Modern multiloop calculations.

Search for new algorithms and fast computer algebra systems

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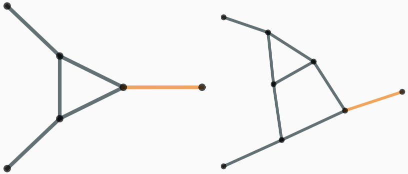
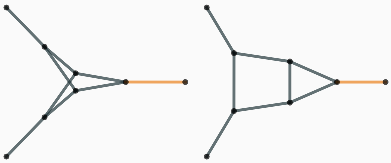
Motivation

- High-precision theoretical description of Standard Model processes is of crucial importance. In particular, the New Physics — new particles and interactions — is likely to appear as small deviations from SM and therefore can be detected only with high precision of theoretical predictions at hand.
- From the computational point of view, our ability to obtain high-precision results depends crucially on multiloop calculation techniques. Complexity grows both qualitatively and quantitatively in an explosive way with the number of loops and/or scales.
- New methods and approaches are always required. Using computer power is a must for at least two last decades. Insights from various fields of mathematics help a lot.

2 loops:





- Dispersion relation
- Feynman parametrization

} [Matsuura, van der Marck, and van Neerven, 1989; Harlander, 2000]

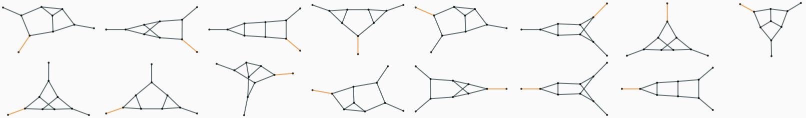
- Mellin-Barnes parametrization
- ${}_pF_q$ expansion in indices, `HypExp`

} [Gehrmann, Huber, and Maitre, 2005]

Example: form factors

3 loops:

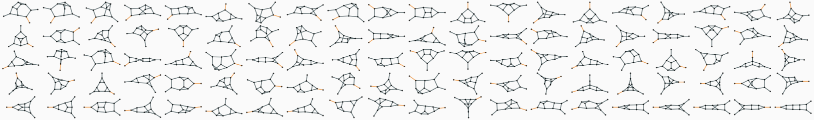
[Gehrmann, Heinrich, Huber, and Studerus, 2006; Heinrich, Huber, and Maître, 2008; RL, Smirnov, and Smirnov, 2010]



- Feynman parametrization
- Mellin-Barnes parametrization, `MB`, `AMBRE` [Czakon, 2006; Gluza et al., 2007]
- Recurrence+analyticity in d, [Tarasov, 1996; RL, 2010]
- PSLQ recognition [Ferguson et al., 1998]

4 loops:

[Henn, Smirnov, Smirnov, and Steinhauser, 2016; RL, Smirnov, Smirnov, and Steinhauser, 2019; RL, von Manteuffel, Schabinger, Smirnov, Smirnov, and Steinhauser, 2021]



- $\sim$ 100 big topologies.
- Linear reducibility, `HyperInt` [Panzer, 2013]
- Parallelization for IBP reduction,
  finite fields reconstruction [von Manteuffel and Schabinger, 2015; Smirnov and Chuharev, 2020]
- Differential equations, reduction to ϵ-form [Henn, 2013; RL, 2015], `Libra` [RL, 2021]
- PSLQ recognition

5 loops:

- $\sim$ 1000 big topologies.
- It looks like no available techniques can help.

NNLO cross sections

- But from the experimental point of view less loops and more scales are even more important. In particular NNLO (two-loop) corrections to the cross sections processes are of a great interest.
- Only very recently multiloop methods have grown to NNLO calculations for more than 2 scales: $2 \to 2$ processes with massive particles, $2 \to 3$ processes with massless particles.
- Partial results start to appear. One example: $e - \mu$ scattering at NNLO [Banerjee et al., 2020].

State of the art

Calculational complexity crucially depends on the number of loops and on the number of scales.

loops / scales	1 loop	2 loops	3 loops	4 loops	5 loops	> 6
1	✓	✓	✓	many	a few	
2	✓	✓	some	a few		
3	✓	some	a few			
> 3	✓	a few				

- Massive internal lines add extra complexity.
- State-of-the-art examples:
  - 5-loop massless propagators [Georgoudis, Gonçalves, Panzer, Pereira, Smirnov, and Smirnov, 2021].
  - 4-loop $g-2$ integrals (onshell massive propagators) [Laporta, 2017]
  - 4-loop $\mathcal{N}=4$ SYM form factors [RL, von Manteuffel, Schabinger, Smirnov, Smirnov, and Steinhauser, 2021]
  - 3-loop massless boxes [Henn, Mistlberger, Smirnov, and Wasser, 2020]
  - 2-loop 5 legs [Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, and Zoia, 2019]
- Massive internal lines add more complexity than just an extra scale:
  - 3-loop massive form factors not yet calculated.
  - results for two-loop boxes with inner massive lines are mostly not available.
    This is basically the minimal complexity of the diagrams required for NNLO precision of differential cross section for $2 \to 2$ processes with massive particles.

