New Method of Fast Calculating Lepton Magnetic Moments in Quantum Electrodynamics

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Motivation&history: AMM of the electron and muon

Experiment:

a_e=0.00115965218073(28) [**2011**, D. Hanneke, S. Fogwell Hoogerheide, G. Gabrielse, Phys. Rev. A 83, 052122]

a_μ=0.00116592061(41) [**2021**, Muon g-2 collaboration, Phys. Rev. Lett. 126, 141801]

Theory:

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geqslant 1} \left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

a_e=0.001159652181606(11)(12)(299)

2019, T. Aoyama, T. Kinoshita, M. Nio, Atoms, 7, 28

Uncertainties come from: $A_1^{(10)}$, hadronic+electroweak, α

α^{-1}=137.035999046(27) [**2018**, R. H. Parker et al., Science, V. 360, Is. 6385, pp. 191-195]

$A_1^{(10)}$=6.737(159) - **not double-checked** (a discrepancy **4.8σ** was revealed: **2019**, S. Volkov, Phys. Rev. D 100, 096004)

a_μ=0.00116591810953(4386)(100)(10) [4.2σ difference with the experiment]

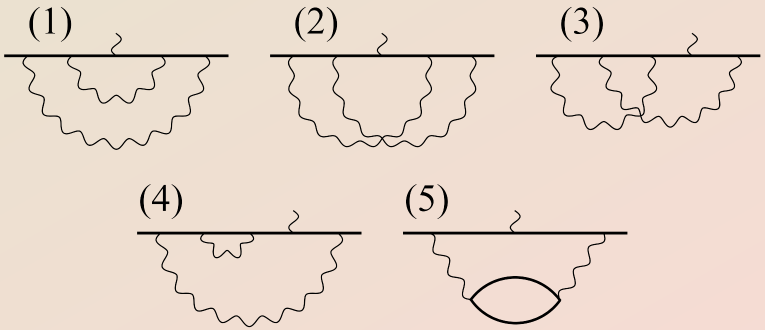
Uncertainties come from: hadronic, electroweak, QED [very optimistic]

2020, T. Aoyama et al., Physics Reports, V. 887, pp. 1-166

$A_2^{(10)}(m_\mu/m_e)$=742.18(87) - **not double-checked**, **a_μ^{12} ~ 5400** - **estimations only** (not confirmed by calculations)

Feynman diagrams, divergences, renormalization

- Lepton anomalous magnetic moment: vertex-like Feynman diagrams (2 external leptons, 1 external photon).
- Integrals in quantum field theory are divergent: UV, IR, mixed UV-IR divergences.
- UV divergences can be subtracted "in-place" using linear operators applied to Feynman amplitudes of UV-divergent subdiagrams. If the operators are defined by the on-shell conditions, it automatically expresses the value through the physical parameters.
- It also removes IR divergences. However, the **IR divergences are cancelled only after summation over all Feynman diagrams**.



Diagrams for $A_1^{(4)}$

№	Value
1	0.77747802
2	-0.46764544
3	-0.564021-(1/2)log(λ^2/m^2)
4	-0.089978+(1/2)log(λ^2/m^2)
5	0.0156874

Contributions to $A_1^{(4)}$ (Petermann, 1957)

Calculations with or without infinitesimal regularization parameters

With infinitesimal regularization parameters:

- a nonzero photon mass or dimensionality shift is OK;
- inter-diagram divergence cancellation is not a problem;
- reduction to finite integrals requires an enormous amount of symbolic manipulations at higher orders (unfeasible for 5 or more loops).

Without infinitesimal regularization parameters:

- finite integrals are required from the beginning;
- feasible at higher orders (with the help of numerical integration);
- the numerical integration can be unprecise due to a steep landscape or oscillating behavior of the integrands.

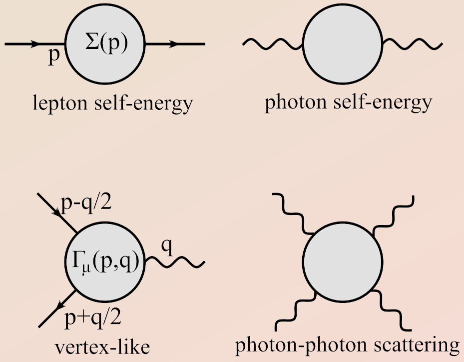
Oscillating behavior: large $\int|f(x)|dx$ with a relatively small $|\int f(x)dx|$.

In-place physical (on-shell) subtractive renormalization in QED

(it is used in different variations since 1950-x: F. Dyson, A. Salam,...)

Prescription (a modification of BPHZ):

- summation over forests F (forest = set of non-overlapping UV-divergent subdiagrams);
- coefficient = $(-1)^{|F|}$;
- Feynman amplitude of each subdiagram in F is replaced with its image under the physical renormalization operator (from smaller to larger subdiagrams); each operator returns a polynomial of degree not greater than the UV degree of divergence).



Types of UV-divergent diagrams in QED

Physical (on-shell) renormalization operators (L and B):

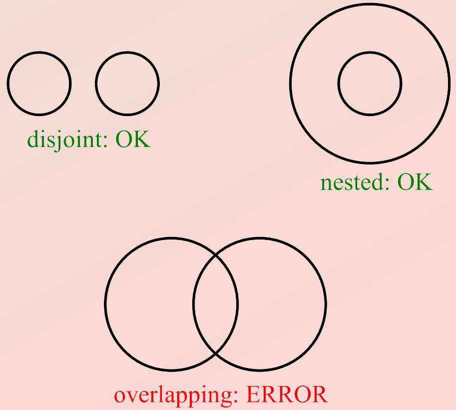
$$L\Gamma_\mu(p,q)=(a(m^2)+b(m^2)m+c(m^2)m^2)\gamma_\mu,$$

$$\Gamma_\mu(p,0)=a(p^2)\gamma_\mu+b(p^2)p_\mu+c(p^2)\hat{p}p_\mu+d(p^2)(\hat{p}\gamma_\mu-\gamma_\mu\hat{p}),$$

$$B\Sigma(p)=r(m^2)+s(m^2)\hat{p}+2m(r'(m^2)+ms'(m^2))(\hat{p}-m),$$

$$\Sigma(p)=r(p^2)+s(p^2)\hat{p},$$

$$\hat{p}=p^\mu\gamma_\mu.$$



Rules for forests

For the other types of subdiagrams the operator takes the Taylor expansion around zero momenta up to the needed order.

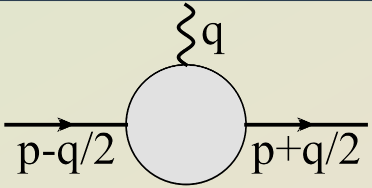
IR divergences in individual Feynman diagrams remain!!!

IR divergences cancellation and renormalization

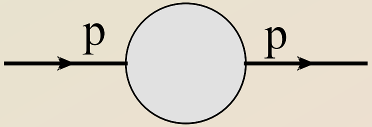
- The physical (on-shell) renormalization cancels all divergences in the QED contributions to the lepton magnetic moments.
- However, IR-divergences remain in individual diagrams after applying the in-place subtractive renormalization.
- It is possible to redistribute IR divergences between diagrams to cancel them (by additional operators).
- The combinatorics of this "redistribution" is an object of a very accurate tuning: it must remove all divergences in each individual diagram and simultaneously it must be equivalent to the on-shell renormalization.
- It is highly appreciated if it immediately leads to finite integrals in Feynman parametric space.
- There are different methods of removing UV and IR divergences simultaneously (in Feynman parametric space):
  M. J. Levine and J. Wright, Phys. Rev. D 8, 3171 (1973);
  R. Carroll and Y. Yao, Phys. Lett. 48B, 125 (1974);
  P. Cvitanović and T. Kinoshita, Phys. Rev. D 10, 3991 (1974);
  T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, Nucl. Phys. B 796, 184 (2008).

Old method (Volkov, 2015)

$\Gamma_\mu(p,q) =$



$\Sigma(p) =$



$$\Gamma_\mu(p,0) = a(p^2)\gamma_\mu + b(p^2)\,p_\mu + c(p^2)\,\hat{p}\,p_\mu + d(p^2)(\hat{p}\gamma_\mu - \gamma_\mu\hat{p})$$

$$\Sigma(p) = r(p^2) + s(p^2)\,\hat{p}$$

$$U\Gamma_\mu = \gamma_\mu a(m^2)$$

$$U\Sigma = r(m^2) + s(m^2)\,\hat{p} \qquad \hat{p} = p^\mu \gamma_\mu$$

- IR divergences are redistributed using the additional linear operator U;
- U preserves the Ward identity, all occurencies of U are cancelled in the final result.

Properties and advantages:

- finite Feynman parametric integral for each individual Feynman diagram;
- the subtraction is equivalent to the on-shell renormalization from the beginning: for obtaining the final result we should only sum up the contributions of individial Feynman diagrams;
- fully automated at any loop order;
- easy for implementation on computers (as easy as BPHZ);
- minimal gauge-invariant classes are preserved.

I developed a new method with the same advantages. The method is more suitable for calculation of mass-dependent contributions and is more flexible (it has more additional operators with some freedom of choice). **The possibility of these degrees of freedom is not trivial!**

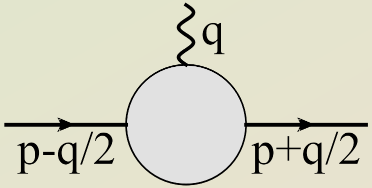
Old method: 5-loop results (Volkov, 2019)

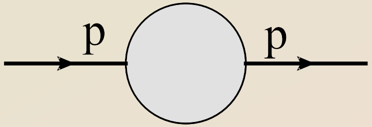
Sergey Volkov, Phys. Rev. D 100, 096004 (2019)

- Volkov, 2019:
  $A_1^{(10)}$[no lepton loops]=6.793(90),
  Aoyama, Hayakawa, Kinoshita, Nio, 2019:
  $A_1^{(10)}$[no lepton loops]=7.668(159)
  (4.8σ DISCREPANCY).
  Combining with different measurements of α: arXiv:2111.00291
- 3213 undirected Feynman diagrams.
- 40000 GPU-hours, NVidia Tesla V100, supercomputer «Govorun» (JINR, Dubna).
- 9 minimal gauge-invariant classes (k,m,n):
  m and n internal photons to the right and to the left from the external photon (or vice versa), k photons with ends on different sides.
  More about gauge-invariant classes: P. Cvitanović, Nucl. Phys. B 127, pp. 176-188 (1977).

Class	Value
(1,4,0)	6.184(65)
(2,3,0)	-0.746(63)
(1,3,1)	0.854(50)
(3,2,0)	-0.399(51)
(2,2,1)	-2.133(53)
(4,1,0)	-1.028(31)
(1,2,2)	0.312(30)
(3,1,1)	2.628(35)
(5,0,0)	1.0929(94)

My new method

$\Gamma_\mu(p,q)=$ 

$\Sigma(p)=$ 

G = whole diagram
I'[G] = the set of all vertex-like UV-divergent subdiagrams (including G) adjoining the external photon of G and lying on the main lepton path of G

Linear operators.

- L is the same as in the standard renormalization.
- A is the anomalous magnetic moment projector.
- U_1, U_2, U_3 instead of U.
- U_0 works as in the usual renormalization for photon self-energy or photon-photon scattering subdiagrams.

Requirements for U_j (j=1,2,3) (tentative):

- U_j extracts the overall UV-divergent part competely;
- U_j preserves the Ward identity (for Γ_μ and Σ);
- U_j cancels IR divergences of the subdiagram;
- U_j extracts the mass part completely:

