Algebraic Geometry and P-Adic Numbers for Scattering Amplitude Ansätze

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The %-Level Precision Frontier

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- In this work consider the rational coefficients $\mathcal{C}_i(\lambda, \tilde{\lambda})$

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  *after univariate partial fractioning [Badger et al. '21]
- Sampling becomes a bottleneck for high-multiplicity

Preview of Results

- Testing ground for current work $\mathcal{R}^{(2)}_{q\bar{q}\to 3\gamma}$

[Chawdhry at al., '19]
Abreu at al. '20]

- Main result: drastically reduced ansatz size (i.e. required samples)

Remainder	$\mathcal{R}^{(2,0)}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{R}^{(2,N_f)}_{\gamma^-\gamma^+\gamma^+}$	$\mathcal{R}^{(2,0)}_{\gamma^+\gamma^+\gamma^+}$	$\mathcal{R}^{(2,N_f)}_{\gamma^+\gamma^+\gamma^+}$
Old Ansatz Size	36401	2315	6665	841
New Ansatz Size	566	20	18	6

Table: Results from the current computation

Plus 317 $\underbrace{\mathbb{Q}_p}_{\text{p-adic numbers}}$ warm-up evaluations - number only dependent on multiplicity

Singular Limits in Complex Kinematics

- Structure of the rational coefficients

$$\mathcal{C}_i(\lambda, \tilde\lambda) = \frac{\mathcal{N}_i(\lambda, \tilde\lambda)}{\prod_j \mathcal{D}_j(\lambda, \tilde\lambda)^{q_{ij}}} \begin{array}{l} \longleftarrow \text{can we say anything about } \mathcal{N}\text{?} \\ \longleftarrow \mathcal{D}_j \text{ are well understood} \end{array}$$

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$$\mathcal{A}^{(0)}_{q^+,g^+,g^+,\bar{q}^-,g^-,g^-} = \frac{\mathcal{N}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}$$

$\longleftarrow$ 143 linear d.o.f.

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Why is there "branching"?

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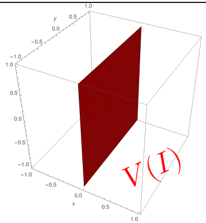
$I = \langle x \rangle = \{a\, x : a \in \mathbb{F}[x, y, z]\}$ In general: $\langle p_1, \dots, p_k \rangle_A = \left\{ \sum_{i=1}^k a_i p_i : a_i \in A \right\}$	

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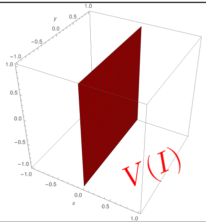
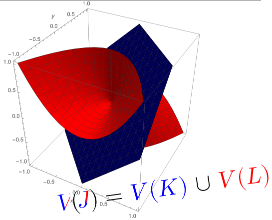
$I = \langle x \rangle = \{a\,x : a \in \mathbb{F}[x, y, z]\}$ In general: $\langle p_1, \dots, p_k \rangle_A = \left\{ \sum_{i=1}^{k} a_i p_i : a_i \in A \right\}$	
$J = \langle x^3 + x^2 y - x^2 z + xyz + y^2 z - yz^2 \rangle$ $= \langle (x + y - z)(x^2 + yz) \rangle$ $= \langle (x + y - z) \rangle \cap \langle (x^2 + yz) \rangle$ $= K \cap L$	

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- counting multiplicities w.r.t. symmetry group we get 317 varieties

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- Finite-field absolute value takes one of two values:

$$|k=0|_{\mathbb{F}_p} = 0\,, \quad |k \neq 0|_{\mathbb{F}_p} = 1$$

$\implies$ either on or away from the variety, cannot be close, since there is no concept of scale-difference in $\mathbb{F}_p$

Definition of P-Adics

- P-Adic numbers (series in p)

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- "Floating-point" representation computers have finite memory

$$x = p^{\nu_p(x)} \left(\sum_{i=0}^{k-1} a_i p^i + \mathcal{O}(p^k) \right)$$

Hilbert's Nullstellensatz

- If a polynomial vanishes everywhere on a variety, then it belongs to the (radical of the) associated ideal

$$\mathcal{N}(\lambda, \tilde{\lambda})|_{\varepsilon \text{ away from } V(I)} \sim \underbrace{\varepsilon}_{p \text{ in } \mathbb{Q}_p}{}^{k>0} \quad \Longrightarrow \quad \mathcal{N} \in \sqrt{I}$$

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Zariski-Nagata

[Zariski '49, Nagata '62, Eisenbud-Hochster '79]

- Vanishing to degree k implies membership to (k^{th} symbolic) power

$$\mathcal{N}(\lambda, \tilde{\lambda})|_{\varepsilon \text{ away from } V(I)} \sim \varepsilon^k \quad \Longrightarrow \quad \mathcal{N} \in \underbrace{\sqrt{I}^{\langle k \rangle}}_{\text{can be computed (normal power + decomposition)}}$$

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- We can then use 1. polynomial division and 2. linear algebra to build the correct vector space

Summary

Some computing references
algebraic geometry: Singular
a python interface: syngular [GDL]
p-adics: flint, sage

We talked about:

- the geometry of varieties in spinor space;
- the decomposition of their algebraic counterparts (ideals);
- p-adic numbers to rescue "closeness" with integer evaluations;
- Zariski-Nagata and symbolic powers to interpret constraints;
- and briefly how to combine the constrains.

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Backup Slides

Additional motivation

- A typical ratio panel from the LHC experiments nowadays [CMS '21]

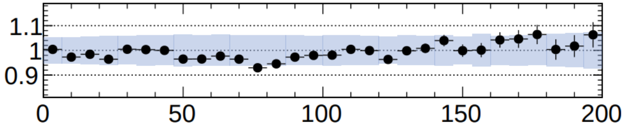

Figure: m_T in $W(\to l\nu)\gamma$ – band: theory; data points: experiment.

- Theory uncertainty larger than experimental one in most bins

Lorentz Ideals & Singular Varieties

- Suppress momentum conservation (always present!)
  (technically, work in quotient ring by mom. cons. ideal)

$$J_{\Lambda_n} = \Big\langle \textstyle\sum_{i=1}^{n} |i\rangle[i| \Big\rangle_{S_n}$$

- At 5-point, use 35 invariants: $\underbrace{\langle ij\rangle}_{10}, \underbrace{[ij]}_{10}, \underbrace{\langle i|j+k|i]}_{15}$
- 11 symmetry-inequivalent pairings (i.e. potentially reducible ideals)

E.g.: $\langle\langle 12\rangle, \langle 23\rangle\rangle_{\text{5pt.}} = \underbrace{\langle |2\rangle\rangle}_{P_1 \sim \text{soft}} \cap \underbrace{\langle\langle 12\rangle, \langle 23\rangle, \langle 13\rangle, [45]\rangle}_{P_2 \sim \text{collinear}} \cap \underbrace{\langle\langle ij\rangle\; \forall i,j\rangle}_{P_3 \sim \text{collinear}}$

- 10 symmetry-inequivalent irreducible varieties (3 shown in above)

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	$\sum$
mult.	10	20	2	30	10	60	120	15	30	20	317

Combining constraints

- Use 1. polynomial division and 2. linear algebra
- Start with the "naive" ansatz $\longleftarrow$ i.e. the unconstrained vector space
- Perform polynomial division by the Gröbner basis of each $P_i^{\langle k_i \rangle}$ (1.)
- The null-space of remainders satisfies the i^{th} constraint (2.)
- Insersect all null-spaces to satisfy all constraints (2.)

Linear Algebra with CUDA

Solving linear systems over FFs (here: $2^{32} - 1$)

Partially pivoted Gaussian elimination to row echelon form

linear size (square matrix)	approx. timings
1024	0.5s
2048	1s
4096	5s
8192	30s
16384	4m
30000	30m

- 32768 is just beyond what fits on my laptop (4gb)
- Can probably be optimized: bit-shit tricks, profiling, etc..

Something like a nvidia quadro rtx 8000 has $12\times$ the memory, $2\times$ the cores, $3\times$ FLOPS