$

$$f_n^{fitted} = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right).$$

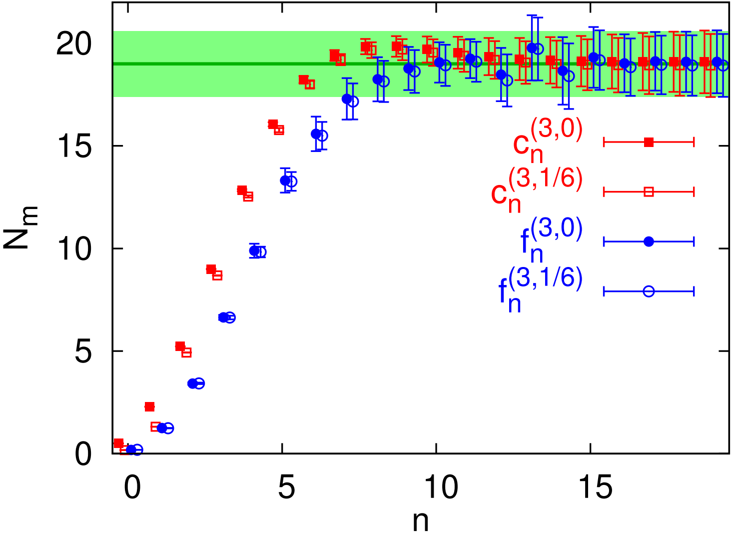

Figure: *N_m, determined from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$ at NNLO. The horizontal band is our final result: $N_m^{\rm latt} = 17.9 \pm 1.2$.*

From lattice to $\overline{\rm MS}$ scheme

$$\alpha_{\overline{\rm MS}}(\mu) = \alpha_{\rm latt}(\mu)\left(1 + d_1\alpha_{\rm latt}(\mu) + d_2\alpha^2_{\rm latt}(\mu) + d_3\alpha^3_{\rm latt}(\mu) + \mathcal{O}(\alpha^4_{\rm latt})\right)\,,$$

$$N^{\overline{\rm MS}}_{m,m_{\tilde g}} = N^{\rm latt}_{m,m_{\tilde g}}\Lambda_{\rm latt}/\Lambda_{\overline{\rm MS}}\,,\quad \text{where}\quad \Lambda_{\overline{\rm MS}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\rm latt} \approx 28.809338139488\,\Lambda_{\rm latt}\,.$$

This yields the numerical values

$$N^{\overline{\rm MS}}_m = 0.620(35)\,,\quad C_F/C_A\,N^{\overline{\rm MS}}_{m_{\tilde g}} = -C_F/C_A\,N^{\overline{\rm MS}}_\Lambda = 0.610(41)\,.$$

Other combinations of interest are

$$N^{\overline{\rm MS}}_{V_s} = -1.240(69)\,,\quad N^{\overline{\rm MS}}_{V_o} = 0.13(12)\,.$$

Assuming that

$$c_{3,\overline{\rm MS}} \simeq N^{\overline{\rm MS}}_m\left(\frac{\beta_0}{2\pi}\right)^3\frac{\Gamma(4+b)}{\Gamma(1+b)}\left(1+\frac{b}{(3+b)}s_1+\frac{b(b-1)}{(3+b)(2+b)}s_2+\cdots\right),$$

and using our central value $c^{(3,0)}_{3,\rm latt} = 794.5$, we obtain

$$d_3 \simeq 352(3)\,,\qquad \beta_3^{\rm latt} \simeq -1.12\times 10^6\,.$$

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CONCLUSIONS

Renormalons go beyond large-β_0 analysis: $\rightarrow$ OPE

Clearly seen in heavy quark physics from $\overline{\mathrm{MS}}$-like computations: Pole mass, static potential, hybrid potential, binding energy, $\cdots$

It is compulsory to take into account renormalon effects in order to do precision computations in heavy quark physics.

For the first time, it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}.
We have (numerically) proven, beyond any reasonable doubt ($\sim$ 20 standard deviations), the existence of the renormalon in QCD.

$$N_m^{\rm latt} = 17.9 \pm 1.0\,, \quad C_F/C_A\, N_\Lambda^{\rm latt} = -17.6 \pm 1.2\,,$$
$$N_m^{\overline{\rm MS}} = 0.620 \pm 0.035\,, \quad C_F/C_A\, N_\Lambda^{\overline{\rm MS}} = -0.610 \pm 0.041\,.$$

Completely consistent with continuum-like determinations ($N_m^{\overline{\rm MS}} = 0.62$).

Nonperturbative quantities ($\bar{\Lambda}$, Λ_H, $\langle G^2\rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.

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Plaquette (Bali, Bauer, Pineda)

$$\langle P \rangle = \sum_{n=0}^{N} p_n \alpha^{n+1}(a^{-1}) + a^4 \frac{\pi^2}{36} \langle G^2 \rangle + \cdots$$

$$d = 1 (n_0 \sim 7) \longrightarrow d = 4 (n_0 \sim 28)$$

$$N + 1 = 35$$

$$p_n = N_P \left(\frac{\beta_0}{d \, 2\pi} \right)^n \frac{\Gamma(n+1+d\,b)}{\Gamma(1+d\,b)} (1 + \cdots) \,.$$

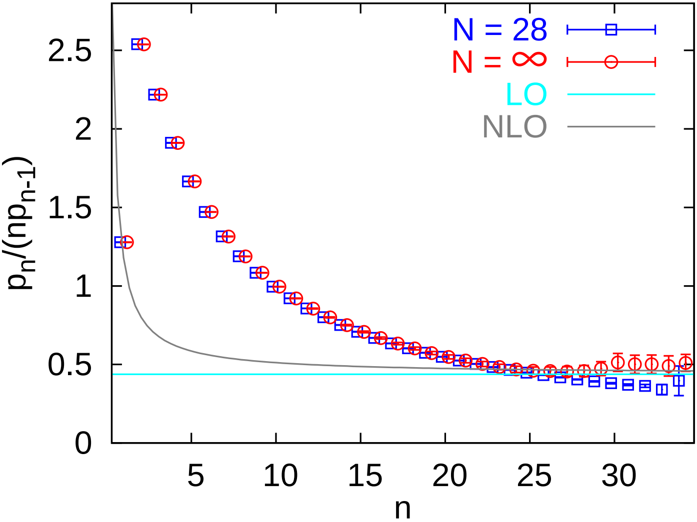

Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients p_n ($N = \infty$, $N = 28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion.