The static self-energy (and the plaquette) at large orders in perturbation theory

Based on

See also hep-ph/0208031, hep-ph/0310130, hep-lat/0509022

Antonio Pineda

Universitat Autònoma de Barcelona & IFAE

ACAT2021, Daejeon, South Korea, November 30th, 2021
**Motivation**

$m_Q$. Fundamental parameter of the Standard Model.

Usual definitions:

- $m_{\text{MS}} \rightarrow$ short distance mass.
- $m_{\text{OS}} \rightarrow$ natural definition for heavy quark physics.

\[
m_{\text{OS}} = m_{\text{MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1},
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Renormalon (OPE) analysis predict $r_n \sim n!$. 
\[ M_B = m_{\text{OS}} + \Lambda_B + \mathcal{O}(1/m_{\text{OS}}), \quad m_\Lambda = m_{\tilde{g},\text{OS}} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},\text{OS}}\right) \]

\( M_B \) is renormalon free. Therefore \( m_{\text{OS}} \) suffers from renormalon ambiguities:

\[ m_{\text{OS}} = m_{\text{MS}} + r_0 \alpha_s + r_1 \alpha_s^2 + \cdots \]

with \( r_n \sim n! \). In other words

\[ \delta^{(\text{pert.})}_{\text{np}} m_{\text{OS}} = \delta^{(\text{pert.})}_{\text{np}} (m_{\text{MS}} + r_0 \alpha_s + r_1 \alpha_s^2 + \cdots) \sim \Lambda_{\text{QCD}}! \]
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\[ m_{\text{OS}} = m_{\text{MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}, \]

\[ m_{\text{OS}} = m_{\text{MS}} + \int_{0}^{\infty} dt \, e^{-t/\alpha_s} \, B[m_{\text{OS}}](t), \quad B[m_{\text{OS}}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}. \]

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform \( (u = \frac{\beta_0 t}{4\pi}) \).

\[ B[m_{\text{OS}}](t) = N_{m} \frac{1}{(1 - 2u)^{1+b}} \left( 1 + c_1(1 - 2u) + c_2(1 - 2u)^2 + \cdots \right) + \text{(analytic term)}, \]

Next renormalon at \( u = 1 \).

\[ r_n \overset{n \to \infty}{=} N_{m} \nu \left( \frac{\beta_0}{2\pi} \right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left( 1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots \right). \]

\[ b = \frac{\beta_1}{2\beta_0^2}, \quad c_1 = \frac{1}{4b\beta_0^3} \left( \frac{\beta_1^2}{\beta_0} - \beta_2 \right), \quad \cdots \]

Beneke; Pineda
Over the years a lot of evidence in favour of the existence of the renormalon. Particularly important for heavy quark physics.

Two that I specially like:

- Static potential: $2m + V_s$ is renormalon free

$$r_n \to \infty \quad m_{\text{MS}} \left( \frac{\beta_0}{2\pi} \right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s [\nu / m_{\text{MS}}]}{s!} \sim \nu$$
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Figure: Plots of the exact ($r_n^{\text{ex}}$) and asymptotic ($r_n^{\text{as}}$) value of $r_n(\nu)$ at different orders in perturbation theory as a function of $\nu/m_{\text{MS}}$. From hep-lat/0509022.
Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon dominance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD (and in heavy quarkonium physics).

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POLYAKOV LOOP versus $\delta m$ (and $m$)

Possible to compute the energy of an static source in the lattice: $\delta m$ of HQET. We use Numerical Stochastic Perturbation Theory (Di Renzo et al.)

$$L^{(R)}(N_S, N_T) = \frac{1}{N_S^3} \sum_n \frac{1}{d_R} \text{tr} \left[ \prod_{n=0}^{N_T-1} U_4^R(n) \right] = e^{-aN_T P^{(R, \rho)}(N_S, N_T)} U^R_\mu(n) \approx e^{iA_\mu(n+1/2) a}$$

We implement triplet and octet representations $R$ ($d_R = 3, 8$).

$$P^{(R, \rho)}(N_S, N_T) = \sum_{n=0}^{\infty} c_n^{(R, \rho)}(N_S, N_T) \alpha^{n+1},$$

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Table: The first arrow states to which order in $\alpha$ the coefficients of $c_n^{(R)}(N_T, N_S)$ have been computed for each specific lattice volume for PBC.