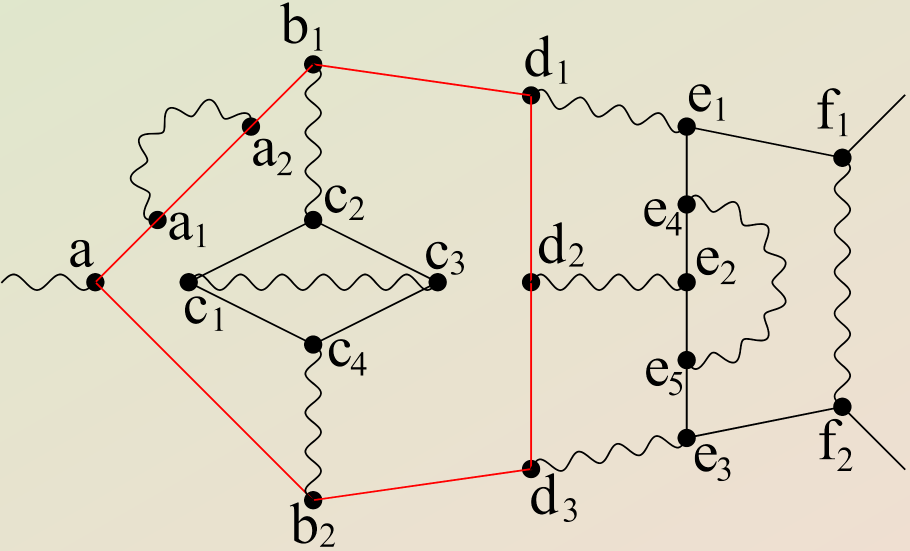
$$U_j(a(p^2)+b(p^2)\hat{p})=a(m^2)+b(m^2)m+(\hat{p}-m)\cdot(\ldots).$$

There is also a possibility to take U_3=L.
U_j can be defined differently for different particles.

Subtraction prescription for G:

- summation over forests F containing G and over M in $F \cap I'[G]$
- coefficient = $(-1)^{|F|-1}$
- the Feynman amplitude of each subdiagram D in F is replaced with its image under the operator:
  - A, if D=M,
  - L-U_1, if D=G, D≠M,
  - L, if D≠G, M is a proper subdiagram of D,
  - U_w, if D in I'[G], D is a proper subdiagram of M, where w=3 if G has its external photon on a lepton loop, w=1 otherwise,
  - U_0, if D is a photon diagram,
  - U_2, if D lies on a lepton loop,
  - U_1 in the other cases.
- take the coefficient before γ_μ

New method: example



$I'[G]=\{G_e, G\}$

$G_e = aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3e_1e_2e_3e_4e_5$

Other UV-divergent subdiagrams:

lepton self-energy – a_1a_2,

vertex-like – $c_1c_2c_3$, $c_1c_3c_4$, $e_2e_4e_5$, $G_c=aa_1a_2b_1b_2c_1c_2c_3c_4$,

photon self-energy – $c_1c_2c_3c_4$,

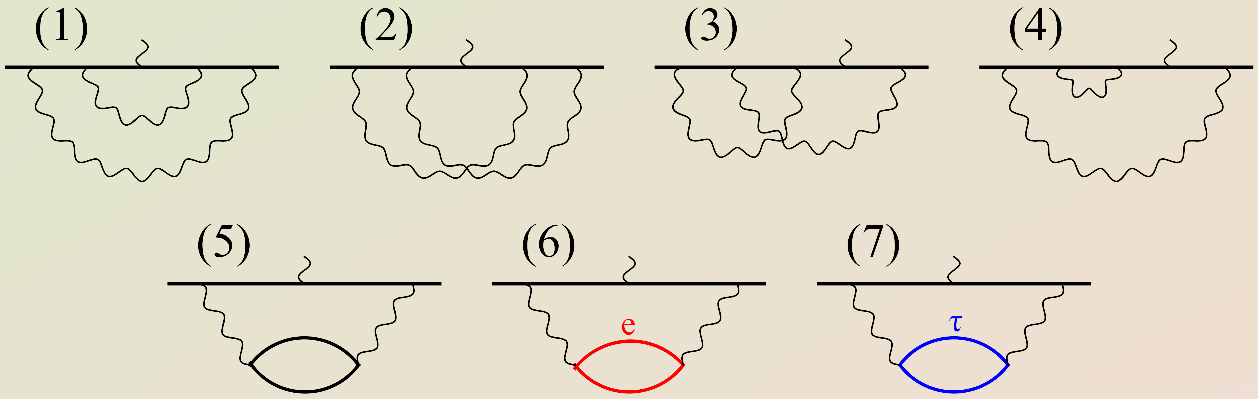
photon-photon scattering – $G_d=aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3$.

