FeynCalc goes multiloop

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Outline

Motivation: Why multiloop with FeynCalc?

- **FEYNCALC** offers a toolbox-oriented approach to symbolic Feynman diagram calculations
- Not foolproof: correctness of the results $\propto$ user's understanding of QFT
- Very useful for people who know what they want to calculate
- FEYNCALC has plenty of tools for calculations at tree- and 1-loop level
- In multiloop setups the package is in general not so useful
- The idea to **substantially** improve on this matured during my work on QCD Energy-energy correlations [Dixon, Luo, VS, Yang and Zhu 2018; Luo, VS, Yang and Zhu 2019; Gao, VS, Yang 2020] and NNLO QCD corrections to B-meson mixing [Gerlach, Nierste, VS, Steinhauser, 2021]
- This talk covers two aspects of the ongoing work in this direction
  - Identification and mapping of loop integral topologies
  - Handling of master integrals
- FEYNCALC 10 featuring this functionality will be released in 2022

1991	**FEYNCALC** 1.0 [Mertig et al., 1991]
1997	**TARCER** [Mertig & Scharf, 1998]
2012	**FEYNCALCFORMLINK** [Feng & Mertig, 2012]
2016	**FEYNCALC** 9.0 [VS, Mertig, Orellana, 2016]
2017	**FEYNHELPERS** [VS, 2016]
2020	**FEYNCALC** 9.3 [VS, Mertig, Orellana, 2020]
2020	**FEYNONIUM** [Brambilla, Chung, VS, Vairo 2020]

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Symanzik polynomials and Pak algorithm

- Feynman parametric representation of an L-loop scalar Minkowskian integral

$$\boxed{\left(\frac{e^{\varepsilon \gamma_E}}{i\pi^{d/2}}\right)^L \int \frac{\left(\prod_{i=1}^L d^d k_i\right)}{P_1^{m_1} \ldots P_N^{m_N}} = \frac{(-1)^{N_m} \Gamma(N_m - \frac{Ld}{2})}{\prod_{j=1}^N \Gamma(m_j)} \int_0^\infty \prod_{j=1}^N dx_j x_j^{m_j - 1} \, \delta\left(1 - \sum_{i=1}^N x_i\right) \frac{\mathcal{U}^{N_m - \frac{(L+1)d}{2}}}{\mathcal{F}^{N_m - \frac{Ld}{2}}}}$$

  with N quadratic/eikonal propagators P_i and $N_m = \sum_{i=1}^N m_i$ with $m_i \geq 0$
- **Properties of the integral encoded in the Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$** (nice summary in [Bogner & Weinzierl, 2010])
- Some combination of $(\mathcal{U}, \mathcal{F})$ and m_i to characterize the given loop integral topology?
- Use this to find mappings between different topologies?
- In principle, yes! But things are not so simple ...

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- Denote $(\mathcal{U}, \mathcal{F})$ as the characteristic polynomial P
- Popular choices: $P = \mathcal{U} \times \mathcal{F}$ or $P = \mathcal{U} + \mathcal{F}$
- P depends on the Feynman parameters x_i and is not unique!
- A new P' from P by permuting x_i (e. g. $x_1 \leftrightarrow x_5, x_3 \leftrightarrow x_7$) still describes the same loop integral
- Enumerating all x_i permutations by brute force highly impractical!
- Need to find some *canonical ordering* of the Feynman parameters x_i in the given P
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- Rough idea: Write P as a matrix, find the canonical form by swapping/sorting rows and columns
- Pak algorithm: canonical ordering of x_i + symmetries between the corresponding lines.
- Very detailed description in the PhD thesis of Jens Hoff [Hoff, 2015]
- Technical implementation in **MATHEMATICA** straightforward
  - Automatic calculation of $\mathcal{U}$ + $\mathcal{F}$ in **UF.M** (now part of **FIESTA** [Smirnov et al., 2021] and **FIRE** [Smirnov & Chuharev, 2020])
  - Many of Pak's ideas implemented in **TOPOID** [Hoff, 2016], `https://github.com/thejensemann/TopoID`

Using FeynCalc in multiloop calculations

- Two new **FEYNCALC** symbols for multiloop calculations
- `FCTopology[id,{propagators}, {loop momenta}, {external momenta}, {kinematics}, {}]` denotes a loop integral family `id`
- `GLI[id,{propagator powers}]` is a loop integral belonging to the integral family `id`
- Syntax inspired by **FIRE** and **LITERED** [Lee, 2014], yet there are important differences
  - `FCTopology`'s are local, can simultaneously work with multiple topologies and/or modify them on the fly
  - No global list of topologies known in the current **MATHEMATICA** session
  - Relevant functions usually take 2 arguments: a list of `GLI`'s and a list of `FCTopology`'s
- In addition to that, dozens of new functions that work with `GLI`'s and `FCTopology`'s:
  - Naming scheme: `FCLoopXYZ`
  - Most new functions also work with integrals in the `(S)FAD`-notation

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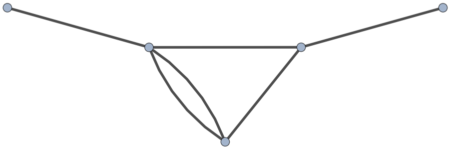
- Example: Integral $G(1,1,0,1,1)$ from the family of fully massive 2-loop on-shell propagators with $q^2 = m_1^2$



- First we need to define the topology (call it `prop2L`)

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In[1]:= topo = FCTopology[prop2L, {FAD[{p1, m1}], FAD[{p2, m2}], FAD[{p1 + q, m3}], FAD[{p2 + q, m4}], FAD[{p1 - p2, m5}]},
        {p1, p2}, {q}, {SPD[q] -> m1^2}, {}];
```

- The integral is just a symbol that doesn't do anything

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In[2]:= GLI[prop2L, {1, 1, 0, 1, 1}]
```

Out[2]= $G^{\text{prop2L}}(1,1,0,1,1)$

- Simplest manipulation: convert a `GLI` symbol to the `(S)FAD`-notation

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In[3]:= FCLoopFromGLI[GLI[prop2L, {1, 1, 0, 1, 1}], {topo}]
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Out[3]= $\dfrac{1}{(p1^2-m1^2)\,(p2^2-m2^2)\,((p2+q)^2-m4^2)\,((p1-p2)^2-m5^2)}$

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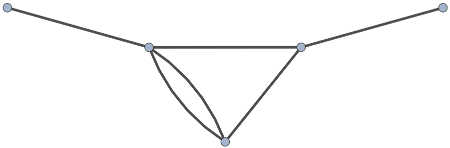

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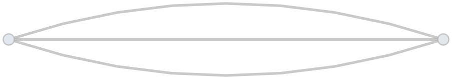
Topology identification

- Coming back to the question of topology identification and canonical ordering ...
- How do we get the $\mathcal{U}$ and $\mathcal{F}$ polynomials of the given integral?
- Example: 2-loop massive tadpole (here we use the `(S)FAD`-notation)



- $\mathcal{U}$ and $\mathcal{F}$: first two entries in the list returned by `FCFeynmanPrepare`

```
In[4]:= FCFeynmanPrepare[SFAD[{p1, m1^2}, {p2, m2^2}, {p1 - p2, m3^2}], {p1, p2}, Names -> x][[1 ;; 2]] // TableForm

Out[4]//TableForm=
        x(1) x(2)+x(3) x(2)+x(1) x(3)
        (x(1) x(2)+x(3) x(2)+x(1) x(3)) (m1^2 x(1)+m2^2 x(3)+m3^2 x(2))
```

- `FCFeynmanPrepare` also returns other building blocks
  - propagator powers
  - M from $\mathcal{U} = \det M$
  - J and Q^μ from $\mathcal{F} = \det\, M(QM^{-1}Q - J)$

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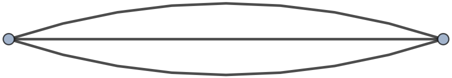

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In[5]:= FCFeynmanPrepare[SFAD[{p1, m1^2}, {p2, m2^2}, {p1 - p2, m3^2}], {p1, p2}, Names -> x][[1 ;; 2]] // TableForm

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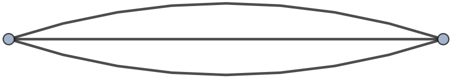

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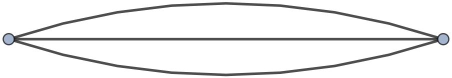

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- What about the characteristic polynomial $P = \mathcal{U} \times \mathcal{F}$?
- Same specimen, the 2-loop tadpole

```
In[8]:= FCLoopToPakForm[FAD[{p1, m1}, {p2, m2}, {p1 - p2, m3}], {p1, p2}, Names -> x][[2]][[1]]
```

Out[8]= $m1^2\, x(1)^2\, x(2)+m1^2\, x(1)^2\, x(3)+m1^2\, x(1)\, x(2)\, x(3)+m2^2\, x(1)\, x(2)^2+m2^2\, x(2)^2\, x(3)+m2^2\, x(1)\, x(2)\, x(3)+m3^2\, x(1)\, x(3)^2+m3^2\, x(2)\, x(3)^2+m3^2\, x(1)\, x(2)\, x(3)+x(1)\, x(2)+x(1)\, x(3)+x(2)\, x(3)$

- The output is already canonically ordered
- Each x_i corresponds to one of the propagators (the function keeps track of that)
- Want to work canonicalize the given polynomial?
- Use `FCLoopPakOrder`

```
In[9]:= poly = -1/4*(x[2]^2*x[3]) - (x[1]^2*x[4])/4 - (x[1]^2*x[5])/4 + (x[1]*x[2]*x[5])/2 - (x[2]^2*x[5])/4 + x[3]*x[4]*x[5];
```

Out[9]= $-\frac{1}{4}\, x(4)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(5)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(5)+x(3)\, x(4)\, x(5)$

```
In[10]:= FCLoopPakOrder[poly, x, Rename -> True]
```

Out[10]= $-\frac{1}{4}\, x(3)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(3)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(4)+x(3)\, x(4)\, x(5)$

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```
In[11]:= FCLoopToPakForm[FAD[{p1, m1}, {p2, m2}, {p1 - p2, m3}], {p1, p2}, Names -> x][[2]][[1]]
```

Out[11]= $m1^2\, x(1)^2\, x(2)+m1^2\, x(1)^2\, x(3)+m1^2\, x(1)\, x(2)\, x(3)+m2^2\, x(1)\, x(2)^2+m2^2\, x(2)^2\, x(3)+m2^2\, x(1)\, x(2)\, x(3)+m3^2\, x(1)\, x(3)^2+m3^2\, x(2)\, x(3)^2+m3^2\, x(1)\, x(2)\, x(3)+x(1)\, x(2)+x(1)\, x(3)+x(2)\, x(3)$

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Out[12]= $-\frac{1}{4}\, x(4)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(5)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(5)+x(3)\, x(4)\, x(5)$

```
In[13]:= FCLoopPakOrder[poly, x, Rename -> True]
```

Out[13]= $-\frac{1}{4}\, x(3)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(3)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(4)+x(3)\, x(4)\, x(5)$

- What about the characteristic polynomial $P = \mathcal{U} \times \mathcal{F}$?
- Same specimen, the 2-loop tadpole

```
In[14]:= FCLoopToPakForm[FAD[{p1, m1}, {p2, m2}, {p1 - p2, m3}], {p1, p2}, Names -> x][[2]][[1]]
```

Out[14]= $m1^2\, x(1)^2\, x(2)+m1^2\, x(1)^2\, x(3)+m1^2\, x(1)\, x(2)\, x(3)+m2^2\, x(1)\, x(2)^2+m2^2\, x(2)^2\, x(3)+m2^2\, x(1)\, x(2)\, x(3)+m3^2\, x(1)\, x(3)^2+m3^2\, x(2)\, x(3)^2+m3^2\, x(1)\, x(2)\, x(3)+x(1)\, x(2)+x(1)\, x(3)+x(2)\, x(3)$

- The output is already canonically ordered
- Each x_i corresponds to one of the propagators (the function keeps track of that)
- Want to work canonicalize the given polynomial?
- Use FCLoopPakOrder

```
In[15]:= poly = -1/4*(x[2]^2*x[3]) - (x[1]^2*x[4])/4 - (x[1]^2*x[5])/4 + (x[1]*x[2]*x[5])/2 - (x[2]^2*x[5])/4 + x[3]*x[4]*x[5];
```

Out[15]= $-\frac{1}{4}\, x(4)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(5)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(5)+x(3)\, x(4)\, x(5)$

```
In[16]:= FCLoopPakOrder[poly, x, Rename -> True]
```

Out[16]= $-\frac{1}{4}\, x(3)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(3)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(4)+x(3)\, x(4)\, x(5)$

- What about the characteristic polynomial $P = \mathcal{U} \times \mathcal{F}$?
- Same specimen, the 2-loop tadpole

```
In[17]:= FCLoopToPakForm[FAD[{p1, m1}, {p2, m2}, {p1 - p2, m3}], {p1, p2}, Names -> x][[2]][[1]]
```

Out[17]= $m1^2\, x(1)^2\, x(2)+m1^2\, x(1)^2\, x(3)+m1^2\, x(1)\, x(2)\, x(3)+m2^2\, x(1)\, x(2)^2+m2^2\, x(2)^2\, x(3)+m2^2\, x(1)\, x(2)\, x(3)+m3^2\, x(1)\, x(3)^2+m3^2\, x(2)\, x(3)^2+m3^2\, x(1)\, x(2)\, x(3)+x(1)\, x(2)+x(1)\, x(3)+x(2)\, x(3)$

- The output is already canonically ordered
- Each x_i corresponds to one of the propagators (the function keeps track of that)
- Want to work canonicalize the given polynomial?
- Use FCLoopPakOrder

```
In[18]:= poly = -1/4*(x[2]^2*x[3]) - (x[1]^2*x[4])/4 - (x[1]^2*x[5])/4 + (x[1]*x[2]*x[5])/2 - (x[2]^2*x[5])/4 + x[3]*x[4]*x[5];
```

Out[18]= $-\frac{1}{4}\, x(4)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(5)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(5)+x(3)\, x(4)\, x(5)$

```
In[19]:= FCLoopPakOrder[poly, x, Rename -> True]
```

Out[19]= $-\frac{1}{4}\, x(3)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(3)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(4)+x(3)\, x(4)\, x(5)$

- What about the characteristic polynomial $P = \mathcal{U} \times \mathcal{F}$?
- Same specimen, the 2-loop tadpole

```
In[20]:= FCLoopToPakForm[FAD[{p1, m1}, {p2, m2}, {p1 - p2, m3}], {p1, p2}, Names -> x][[2]][[1]]
```

Out[20]= $m1^2\, x(1)^2\, x(2)+m1^2\, x(1)^2\, x(3)+m1^2\, x(1)\, x(2)\, x(3)+m2^2\, x(1)\, x(2)^2+m2^2\, x(2)^2\, x(3)+m2^2\, x(1)\, x(2)\, x(3)+m3^2\, x(1)\, x(3)^2+m3^2\, x(2)\, x(3)^2+m3^2\, x(1)\, x(2)\, x(3)+x(1)\, x(2)+x(1)\, x(3)+x(2)\, x(3)$

- The output is already canonically ordered
- Each x_i corresponds to one of the propagators (the function keeps track of that)
- Want to work canonicalize the given polynomial?
- Use `FCLoopPakOrder`

```
In[21]:= poly = -1/4*(x[2]^2*x[3]) - (x[1]^2*x[4])/4 - (x[1]^2*x[5])/4 + (x[1]*x[2]*x[5])/2 - (x[2]^2*x[5])/4 + x[3]*x[4]*x[5];
```

Out[21]= $-\frac{1}{4}\, x(4)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(5)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(5)+x(3)\, x(4)\, x(5)$

```
In[22]:= FCLoopPakOrder[poly, x, Rename -> True]
```

Out[22]= $-\frac{1}{4}\, x(3)\, x(1)^2-\frac{1}{4}\, x(5)\, x(1)^2+\frac{1}{2}\, x(2)\, x(3)\, x(1)-\frac{1}{4}\, x(2)^2\, x(3)-\frac{1}{4}\, x(2)^2\, x(4)+x(3)\, x(4)\, x(5)$

- Suppose that we have a set of source topologies that can be mapped into a set of target topologies
- Working with amplitudes, not graphs: need explicit momentum shifts that describe these mappings
- Example: 3-loop propagator-type massless integrals, 2 source and 1 target topologies

```
In[23]:= source = {FCTopology[topo1, {FAD[p1], FAD[p2], FAD[p3], FAD[p1 + p3], FAD[p2 + p3], FAD[p2 - Q],
   FAD[p1 + p3 - Q], FAD[p2 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}],
   FCTopology[topo2, {FAD[p1], FAD[p2], FAD[p3], FAD[p1 + p3], FAD[p2 + p3], FAD[p1 - Q], FAD[p2 - Q],
   FAD[p1 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}]}
```

Out[23]= $\{\text{FCTopology}(\text{topo1},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\}),$
$\text{FCTopology}(\text{topo2},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p1-Q)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\})\}$

```
In[24]:= target = {FCTopology[prop3L, {FAD[p1], FAD[p2], FAD[p3], FAD[p2 + p3], FAD[p1 - Q], FAD[p2 - Q],
   FAD[p1 + p3 - Q], FAD[p2 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}]}
```

Out[24]= $\{\text{FCTopology}(\text{prop3L},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p2+p3)^2},\frac{1}{(p1-Q)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\})\}$

- We can get the mappings to the target topology with just one command

```
In[25]:= mappings = FCLoopFindTopologyMappings[Join[source, target], PreferredTopologies -> prop3L]
```

- Suppose that we have a set of source topologies that can be mapped into a set of target topologies
- Working with amplitudes, not graphs: need explicit momentum shifts that describe these mappings
- Example: 3-loop propagator-type massless integrals, 2 source and 1 target topologies

```
In[26]:= source = {FCTopology[topo1, {FAD[p1], FAD[p2], FAD[p3], FAD[p1 + p3], FAD[p2 + p3], FAD[p2 - Q],
    FAD[p1 + p3 - Q], FAD[p2 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}],
    FCTopology[topo2, {FAD[p1], FAD[p2], FAD[p3], FAD[p1 + p3], FAD[p2 + p3], FAD[p1 - Q], FAD[p2 - Q],
    FAD[p1 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}]}
```

Out[26]= $\{\mathrm{FCTopology}(\mathrm{topo1},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\}),$
$\mathrm{FCTopology}(\mathrm{topo2},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p1-Q)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\})\}$

```
In[27]:= target = {FCTopology[prop3L, {FAD[p1], FAD[p2], FAD[p3], FAD[p2 + p3], FAD[p1 - Q], FAD[p2 - Q],
    FAD[p1 + p3 - Q], FAD[p2 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}]}
```

Out[27]= $\{\mathrm{FCTopology}(\mathrm{prop3L},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p2+p3)^2},\frac{1}{(p1-Q)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\})\}$

- We can get the mappings to the target topology with just one command

```
In[28]:= mappings = FCLoopFindTopologyMappings[Join[source, target], PreferredTopologies -> prop3L]
```

- Suppose that we have a set of source topologies that can be mapped into a set of target topologies
- Working with amplitudes, not graphs: need explicit momentum shifts that describe these mappings
- Example: 3-loop propagator-type massless integrals, 2 source and 1 target topologies

```
In[29]:= source = {FCTopology[topo1, {FAD[p1], FAD[p2], FAD[p3], FAD[p1 + p3], FAD[p2 + p3], FAD[p2 - Q],
   FAD[p1 + p3 - Q], FAD[p2 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}],
   FCTopology[topo2, {FAD[p1], FAD[p2], FAD[p3], FAD[p1 + p3], FAD[p2 + p3], FAD[p1 - Q], FAD[p2 - Q],
   FAD[p1 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}]}
```

Out[29]= $\{\mathrm{FCTopology}(\mathrm{topo1},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\}),$
$\mathrm{FCTopology}(\mathrm{topo2},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p1-Q)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\})\}$

```
In[30]:= target = {FCTopology[prop3L, {FAD[p1], FAD[p2], FAD[p3], FAD[p2 + p3], FAD[p1 - Q], FAD[p2 - Q],
   FAD[p1 + p3 - Q], FAD[p2 + p3 - Q], FAD[p1 + p2 + p3 - Q]}, {p1, p2, p3}, {Q}, {}, {}]}
```

Out[30]= $\{\mathrm{FCTopology}(\mathrm{prop3L},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p2+p3)^2},\frac{1}{(p1-Q)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\})\}$

- We can get the mappings to the target topology with just one command

```
In[31]:= mappings = FCLoopFindTopologyMappings[Join[source, target], PreferredTopologies -> prop3L]
```

- Let us examine the output of

  In[32]:= **mappings = FCLoopFindTopologyMappings[Join[source, target], PreferredTopologies -> {prop3L}]**

  in more details
- The first entry maps `topo1` to `prop3L`

  In[33]:= **mappings[[1]]**

  Out[33]= $\{\mathrm{FCTopology}(\mathrm{topo1},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\}),$
  $\{p1\to -p1-p3+Q,p2\to -p2-p3+Q,p3\to p3\},\ G^{\mathrm{topo1}}(n7\_,n8\_,n3\_,n5\_,n6\_,n4\_,n1\_,n2\_,n9\_):\to G^{\mathrm{prop3L}}(n1,n2,n3,n4,n5,n6,n7,n8,n9)\}$
- Content: original source topology, a list of momenta shifts and a replacement rule for scalar integrals
- All information that you need to convert scalar and tensor integrals from `topo1` to integrals from `prop3L`!

- Let us examine the output of

In[34]:= **mappings = FCLoopFindTopologyMappings[Join[source, target], PreferredTopologies -> {prop3L}]**

in more details

- The first entry maps `topo1` to `prop3L`

In[35]:= **mappings[[1]]**

Out[35]= $\{\text{FCTopology}(\text{topo1},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\}),$
$\{p1\to -p1-p3+Q, p2\to -p2-p3+Q, p3\to p3\}, G^{\text{topo1}}(n7\_,n8\_,n3\_,n5\_,n6\_,n4\_,n1\_,n2\_,n9\_) :\to G^{\text{prop3L}}(n1,n2,n3,n4,n5,n6,n7,n8,n9)\}$

- Content: original source topology, a list of momenta shifts and a replacement rule for scalar integrals
- All information that you need to convert scalar and tensor integrals from `topo1` to integrals from `prop3L`!

- Let us examine the output of

```
In[36]:= mappings = FCLoopFindTopologyMappings[Join[source, target], PreferredTopologies -> {prop3L}]
```

  in more details

- The first entry maps `topo1` to `prop3L`

```
In[37]:= mappings[[1]]
```

Out[37]= $\{\mathrm{FCTopology}(\mathrm{topo1},\{\frac{1}{p1^2},\frac{1}{p2^2},\frac{1}{p3^2},\frac{1}{(p1+p3)^2},\frac{1}{(p2+p3)^2},\frac{1}{(p2-Q)^2},\frac{1}{(p1+p3-Q)^2},\frac{1}{(p2+p3-Q)^2},\frac{1}{(p1+p2+p3-Q)^2}\},\{p1,p2,p3\},\{Q\},\{\},\{\}),$
$\{p1\to -p1-p3+Q,p2\to -p2-p3+Q,p3\to p3\},\ G^{\mathrm{topo1}}(n7\_,n8\_,n3\_,n5\_,n6\_,n4\_,n1\_,n2\_,n9\_):\ \to G^{\mathrm{prop3L}}(n1,n2,n3,n4,n5,n6,n7,n8,n9)\}$

- Content: original source topology, a list of momenta shifts and a replacement rule for scalar integrals
- All information that you need to convert scalar and tensor integrals from `topo1` to integrals from `prop3L`!

- Applying this machinery to **amplitudes** is already possible, but somewhat cumbersome
  - `FCLoopFindTopologies[exp, q1, q2, ...]` to identify distinct topologies in the expression
  - `FCLoopFindTopologyMappings[topo1, topo2, ...]` to minimize the number of topologies
  - `FCLoopApplyTopologyMappings[exp, mappings]` to apply the so-obtained mappings
- Some aspects that require further attention/optimization
  - Handling of tensor integrals
  - Automatically augmenting incomplete topologies to fit them into existing (or new) complete topologies
  - Performance optimizations
- **What we actually want is to calculate multiloop amplitudes with FORM, not Mathematica!**
- **The goal is to have a hybrid framework: toolbox-like FORM library for heavy computations and a Mathematica library (FeynCalc) for everything else (e. g. topology identification, R&D, etc.)**

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Master integrals

- Suppose that you are done with the IBP-reduction [Chetyrkin & Tkachov, 1981; Tkachov, 1981] of your loop integrals, what are the next steps?
  - Find mappings between master integrals from different integral families
  - Visualize your unique master integrals
  - Try to calculate some of them analytically?
- Let us see how FEYNCALC can help us in tackling those tasks!

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  - Find mappings between master integrals from different integral families
  - Visualize your unique master integrals
  - Try to calculate some of them analytically?
- Let us see how **FeynCalc** can help us in tackling those tasks!

- With Pak algorithm at our disposal finding mappings between master integrals is straightforward
- Example: 2-loop self-energies with one or two massive lines

```
In[38]:= topos = {FCTopology[prop2Lv1, {FAD[p1], FAD[{p1 + q1, m}], FAD[p2], FAD[p2 + q1], FAD[p1 - p2]}, {p1, p2}, {q1}, {}, {}],
          FCTopology[prop2Lv2, {FAD[p1], FAD[{p2, m}], FAD[{p2 + q1, m}], FAD[p1 + q1], FAD[-p1 + p2]}, {p1, p2}, {q1}, {}, {}]};
```

- We have a list of master integrals that are not all unique

```
In[39]:= glis = {GLI[prop2Lv1, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {1, 0, 1, 1, 1}],
           GLI[prop2Lv2, {1, 1, 0, 1, 1}], GLI[prop2Lv2, {1, 1, 1, 0, 1}]}
```

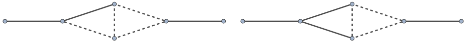
- FeynCalc identifies three mappings between our masters

```
In[40]:= FCLoopFindIntegralMappings[glis, topos][[1]]
```

Out[40]= $\{G^{\text{prop2Lv2}}(1,0,1,1,1)\to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,0,1,1)\to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,1,0,1)\to G^{\text{prop2Lv2}}(0,1,1,1,1)\}$

- Working principle based on `FindRules` from **FIRE**
- Can specify a set of preferred master integrals via the `PreferredIntegrals` option

- With Pak algorithm at our disposal finding mappings between master integrals is straightforward
- Example: 2-loop self-energies with one or two massive lines



```
In[41]:= topos = {FCTopology[prop2Lv1, {FAD[p1], FAD[{p1 + q1, m}], FAD[p2], FAD[p2 + q1], FAD[p1 - p2]}, {p1, p2}, {q1}, {}, {}],
         FCTopology[prop2Lv2, {FAD[p1], FAD[{p2, m}], FAD[{p2 + q1, m}], FAD[p1 + q1], FAD[-p1 + p2]}, {p1, p2}, {q1}, {}, {}]};
```

- We have a list of master integrals that are not all unique

```
In[42]:= glis = {GLI[prop2Lv1, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {1, 0, 1, 1, 1}],
           GLI[prop2Lv2, {1, 1, 0, 1, 1}], GLI[prop2Lv2, {1, 1, 1, 0, 1}]}
```

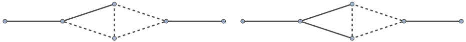
- FEYNCALC identifies three mappings between our masters

```
In[43]:= FCLoopFindIntegralMappings[glis, topos][[1]]
```

Out[43]= $\{G^{\text{prop2Lv2}}(1,0,1,1,1) \to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,0,1,1) \to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,1,0,1) \to G^{\text{prop2Lv2}}(0,1,1,1,1)\}$

- Working principle based on `FindRules` from **FIRE**
- Can specify a set of preferred master integrals via the `PreferredIntegrals` option

- With Pak algorithm at our disposal finding mappings between master integrals is straightforward
- Example: 2-loop self-energies with one or two massive lines



```
In[44]:= topos = {FCTopology[prop2Lv1, {FAD[p1], FAD[{p1 + q1, m}], FAD[p2], FAD[p2 + q1], FAD[p1 - p2]}, {p1, p2}, {q1}, {}, {}],
         FCTopology[prop2Lv2, {FAD[p1], FAD[{p2, m}], FAD[{p2 + q1, m}], FAD[p1 + q1], FAD[-p1 + p2]}, {p1, p2}, {q1}, {}, {}]};
```

- We have a list of master integrals that are not all unique

```
In[45]:= glis = {GLI[prop2Lv1, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {1, 0, 1, 1, 1}],
           GLI[prop2Lv2, {1, 1, 0, 1, 1}], GLI[prop2Lv2, {1, 1, 1, 0, 1}]}
```

- **FeynCalc** identifies three mappings between our masters

```
In[46]:= FCLoopFindIntegralMappings[glis, topos][[1]]
```

Out[46]= $\{G^{\text{prop2Lv2}}(1,0,1,1,1)\to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,0,1,1)\to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,1,0,1)\to G^{\text{prop2Lv2}}(0,1,1,1,1)\}$

- Working principle based on `FindRules` from **FIRE**
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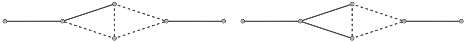


```
In[47]:= topos = {FCTopology[prop2Lv1, {FAD[p1], FAD[{p1 + q1, m}], FAD[p2], FAD[p2 + q1], FAD[p1 - p2]}, {p1, p2}, {q1}, {}, {}],
         FCTopology[prop2Lv2, {FAD[p1], FAD[{p2, m}], FAD[{p2 + q1, m}], FAD[p1 + q1], FAD[-p1 + p2]}, {p1, p2}, {q1}, {}, {}]};
```

- We have a list of master integrals that are not all unique

```
In[48]:= glis = {GLI[prop2Lv1, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {0, 1, 1, 1, 1}], GLI[prop2Lv2, {1, 0, 1, 1, 1}],
           GLI[prop2Lv2, {1, 1, 0, 1, 1}], GLI[prop2Lv2, {1, 1, 1, 0, 1}]}
```

- **FeynCalc** identifies three mappings between our masters

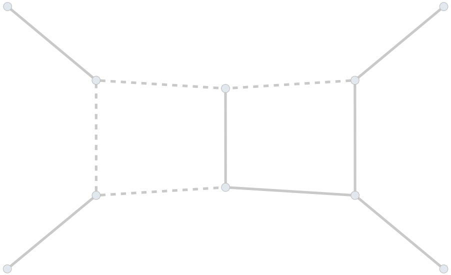
```
In[49]:= FCLoopFindIntegralMappings[glis, topos][[1]]
```

Out[49]= $\{G^{\text{prop2Lv2}}(1,0,1,1,1)\to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,0,1,1)\to G^{\text{prop2Lv1}}(0,1,1,1,1), G^{\text{prop2Lv2}}(1,1,1,0,1)\to G^{\text{prop2Lv2}}(0,1,1,1,1)\}$

- Working principle based on `FindRules` from **FIRE**
- Can specify a set of preferred master integrals via the `PreferredIntegrals` option

- Given a propagator representation of a loop integral, how to obtain its graph representation?
- Can be done using AZURITE [Georgoudis et al., 2017], PlanarityTest, [Bielas et al., 2013], LiteRed and now also FeynCalc
- Two-step approach: `FCLoopIntegralToGraph` reconstructs the graph, `FCLoopGraphPlot` plots it.