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<thead>
<tr>
<th>$\mathcal{O}(\alpha^3)$</th>
<th>$N_S(N_T)$</th>
<th>5(5, 6, 7, 8, 10)</th>
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<tbody>
<tr>
<td>$\mathcal{O}(\alpha^4)$</td>
<td>$N_S(N_T)$</td>
<td>4(5, 6, 7, 8, 10, 12, 16, 20, 24)</td>
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<tr>
<td>$\mathcal{O}(\alpha^{12})$</td>
<td>$N_S(N_T)$</td>
<td>6(6, 8, 10, 12, 16)</td>
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<td>$N_S(N_T)$</td>
<td>10(8, 12, 16, 20)</td>
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<td>7(7, 8)</td>
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<td>$N_S(N_T)$</td>
<td>11(16)</td>
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Table: The first column states to which order in $\alpha$ the coefficients of $c_n^{(R)}(N_T, N_S)$ and the associated ratios have been computed for each specific lattice volume for TBC.
Figure: $c_{1,2,3}^{(3,0)}(4, N_T)$ as a function of $1/N_T$, in comparison to a constant plus linear fit, a constant plus cubic fit, and a constant fitted only to the $N_T > 10$ points.
Perturbative OPE at finite volume

\[
\delta m(N_S) = \lim_{N_T \to \infty} P(N_S, N_T) \quad \text{and} \quad c_n(N_S) = \lim_{N_T \to \infty} c_n(N_S, N_T).
\]

For large \( N_S \), we write \((\text{OPE: } \frac{1}{a} \gg \frac{1}{N_S a} \gg \Lambda_{\text{QCD}})\)

\[
\delta m(N_S) = \frac{1}{a} \sum_{n=0}^{\infty} c_n a^{n+1} (a^{-1}) - \frac{1}{a N_S} \sum_{n=0}^{\infty} f_n a^{n+1} ((aN_S)^{-1}) + O \left( \frac{1}{N_S^2} \right).
\]

Taylor expansion of \( \alpha((aN_S)^{-1}) \) in powers of \( \alpha(a^{-1}) \):

\[
c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + O \left( \frac{1}{N_S^2} \right); \quad f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S),
\]

\( f_n^{(0)} = f_n \) and the coefficients \( f_n^{(i)} \) for \( i > 0 \) are determined by \( f_m \) with \( m < n \) and \( \beta_j \) with \( j \leq n - 1 \).

\[
f_1(N_S) = f_1 + f_0 \frac{\beta_0}{2\pi} \ln(N_S),
\]

\[
f_2(N_S) = f_2 + \left[ 2f_1 \frac{\beta_0}{2\pi} + f_0 \frac{\beta_1}{8\pi^2} \right] \ln(N_S) + f_0 \left( \frac{\beta_0}{2\pi} \right)^2 \ln^2(N_S),
\]

and so on.
"Physical interpretation"

Figure: Self-interactions with replicas producing $1/L = 1/(aN_S)$ Coulomb terms.

$$P \propto \int_{1/(aN_S)}^{1/a} dk \alpha(k) \sim \frac{1}{a} \sum_n c_n \alpha^{n+1}(a^{-1}) - \frac{1}{aN_S} \sum_n c_n \alpha^{n+1}((aN_S)^{-1})$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n!,$$

$$f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!}.$$
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<tr>
<th>$c_n$</th>
<th>$C_n^{(3,0)}$</th>
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<th>$C_n^{(8,0)}$</th>
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<td>$c_0$</td>
<td>2.117274357</td>
<td>0.72181(99)</td>
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</tr>
<tr>
<td>$c_1$</td>
<td>11.136(11)</td>
<td>6.385(10)</td>
<td>11.140(12)</td>
<td>6.387(10)</td>
</tr>
<tr>
<td>$c_2/10$</td>
<td>8.610(13)</td>
<td>8.124(12)</td>
<td>8.587(14)</td>
<td>8.129(12)</td>
</tr>
<tr>
<td>$c_3/10^2$</td>
<td>7.945(16)</td>
<td>7.670(13)</td>
<td>7.917(20)</td>
<td>7.682(15)</td>
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<tr>
<td>$c_4/10^3$</td>
<td>8.215(34)</td>
<td>8.017(33)</td>
<td>8.197(42)</td>
<td>8.017(36)</td>
</tr>
<tr>
<td>$c_5/10^4$</td>
<td>9.322(59)</td>
<td>9.160(59)</td>
<td>9.295(76)</td>
<td>9.139(64)</td>
</tr>
<tr>
<td>$c_6/10^6$</td>
<td>1.153(11)</td>
<td>1.138(11)</td>
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<td>$c_7/10^7$</td>
<td>1.558(21)</td>
<td>1.541(22)</td>
<td>1.533(25)</td>
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<td>$c_8/10^8$</td>
<td>2.304(43)</td>
<td>2.284(45)</td>
<td>2.254(51)</td>
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<td>$c_9/10^9$</td>
<td>3.747(95)</td>
<td>3.717(97)</td>
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<td>$c_{10}/10^{10}$</td>
<td>6.70(22)</td>
<td>6.65(22)</td>
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<td>$c_{11}/10^{12}$</td>
<td>1.316(52)</td>
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<td>$c_{12}/10^{13}$</td>
<td>2.81(13)</td>
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<td>$c_{13}/10^{14}$</td>
<td>6.51(35)</td>
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<td>$c_{14}/10^{16}$</td>
<td>1.628(96)</td>
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<td>$c_{16}/10^{19}$</td>
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<td>$c_{18}/10^{22}$</td>
<td>1.215(93)</td>
<td>1.204(94)</td>
<td>1.176(95)</td>
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<tr>
<td>$c_{19}/10^{23}$</td>
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<tr>
<td>$f_3/10^2$</td>
<td>5.87(11)</td>
<td>5.858(76)</td>
<td>6.00(18)</td>
<td>5.946(91)</td>
</tr>
<tr>
<td>$f_4/10^3$</td>
<td>6.33(22)</td>
<td>6.29(17)</td>
<td>6.57(40)</td>
<td>6.26(23)</td>
</tr>
<tr>
<td>$f_5/10^4$</td>
<td>7.73(35)</td>
<td>7.71(26)</td>
<td>7.67(66)</td>
<td>7.78(42)</td>
</tr>
<tr>
<td>$f_6/10^5$</td>
<td>9.86(53)</td>
<td>9.80(42)</td>
<td>9.68(99)</td>
<td>9.79(69)</td>
</tr>
<tr>
<td>$f_7/10^7$</td>
<td>1.388(81)</td>
<td>1.378(71)</td>
<td>1.35(15)</td>
<td>1.38(11)</td>
</tr>
<tr>
<td>$f_8/10^8$</td>
<td>2.12(12)</td>
<td>2.11(12)</td>
<td>2.06(22)</td>
<td>2.10(17)</td>
</tr>
<tr>
<td>$f_9/10^9$</td>
<td>3.54(20)</td>
<td>3.52(20)</td>
<td>3.40(37)</td>
<td>3.51(27)</td>
</tr>
<tr>
<td>$f_{10}/10^{10}$</td>
<td>6.49(33)</td>
<td>6.44(34)</td>
<td>6.23(67)</td>
<td>6.44(43)</td>
</tr>
<tr>
<td>$f_{11}/10^{12}$</td>
<td>1.296(64)</td>
<td>1.286(66)</td>
<td>1.24(13)</td>
<td>1.286(74)</td>
</tr>
<tr>
<td>$f_{12}/10^{13}$</td>
<td>2.68(19)</td>
<td>2.64(18)</td>
<td>2.65(33)</td>
<td>2.65(21)</td>
</tr>
<tr>
<td>$f_{13}/10^{14}$</td>
<td>6.70(54)</td>
<td>6.68(52)</td>
<td>6.36(90)</td>
<td>6.66(57)</td>
</tr>
<tr>
<td>$f_{14}/10^{16}$</td>
<td>1.58(14)</td>
<td>1.56(14)</td>
<td>1.55(22)</td>
<td>1.57(15)</td>
</tr>
<tr>
<td>$f_{15}/10^{17}$</td>
<td>4.41(34)</td>
<td>4.37(33)</td>
<td>4.24(47)</td>
<td>4.37(35)</td>
</tr>
<tr>
<td>$f_{16}/10^{19}$</td>
<td>1.241(92)</td>
<td>1.230(91)</td>
<td>1.20(11)</td>
<td>1.231(94)</td>
</tr>
<tr>
<td>$f_{17}/10^{20}$</td>
<td>3.79(28)</td>
<td>3.75(28)</td>
<td>3.67(30)</td>
<td>3.76(28)</td>
</tr>
<tr>
<td>$f_{18}/10^{22}$</td>
<td>1.215(94)</td>
<td>1.204(94)</td>
<td>1.176(97)</td>
<td>1.205(94)</td>
</tr>
<tr>
<td>$f_{19}/10^{23}$</td>
<td>4.12(33)</td>
<td>4.08(33)</td>
<td>3.99(34)</td>
<td>4.08(33)</td>
</tr>
</tbody>
</table>

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Figure: $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ for $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ (top to bottom). For each value of $N_S$ we have plotted the data point with the maximum value of $N_T$. The curves represent the global fit. $-(1/N_S)f_0^{(3,0)}DLPT/c_0^{(3,0)}DLPT$ is shown for $n = 0$. The static self-energy (and the plaquette) at large orders in perturbation theory Antonio Pineda
Figure: $c_9^{(3,0)}(N_S)/c_9^{(3,0)} - 1$. For each value of $N_S$ we have plotted the data point with the maximum value of $N_T$. The curve represents the global fit.
Ratios

\[
\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}} \frac{1}{n} = \frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}} \frac{1}{n}
\]

\[
= \frac{\beta_0}{2\pi} \left\{ 1 + \frac{b}{n} - \frac{bs_1}{n^2} + \frac{1}{n^3} \left[ b^2 s_1^2 + b(b-1)(s_1 - 2s_2) \right] + \mathcal{O}\left(\frac{1}{n^4}\right) \right\}.
\]
Figure: Ratios $c_n/(nc_{n-1})$ of the smeared (blue) and unsmeared (red) triplet static self-energy coefficients $c_n$ in comparison to the theoretical prediction at different orders in the $1/n$ expansion.
The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet fundamental static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.
\[ N_m \]

\[
c_n^{\text{fitted}} = N_m \left( \frac{\beta_0}{2\pi} \right)^n \Gamma(n + 1 + b) \frac{\Gamma(n + 1 + b)}{\Gamma(1 + b)} \left( 1 + \frac{b}{n + b} c_1 + \frac{b(b - 1)}{(n + b)(n + b - 1)} c_2 + \cdots \right).\]

\[
f_n^{\text{fitted}} = N_m \left( \frac{\beta_0}{2\pi} \right)^n \Gamma(n + 1 + b) \frac{\Gamma(n + 1 + b)}{\Gamma(1 + b)} \left( 1 + \frac{b}{n + b} c_1 + \frac{b(b - 1)}{(n + b)(n + b - 1)} c_2 + \cdots \right).\]
Figure: $N_m$, determined from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$ at NNLO. The horizontal band is our final result: $N_m^{\text{latt}} = 17.9 \pm 1.2$. 

The static self-energy (and the plaquette) at large orders in perturbation theory

Antonio Pineda
From lattice to $\overline{\text{MS}}$ scheme

$$\alpha_{\overline{\text{MS}}} (\mu) = \alpha_{\text{latt}} (\mu) \left( 1 + d_1 \alpha_{\text{latt}} (\mu) + d_2 \alpha_{\text{latt}}^2 (\mu) + d_3 \alpha_{\text{latt}}^3 (\mu) + O(\alpha_{\text{latt}}^4) \right),$$

$$N_{m, \tilde{g}}^{\overline{\text{MS}}} = N_{m, \tilde{g}}^{\text{latt}} \Lambda_{\text{latt}} / \Lambda_{\overline{\text{MS}}}, \quad \text{where} \quad \Lambda_{\overline{\text{MS}}} = e^{2\pi d_1 / \beta_0} \Lambda_{\text{latt}} \approx 28.809338139488 \Lambda_{\text{latt}}.$$

This yields the numerical values

$$N_{m}^{\overline{\text{MS}}} = 0.620(35), \quad C_F / C_A N_{m, \tilde{g}}^{\overline{\text{MS}}} = -C_F / C_A N_{\Lambda}^{\overline{\text{MS}}} = 0.610(41).$$

Other combinations of interest are

$$N_{V_s}^{\overline{\text{MS}}} = -1.240(69), \quad N_{V_0}^{\overline{\text{MS}}} = 0.13(12).$$

Assuming that

$$c_{3, \overline{\text{MS}}} \simeq N_{m}^{\overline{\text{MS}}} \left( \frac{\beta_0}{2\pi} \right)^3 \frac{\Gamma(4 + b)}{\Gamma(1 + b)} \left( 1 + \frac{b}{3 + b} s_1 + \frac{b(b - 1)}{(3 + b)(2 + b)} s_2 + \cdots \right),$$

and using our central value $c_{3, \text{latt}}^{(3,0)} = 794.5$, we obtain

$$d_3 \simeq 352(3), \quad \beta_3^{\text{latt}} \simeq -1.12 \times 10^6.$$
From lattice to $\overline{\text{MS}}$ scheme

\[ \alpha_{\overline{\text{MS}}} (\mu) = \alpha_{\text{latt}} (\mu) \left( 1 + d_1 \alpha_{\text{latt}} (\mu) + d_2 \alpha_{\text{latt}}^2 (\mu) + d_3 \alpha_{\text{latt}}^3 (\mu) + O(\alpha_{\text{latt}}^4) \right), \]

\[ N_{m, m_g}^{\overline{\text{MS}}} = N_{m, m_g}^{\text{latt}} \Lambda_{\text{latt}} / \Lambda_{\overline{\text{MS}}}, \quad \text{where} \quad \Lambda_{\overline{\text{MS}}} = e^{\frac{2\pi d_1}{\beta_0}} \Lambda_{\text{latt}} \approx 28.809338139488 \Lambda_{\text{latt}}. \]