Expression:

$$\left[A_G(1-(U_3)_{G_e})-(L_G-(U_1)_G)A_{G_e}\right]\times(1-(U_2)_{G_c})\times(1-(U_1)_{e_2e_4e_5})$$
$$\times(1-(U_0)_{G_d})(1-(U_0)_{c_1c_2c_3c_4})(1-(U_2)_{c_1c_2c_3}-(U_2)_{c_1c_3c_4})(1-(U_2)_{a_1a_2})$$

Comparison of the old and new methods: arXiv:2111.00291

Tests: muon, contributions to a_μ^4, U is used

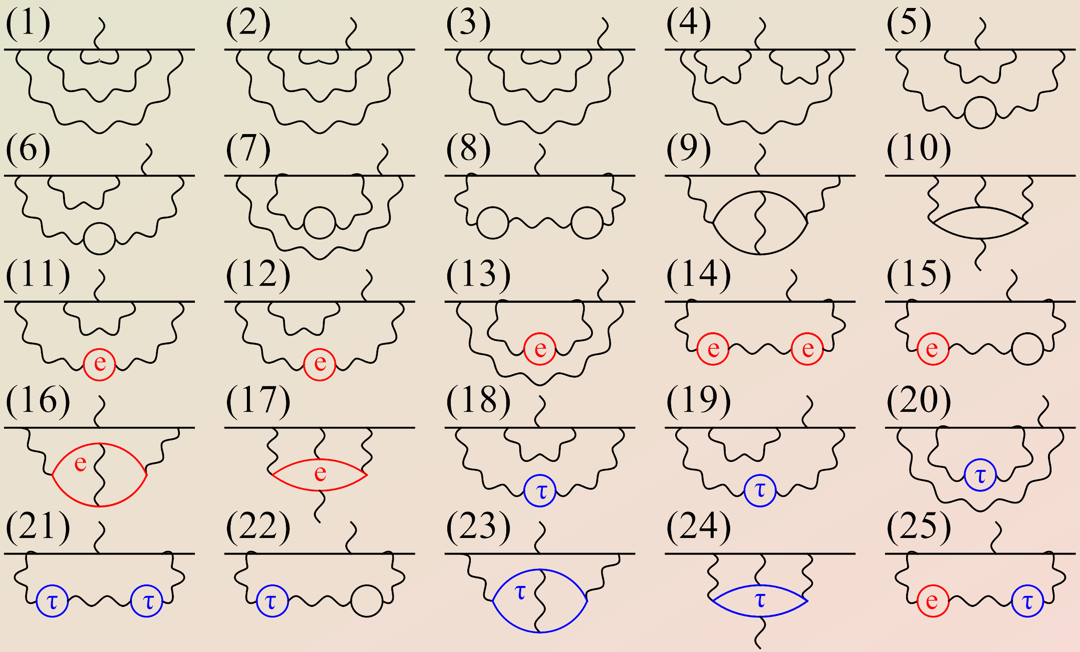


№	My value	Analyt
1	0.77747774(18)	0.77747802
2	-0.4676475(17)	-0.46764544
3	-0.0640193(19)	-0.564021 $-(1/2)\log(\lambda^2/m^2)$
4	-0.5899758(14)	-0.089978 $+(1/2)\log(\lambda^2/m^2)$
5	0.01568736(15)	0.0156874
1-5	-0.3284777(34)	-0.328478966
6	1.0942613(60)	1.0942583
7	0.0000780864(51)	0.000078076(11)

- The masses m_μ=105.6583745 MeV, m_e=0.51099895 MeV, m_τ=1776.82 MeV are FIXED (the uncertainty is not taken into account; test only).
- Self-made Monte Carlo integration.
- Analytical results:
  1-5: Petermann, 1957;
  6-7: Elend, 1966 (with the recent masses).

Tests: muon, contributions to a_μ^6, 25 gauge-invariant classes

- Each class is obtained by moving internal photons along lepton paths and loops without jumping over the external photon.
- The masses m_μ=105.6583745 MeV, m_e=0.51099895 MeV, m_τ=1776.82 MeV are FIXED (the uncertainty is not taken into account; test only).
- Self-made Monte Carlo integration, ~2 days on GPU NVidia Tesla V100.