```
In[50]:= FCLoopIntegralToGraph[FAD[{p1, m1}, {p2, m2}, {Q1 + p1, m3}, Q2 - p1, Q1 + p1 + p2, Q2 - p1 - p2, Q2 + Q3 - p1 - p2], {p1, p2}];
         FCLoopGraphPlot[%]
```



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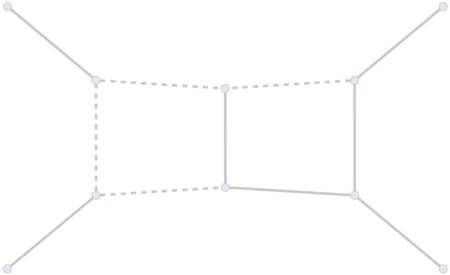
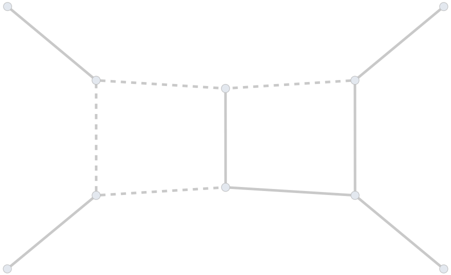
- Two-step approach: `FCLoopIntegralToGraph` reconstructs the graph, `FCLoopGraphPlot` plots it.

```
In[51]:= FCLoopIntegralToGraph[FAD[{p1, m1}, {p2, m2}, {Q1 + p1, m3}, Q2 - p1, Q1 + p1 + p2, Q2 - p1 - p2, Q2 + Q3 - p1 - p2], {p1, p2}];
         FCLoopGraphPlot[%]
```



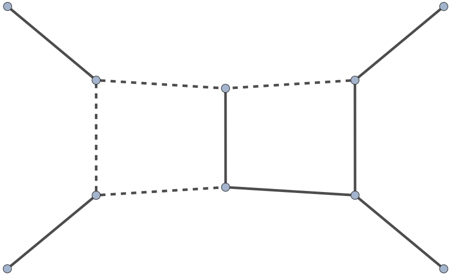
- Given a propagator representation of a loop integral, how to obtain its graph representation?
- Can be done using **AZURITE** [Georgoudis et al., 2017], **PlanarityTest**, [Bielas et al., 2013], **LiteRed** and now also **FeynCalc**
- Two-step approach: `FCLoopIntegralToGraph` reconstructs the graph, `FCLoopGraphPlot` plots it.

```
In[52]:= FCLoopIntegralToGraph[FAD[{p1, m1}, {p2, m2}, {Q1 + p1, m3}, Q2 - p1, Q1 + p1 + p2, Q2 - p1 - p2, Q2 + Q3 - p1 - p2], {p1, p2}];
FCLoopGraphPlot[%]
```



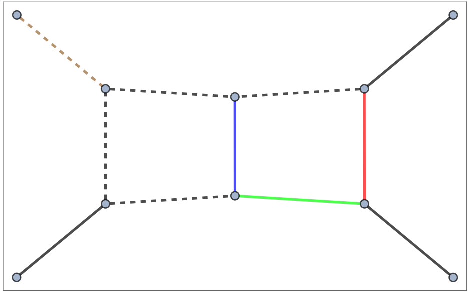
- Given a propagator representation of a loop integral, how to obtain its graph representation?
- Can be done using **AZURITE** [Georgoudis et al., 2017], **PLANARITYTEST**, [Bielas et al., 2013], **LITERED** and now also **FEYNCALC**
- Two-step approach: `FCLoopIntegralToGraph` reconstructs the graph, `FCLoopGraphPlot` plots it.

```
In[53]:= FCLoopIntegralToGraph[FAD[{p1, m1}, {p2, m2}, {Q1 + p1, m3}, Q2 - p1, Q1 + p1 + p2, Q2 - p1 - p2, Q2 + Q3 - p1 - p2], {p1, p2}];
         FCLoopGraphPlot[%]
```



- The plot can be styled to make it more visually appealing

```
In[54]:= FCLoopIntegralToGraph[FAD[{p1, m1}, {p2, m2}, {Q1 + p1, m3}, Q2 - p1, Q1 + p1 + p2, Q2 - p1 - p2, Q2 + Q3 - p1 - p2], {p1, p2}];
FCLoopGraphPlot[%, GraphPlot -> {MultiedgeStyle -> 0.35, Frame -> True}, Style -> {
{"InternalLine", _, _, mm_ /; ! FreeQ[mm, m1]} -> {Red, Thick},
{"InternalLine", _, _, mm_ /; ! FreeQ[mm, m2]} -> {Blue, Thick},
{"InternalLine", _, _, mm_ /; ! FreeQ[mm, m3]} -> {Green, Thick},
{"ExternalLine", Q1} -> {Brown, Thick, Dashed}}]
```



- Last but not least: Automatic derivation of Feynman parametrizations for the given integral/topology
- Useful if you want to calculate the integral directly e. g. using **HyperInt** [Panzer, 2015]
- Example: 2-loop self-energy with 2 eikonal propagators

```
In[55]:= int = SFAD[{p1, m^2}], SFAD[{p3, m^2}], SFAD[{(p3 - p1), m^2}], SFAD[{{0, 2 p1 . n}}], SFAD[{{0, 2 p3 . n}}]
```

Out[55]= $\left\{\frac{1}{p1^2-m2^2}, \frac{1}{p3^2-m2^2}, \frac{1}{(p3-p1)^2-m2^2}, \frac{1}{(2\,(n\cdot p1))}, \frac{1}{(2\,(n\cdot p3))}\right\}$

```
In[56]:= FCFeynmanParametrize[int, {p1, p3}, Names -> x, FCReplaceD -> {D -> 4 - 2 ep},
FinalSubstitutions -> {SPD[n] -> 1, m -> 1}]
```

Out[56]= $\{(x(3)\,x(4)+x(5)\,x(4)+x(3)\,x(5))^{3\,ep-1}\,(m2^2\,x(3)\,x(4)^2+m2^2\,x(3)\,x(5)^2+$
$m2^2\,x(4)\,x(5)^2+m2^2\,x(3)^2\,x(4)+m2^2\,x(3)^2\,x(5)+m2^2\,x(4)^2\,x(5)+$
$3\,m2^2\,x(3)\,x(4)\,x(5)+x(2)^2\,x(3)+x(1)^2\,x(4)+x(1)^2\,x(5)+x(2)^2\,x(5)+2\,x(1)\,x(2)\,x(5))^{-2\,ep-1},$
$-\Gamma(2\,ep+1),$
$\{x(1),x(2),x(3),x(4),x(5)\}\}$

- 3-element output: integrand, prefactor, list of integration variables
- Many options to fine-tune the output ...
- Supports tensor integrals, scalar products in numerators, symbolic propagator powers, Euclidean metric, Cartesian integrals, ...
- Some tricks borrowed from pySecDec [Heinrich et al., 2021], c. f. also PhD thesis of Stefan Jahn [Jahn, 2020]

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- Example: 2-loop self-energy with 2 eikonal propagators