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Other combinations of interest are

\[ N_{V_s}^{\overline{\text{MS}}} = -1.240(69), \quad N_{V_\sigma}^{\overline{\text{MS}}} = 0.13(12). \]

Assuming that

\[ c_{3, \overline{\text{MS}}} \simeq N_{m}^{\overline{\text{MS}}} \left( \frac{\beta_0}{2\pi} \right)^3 \frac{\Gamma(4 + b)}{\Gamma(1 + b)} \left( 1 + \frac{b}{(3 + b)} s_1 + \frac{b(b - 1)}{(3 + b)(2 + b)} s_2 + \cdots \right), \]

and using our central value \( c_{3, \text{latt}}^{(3,0)} = 794.5 \), we obtain

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CONCLUSIONS

Renormalons go beyond large-\(\beta_0\) analysis: \(\to\) OPE

Clearly seen in heavy quark physics from \(\overline{\text{MS}}\)-like computations: Pole mass, static potential, hybrid potential, binding energy, \(\cdots\)

It is compulsory to take into account renormalon effects in order to do precision computations in heavy quark physics.

For the first time, it was possible to follow the factorial growth of the coefficients over many orders, from around \(\alpha^9\) up to \(\alpha^{20}\).

We have (numerically) proven, beyond any reasonable doubt (\(\sim 20\) standard deviations), the existence of the renormalon in QCD.

\[
\begin{align*}
N^{\text{latt}}_m &= 17.9 \pm 1.0, & C_F/C_A \, N^{\text{latt}}_\Lambda &= -17.6 \pm 1.2, \\
N^{\overline{\text{MS}}}^m &= 0.620 \pm 0.035, & C_F/C_A \, N^{\overline{\text{MS}}}_\Lambda &= -0.610 \pm 0.041.
\end{align*}
\]

Completely consistent with continuum-like determinations (\(N^{\overline{\text{MS}}}_m = 0.62\)).

Nonperturbative quantities (\(\bar{\Lambda}, \Lambda_H, \langle G^2 \rangle, \cdots\)) can only be defined after subtracting the divergent perturbative series.
CONCLUSIONS

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Clearly seen in heavy quark physics from $\overline{\text{MS}}$-like computations: Pole mass, static potential, hybrid potential, binding energy, \ldots

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\[
N_{\text{latt}} = 17.9 \pm 1.0, \quad C_F/C_A N_{\Lambda} = -17.6 \pm 1.2, \\
N_{\overline{\text{MS}}} = 0.620 \pm 0.035, \quad C_F/C_A N_{\Lambda} = -0.610 \pm 0.041.
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\[
N_{m}^{\text{latt}} = 17.9 \pm 1.0, \quad C_F / C_A N_{\Lambda}^{\text{latt}} = -17.6 \pm 1.2, \\
N_{m}^{\overline{\text{MS}}} = 0.620 \pm 0.035, \quad C_F / C_A N_{\Lambda}^{\overline{\text{MS}}} = -0.610 \pm 0.041.
\]

Completely consistent with continuum-like determinations ($N_{m}^{\overline{\text{MS}}} = 0.62$).

Nonperturbative quantities ($\overline{\Lambda}$, $\Lambda_H$, $\langle G^2 \rangle$, $\cdots$) can only be defined after subtracting the divergent perturbative series.
Plaquette (Bali, Bauer, Pineda)

\[
\langle P \rangle = \sum_{n=0}^{N} p_n \alpha^{n+1} (a^{-1}) + a^4 \frac{\pi^2}{36} \langle G^2 \rangle + \cdots
\]

\[
d = 1(n_0 \sim 7) \rightarrow d = 4(n_0 \sim 28)
\]

\[
N + 1 = 35
\]

\[
p_n = N_P \left( \frac{\beta_0}{d \ 2\pi} \right)^n \frac{\Gamma(n + 1 + d \ b)}{\Gamma(1 + d \ b)} (1 + \cdots).
\]
Figure: Ratios $p_n/(np_{n-1})$ of the plaquette coefficients $p_n$ ($N = \infty$, $N = 28$) in comparison to the theoretical prediction at different orders in the $1/n$ expansion.