Analytical results:
1969-1999
R. Barbieri
D. Billi
M. Caffo
A. Czarnecki
L. L. DeRaad
B. Krause
S. Laporta
M. Levine
G. Li
J. Mignaco
K. A. Milton
R. Perisho
E. Remiddi
R. Roskies
M. A. Samuel
M. Skrzypek
W. Tsai
......

Class	My result	Analytical
1	0.448703(35)	0.44870
2	-0.498224(67)	-0.49825
3	0.533289(54)	0.53336
4	0.421080(43)	0.42117
5	0.050178(16)	0.0501487
6	-0.112324(21)	-0.112336
7	-0.087987(12)	-0.0879847
8	0.0025598(15)	0.0025585
9	0.052865(11)	0.05287
10	0.37094(15)	0.371005
11	1.61752(28)	?
12	-2.06183(39)	?
13	-1.94880(28)	?
11-13	-2.39311(56)	-2.39239181(7)
14	2.71885(62)	2.7186557(2)
15	0.10038(12)	0.100519296(3)
16	1.49545(92)	1.49367180(4)
17	20.9475(13)	20.9471(29)
18	0.00054410(20)	?
19	-0.00160008(30)	?
20	-0.00106137(17)	?
18-20	-0.00211735(40)	-0.00211713
21	0.0000002766(30)	0.000000277833
22	0.000038704(63)	0.0000386875
23	0.000295496(73)	0.000295557
24	0.0021443(12)	0.00214331
25	0.0005245(70)	0.000527761

$A_1^{(6)}$, Set 3, different subtraction points

New 1:

$$U_1[r(p^2)+s(p^2)\hat{p}]=r(m^2)+s(m^2)\hat{p},$$

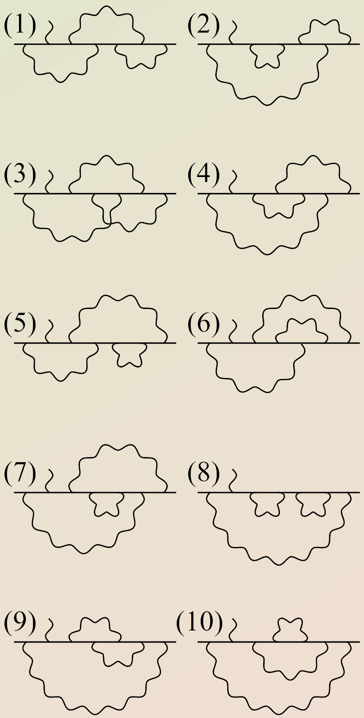
$$U_1\Gamma_\mu(p,q)=a(m^2)\gamma_\mu$$

New 2:

$$U_1[r(p^2)+s(p^2)\hat{p}]=r(m^2)+s(m^2)m+s(-m^2)(\hat{p}-m),$$

$$U_1\Gamma_\mu(p,q)=a(-m^2)\gamma_\mu$$

$$\Gamma_\mu(p,0)=a(p^2)\gamma_\mu+b(p^2)p_\mu+c(p^2)\hat{p}\,p_\mu+d(p^2)(\hat{p}\gamma_\mu-\gamma_\mu\hat{p}),\qquad \hat{p}=p^\mu\gamma_\mu$$