```
In[57]:= int = SFAD[{p1, m^2}], SFAD[{p3, m^2}], SFAD[{(p3 - p1), m^2}], SFAD[{{0, 2 p1 . n}}], SFAD[{{0, 2 p3 . n}}]
```

Out[57]= $\left\{\frac{1}{p1^2-m2^2}, \frac{1}{p3^2-m2^2}, \frac{1}{(p3-p1)^2-m2^2}, \frac{1}{(2\,(n\cdot p1))}, \frac{1}{(2\,(n\cdot p3))}\right\}$

```
In[58]:= FCFeynmanParametrize[int, {p1, p3}, Names -> x, FCReplaceD -> {D -> 4 - 2 ep},
FinalSubstitutions -> {SPD[n] -> 1, m -> 1}]
```

Out[58]= $\{(x(3)\,x(4)+x(5)\,x(4)+x(3)\,x(5))^{3\,ep-1}\,(m2^2\,x(3)\,x(4)^2+m2^2\,x(3)\,x(5)^2+$
$m2^2\,x(4)\,x(5)^2+m2^2\,x(3)^2\,x(4)+m2^2\,x(3)^2\,x(5)+m2^2\,x(4)^2\,x(5)+$
$3\,m2^2\,x(3)\,x(4)\,x(5)+x(2)^2\,x(3)+x(1)^2\,x(4)+x(1)^2\,x(5)+x(2)^2\,x(5)+2\,x(1)\,x(2)\,x(5))^{-2\,ep-1},$
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```
In[59]:= int = SFAD[{p1, m^2}], SFAD[{p3, m^2}], SFAD[{(p3 - p1), m^2}], SFAD[{{0, 2 p1 . n}}], SFAD[{{0, 2 p3 . n}}]
```

Out[59]= $\left\{\frac{1}{p1^2-m2^2}, \frac{1}{p3^2-m2^2}, \frac{1}{(p3-p1)^2-m2^2}, \frac{1}{(2\,(n\cdot p1))}, \frac{1}{(2\,(n\cdot p3))}\right\}$

```
In[60]:= FCFeynmanParametrize[int, {p1, p3}, Names -> x, FCReplaceD -> {D -> 4 - 2 ep},
         FinalSubstitutions -> {SPD[n] -> 1, m -> 1}]
```

Out[60]= $\{(x(3)\,x(4)+x(5)\,x(4)+x(3)\,x(5))^{3\,ep-1}\,(m2^2\,x(3)\,x(4)^2+m2^2\,x(3)\,x(5)^2+$
$m2^2\,x(4)\,x(5)^2+m2^2\,x(3)^2\,x(4)+m2^2\,x(3)^2\,x(5)+m2^2\,x(4)^2\,x(5)+$
$3\,m2^2\,x(3)\,x(4)\,x(5)+x(2)^2\,x(3)+x(1)^2\,x(4)+x(1)^2\,x(5)+x(2)^2\,x(5)+2\,x(1)\,x(2)\,x(5))^{-2\,ep-1},$
$-\Gamma(2\,ep+1),$
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Summary and Outlook

Summary

- **FEYNCALC** 10 will feature many flexible and easy-to-use routines for multiloop calculations
- This functionality can be quite useful for your **FORM**-based setups
- Standing on the shoulders of giants: **FIRE**, **FIESTA**, **TOPOID**, **LITERED**, **PYSECDEC**, ...
- Sincere gratitude to the authors of these open-source tools!

Public preview

- Interested? Try it out

```
In[1]:= Import["https://raw.githubusercontent.com/FeynCalc/feyncalc/master/install.m"]
        InstallFeynCalc[InstallFeynCalcDevelopmentVersion -> True]
```

- Everything already documented: `https://feyncalc.github.io/referenceDev`
- Questions or comments? Visit our forum: `https://github.com/FeynCalc/feyncalc/discussions`

Outlook

- Topology identification is *mostly* not a performance bottleneck, **MATHEMATICA** sufficient for the time being.
- Multiloop amplitude evaluation: need a **FORM**-based library with **FEYNCALC** characteristics (WIP)
- Next milestone: support helicity amplitude methods (**FEYNCALC** 11?)

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Backup

Brief summary of Pak's algorithm:

- Char. polynomial P (l terms, n Feynman parameters x_i) as an $l \times (n+1)$ matrix
- Set $i = 1$, switch the $(i+1)$-th column with each of the next columns keep track of the permutations $\Rightarrow$ new matrices
- looking only at the first $(i+1)$-columns of the new matrices, sort their rows (e. g. lexicographically)
- sorted matrices: extract the i-th column $\Rightarrow$ list of vectors
- sort the vectors and select the first one in the sorted list
- keep all matrices containing the selected column, discard the rest $\Rightarrow$ final set of matrices in this iteration
- $i++$ while $i < n-1$, otherwise return the final permutations σ.

$$P = c_2 x_2 x_3 + c_1 x_2^2 + c_2 x_1 x_3 + c_1 x_1^2 \Rightarrow \begin{pmatrix} c_2 & 0 & 1 & 1 \\ c_1 & 0 & 2 & 0 \\ c_2 & 1 & 0 & 1 \\ c_1 & 2 & 0 & 0 \end{pmatrix} \equiv M_0^{(123)}$$

1st iteration: $i = 1$, start with $\{M_0^{(123)}\}$ permute the 2nd column

$$\{M_0^{(123)} = \begin{pmatrix} c_2 & 0 & 1 & 1 \\ c_1 & 0 & 2 & 0 \\ c_2 & 1 & 0 & 1 \\ c_1 & 2 & 0 & 0 \end{pmatrix}, M_0^{(213)} = \begin{pmatrix} c_2 & 1 & 0 & 1 \\ c_1 & 2 & 0 & 0 \\ c_2 & 0 & 1 & 1 \\ c_1 & 0 & 2 & 0 \end{pmatrix},$$

$$M_0^{(321)} = \begin{pmatrix} c_2 & 1 & 1 & 0 \\ c_1 & 0 & 2 & 0 \\ c_2 & 1 & 0 & 1 \\ c_1 & 0 & 0 & 2 \end{pmatrix}\}$$

After sorting rows w.r.t to the first 2 columns

$$\{\tilde{M}_0^{(123)} = \begin{pmatrix} \boldsymbol{c_1} & \mathbf{0} & 2 & 0 \\ \boldsymbol{c_1} & \mathbf{2} & 0 & 0 \\ \boldsymbol{c_2} & \mathbf{0} & 1 & 1 \\ \boldsymbol{c_2} & \mathbf{1} & 0 & 1 \end{pmatrix}, \tilde{M}_0^{(213)} = \begin{pmatrix} \boldsymbol{c_1} & \mathbf{0} & 2 & 0 \\ \boldsymbol{c_1} & \mathbf{2} & 0 & 0 \\ \boldsymbol{c_2} & \mathbf{0} & 1 & 1 \\ \boldsymbol{c_2} & \mathbf{1} & 0 & 1 \end{pmatrix},$$

$$\tilde{M}_0^{(321)} = \begin{pmatrix} \boldsymbol{c_1} & \mathbf{0} & 2 & 0 \\ \boldsymbol{c_1} & \mathbf{0} & 0 & 2 \\ \boldsymbol{c_2} & \mathbf{1} & 1 & 0 \\ \boldsymbol{c_2} & \mathbf{1} & 0 & 1 \end{pmatrix}\}$$

Max vector among the 2nd columns : $(\mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{1})^T \Rightarrow$ keep $\tilde{M}_0^{(123)}$ and $\tilde{M}_0^{(213)}$

Brief summary of Pak's algorithm:

- Char. polynomial P (l terms, n Feynman parameters x_i) as an $l \times (n+1)$ matrix
- Set $i = 1$, switch the $(i+1)$-th column with each of the next columns keep track of the permutations $\Rightarrow$ new matrices
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- sort the vectors and select the first one in the sorted list
- keep all matrices containing the selected column, discard the rest $\Rightarrow$ final set of matrices in this iteration
- $i++$ while $i < n-1$, otherwise return the final permutations σ.

2nd iteration: $i = 2$, start with

$$\{\tilde{M}_0^{(123)} = \begin{pmatrix} c_1 & 0 & 2 & 0 \\ c_1 & 2 & 0 & 0 \\ c_2 & 0 & 1 & 1 \\ c_2 & 1 & 0 & 1 \end{pmatrix}, \tilde{M}_0^{(213)} = \begin{pmatrix} c_1 & 0 & 2 & 0 \\ c_1 & 2 & 0 & 0 \\ c_2 & 0 & 1 & 1 \\ c_2 & 1 & 0 & 1 \end{pmatrix}\}$$

Permute the 3rd column

$$\{M_1^{(123)} = \begin{pmatrix} c_1 & 0 & 2 & 0 \\ c_1 & 2 & 0 & 0 \\ c_2 & 0 & 1 & 1 \\ c_2 & 1 & 0 & 1 \end{pmatrix}, M_1^{(132)} = \begin{pmatrix} c_1 & 0 & 0 & 2 \\ c_1 & 2 & 0 & 0 \\ c_2 & 0 & 1 & 1 \\ c_2 & 1 & 1 & 0 \end{pmatrix},$$

$$M_1^{(213)} = \begin{pmatrix} c_1 & 0 & 2 & 0 \\ c_1 & 2 & 0 & 0 \\ c_2 & 0 & 1 & 1 \\ c_2 & 1 & 0 & 1 \end{pmatrix} M_1^{(231)} = \begin{pmatrix} c_1 & 0 & 0 & 2 \\ c_1 & 2 & 0 & 0 \\ c_2 & 0 & 1 & 1 \\ c_2 & 1 & 1 & 0 \end{pmatrix}\}$$

Brief summary of Pak's algorithm:

- Char. polynomial P (l terms, n Feynman parameters x_i) as an $l \times (n+1)$ matrix
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- sort the vectors and select the first one in the sorted list
- keep all matrices containing the selected column, discard the rest $\Rightarrow$ final set of matrices in this iteration
- $i++$ while $i < n-1$, otherwise return the final permutations σ.

After sorting rows w.r.t the first 3 columns (no changes here)

$$\{\tilde{M}_1^{(123)} = \begin{pmatrix} \boldsymbol{c}_1 & \mathbf{0} & \mathbf{2} & 0 \\ \boldsymbol{c}_1 & \mathbf{2} & \mathbf{0} & 0 \\ \boldsymbol{c}_2 & \mathbf{0} & \mathbf{1} & 1 \\ \boldsymbol{c}_2 & \mathbf{1} & \mathbf{0} & 1 \end{pmatrix}, \tilde{M}_1^{(132)} = \begin{pmatrix} \boldsymbol{c}_1 & \mathbf{0} & \mathbf{0} & 2 \\ \boldsymbol{c}_1 & \mathbf{2} & \mathbf{0} & 0 \\ \boldsymbol{c}_2 & \mathbf{0} & \mathbf{1} & 1 \\ \boldsymbol{c}_2 & \mathbf{1} & \mathbf{1} & 0 \end{pmatrix},$$

$$\tilde{M}_1^{(213)} = \begin{pmatrix} \boldsymbol{c}_1 & \mathbf{0} & \mathbf{2} & 0 \\ \boldsymbol{c}_1 & \mathbf{2} & \mathbf{0} & 0 \\ \boldsymbol{c}_2 & \mathbf{0} & \mathbf{1} & 1 \\ \boldsymbol{c}_2 & \mathbf{1} & \mathbf{0} & 1 \end{pmatrix} \tilde{M}_1^{(231)} = \begin{pmatrix} \boldsymbol{c}_1 & \mathbf{0} & \mathbf{0} & 2 \\ \boldsymbol{c}_1 & \mathbf{2} & \mathbf{0} & 0 \\ \boldsymbol{c}_2 & \mathbf{0} & \mathbf{1} & 1 \\ \boldsymbol{c}_2 & \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}\}$$

Max vector among the 3rd columns : $(\mathbf{2}, \mathbf{0}, \mathbf{1}, \mathbf{0})^T$
$\Rightarrow$ keep $\tilde{M}_1^{(123)}$ and $\tilde{M}_1^{(213)}$

Since $i = 3 \geq n - 1 = 3$ the algorithm terminates here.

Symmetries under renamings of x_i: $\sigma = \{(123), (213)\}$

Meaning that

$$P^{(123)} = c_2 x_2 x_3 + c_1 x_2^2 + c_2 x_1 x_3 + c_1 x_1^2$$
$$P^{(213)} = c_2 x_1 x_3 + c_1 x_1^2 + c_2 x_2 x_3 + c_1 x_2^2$$

are equivalent (obviously)