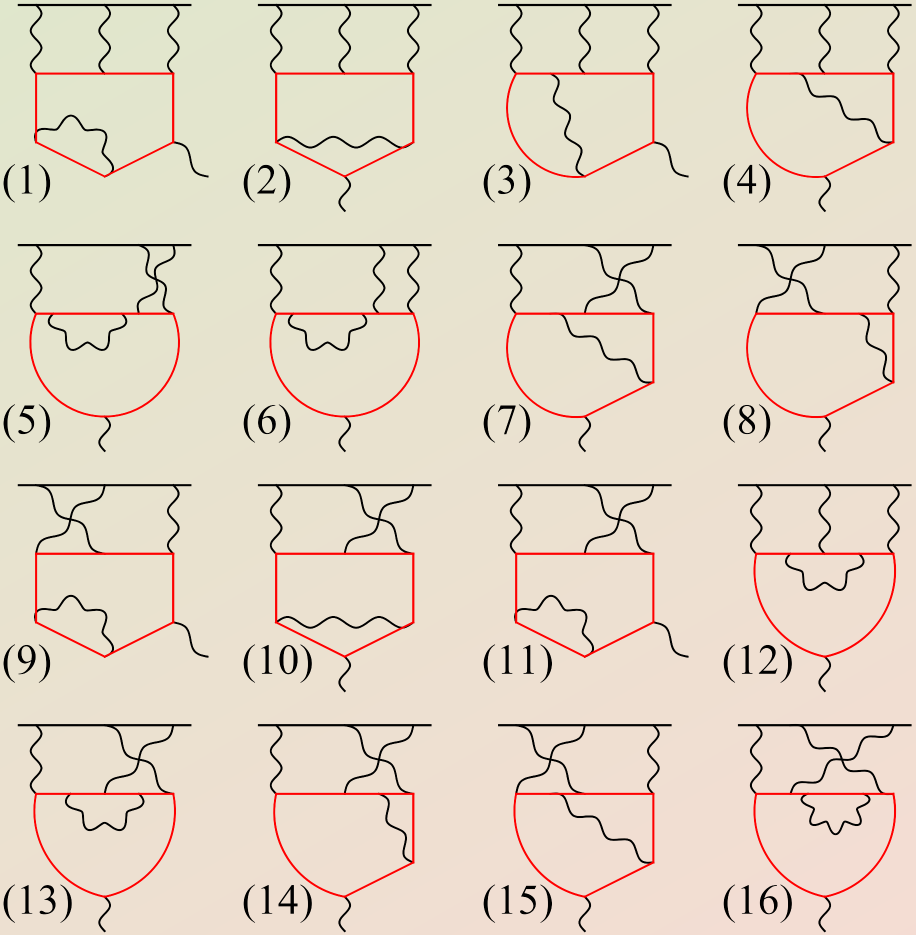
Values:

№	New 1	New 2
1	-1.68013(35)	-0.97296(81)
2	-0.09677(49)	-0.6103(18)
3	0.21489(27)	-0.48545(34)
4	0.14480(27)	0.8527(27)
5	0.83283(40)	1.1325(19)
6	-0.02875(23)	0.2158(25)
7	0.80395(54)	1.10398(60)
8	-2.12293(27)	-1.2838(17)
9	2.52480(29)	1.1537(41)
10	-0.05880(20)	-0.5792(27)
Σ	0.5339(11)	0.5272(70)

Integrals of the abs. values:

№	New 1	New 2
1	2.105	1.394
2	1.667	1.311
3	0.819	0.86
4	1.111	1.01
5	1.33	1.251
6	0.666	0.496
7	1.31	1.173
8	2.123	1.318
9	2.587	1.413
10	1.084	1.269
Σ	14.8	11.49

Tests: a_μ^8, Kinoshita's IV(b), muon with e-loop, different subtraction points



№	$M^2=m^2=(m_e)^2$	$M^2=-m_e m_\mu$
1	32.712(73)	-2.753(13)
2	-7.118(63)	10.580(11)
3	-27.870(89)	7.688(17)
4	-23.811(15)	-23.791(11)
5	26.517(93)	11.071(18)
6	39.12(10)	3.642(19)
7	-23.529(21)	-23.522(13)
8	-20.30(13)	-4.753(22)
9	16.70(11)	1.221(16)
10	4.92(12)	20.465(20)
11	20.817(95)	5.318(18)
12	-7.745(98)	9.902(17)
13	-11.21(13)	4.371(27)
14	-19.36(11)	-3.791(22)
15	-26.433(25)	-26.417(11)
16	25.86(10)	10.431(22)
Σ	**-0.73(37)**	**-0.340(72)**

$$U_2[a(p^2)+b(p^2)\hat{p}] = a(m^2)+b(m^2)m+b(M^2)(\hat{p}-m),$$

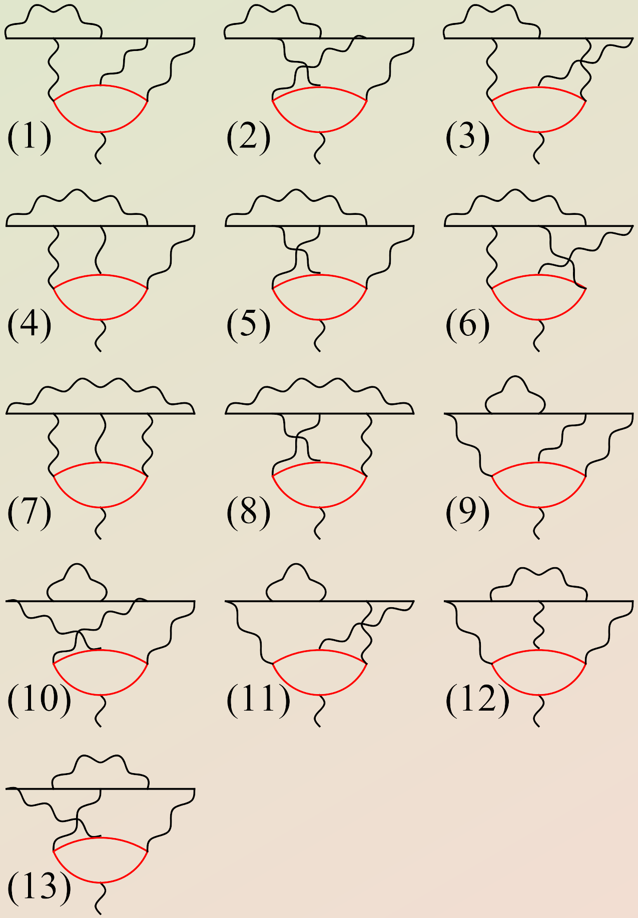
$$U_2[a(p^2)\gamma_\mu+\ldots] = a(M^2)\gamma_\mu,$$

$$\hat{p}=p^\mu\gamma_\mu$$

T. Aoyama,
M.Hayakawa,
T. Kinoshita,
M. Nio [2012]:
-0.4170(37)

A. Kurz,
T. Liu,
P. Marquard,
A. V. Smirnov,
V. A. Smirnov,
M. Steinhauser [2016]:
-0.38(8)

Tests: a_μ^8, Kinoshita's IV(c), muon with electron loop



№	My new 1	My new 2
1	-118.179(22)	-118.167(22)
2	91.841(21)	91.778(21)
3	-80.820(20)	-80.839(21)
4	2.620(22)	2.612(25)
5	-74.730(26)	-74.694(29)
6	37.016(22)	37.015(24)
7	48.934(19)	64.810(19)
8	20.775(32)	4.920(30)
9	100.475(24)	100.514(25)
10	-76.924(32)	-76.923(33)
11	89.140(31)	89.186(35)
12	-26.766(12)	-26.763(12)
13	-10.543(21)	-10.550(20)
Σ	**2.840(86)**	**2.899(90)**

New 1: $U_1=U_2=U$, $U_3=L$
New 2: $U_1=U_2=U_3=U$

T. Aoyama,
M.Hayakawa,
T. Kinoshita,
M. Nio [2012]:
2.9072(44)

A. Kurz,
T. Liu,
P. Marquard,
A. V. Smirnov,
V. A. Smirnov,
M. Steinhauser [2016]:
2.94(30)

Thank you for your attention!

Summary:

- High-order QED calculations are still important.
- Point-by-point subtraction of divergences in Feynman parametric space is a very effective approach.
- Infrared divergences are very insidious; there is no universal method of handling them so far.
- I developed a family of elegant methods of removing all divergences in Feynman parametric integrals; the methods allow us to calculate the QED contributions to the lepton magnetic moments at any loop order.
- Both old and new methods are able to work with mass-dependent contributions and to preserve minimal gauge-invariant classes.
- The old method was applied to the calculation of the total 5-loop QED contribution without lepton loops (a discrepancy was revealed; the reason is unknown, but the method is not guilty).
- The new method is flexible and nonredundant; the flexibility can be used for improving the precision of calculations (affected by large logarithms, for example).