Calculation path

1. Diagram generation

Generate diagrams contributing to the chosen order of perturbation theory.

Tools: `qgraf` [Nogueira, 1993], `FeynArts` [Hahn, 2001],...

2. IBP reduction

Setup IBP reduction, derive differential system for master integrals.

Tools: `FIRE6` [Smirnov and Chuharev, 2020], `Kira2` [Klappert et al., 2021], `LiteRed` [RL, 2012], `Reduze2` [von Manteuffel and Studerus, 2012],...

3. DE Solution

Reduce the system to ϵ-form, write down solution in terms of polylogarithms. Fix boundary conditions by auxiliary methods.

Tools: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], `Libra` [RL, 2021]

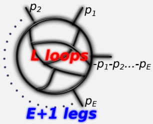
IBP reduction: new ideas

IBP identities

Given a Feynman diagram, consider a family

$$j(\boldsymbol{n}) = \int d\mu_L \prod_{k=1}^{N} D_k^{-n_k}, \quad d\mu_L = \prod_{k=i}^{L} d^d l_i$$

$D_1, \ldots, D_M$ — denominators of the diagram,
$D_{M+1}, \ldots, D_N$ — irreducible numerators, such that
$N = L(L+1)/2 + L \cdot E$.



From $0 = \int d\mu_L \frac{\partial}{\partial l_i} \cdot q_m \prod_{k=1}^{N} D_k^{-n_k}$ one obtains

IBP identities

$$[c_{kl} B_k A_l + c_l A_l]\, j(\boldsymbol{n}) = 0.$$

Here c_{kl}, c_l are some coefficients.

$$A_l j(n_l) = n_l j(n_l + 1),$$

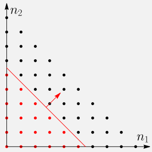
$$B_l j(n_l) = j(n_l - 1)$$

IBP identities allow one to express any integral in the family via a finite # of master integrals. They also allow to construct differential and difference equations for the latter.

IBP reduction

Laporta algorithm (FIRE, Kira, Reduze, ...)

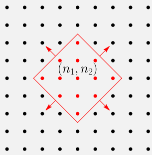
- generate identities for many numeric $\boldsymbol{n} \in \mathbb{Z}^N$.
- use Gauss elimination and collect reduction rules to database.
- twist: mapping to finite fields $\mathbb{F}_p$ + reconstruction. $\Longleftarrow$ naturally parallelizable



Heuristic search (LiteRed)

1. Generate identities for shifts around $\boldsymbol{n}$ with *symbolic* entries.
2. Use Gauss elimination until acceptable rule is found.
3. Solve Diophantine equations to derive applicability condition.

Observation: only a small fraction of identities finally contribute to the reduction rule.



IBP reduction in parametric representation

Note that $N = L(L+1)/2 + L \cdot E$ grows quadratically with L, while M, the # of lines in the diagram, grows only linearly. Parametric representation: only M indices.

Parametric representation

$$\tilde{j}^{(d)}(n_1, \ldots n_M) = \int \frac{\prod_{k=1}^{M} dx_k x_k^{n_k - 1}}{G(\boldsymbol{x})^{d/2}}$$

$G = U + F$, where U and F are Feynman graph polynomials.

IBP identities relating integrals with the same d require constructing *syzygy module* for ideal generated by $\langle G, \partial_1 G, \partial_M G \rangle$.

IBP identities from syzygies [RL, 2014]. Baikov rep.: [Zhang, 2014]

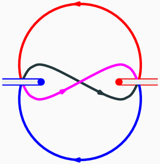
Syzygy $QG + Q_1 \partial_1 G + \ldots + Q_M \partial_M G = 0$ leads to IBP identity

$$[\tfrac{d}{2} Q(\boldsymbol{A}) + Q_k(\boldsymbol{A}) B_k] \tilde{j}(\boldsymbol{n}) = 0$$

Quite promising, but a fast algorithm for constructing a *minimal* (rather than Groebner) basis of syzygy module is very desirable.

IBP reduction with intersection theory?

[Mastrolia and Mizera, 2019]: use intersection theory for IBP reduction.



- Integral in parametric representation is understood as bilinear pairing between integration cycle C and differential form ϕ.

$$\int_C G^{-\nu}\phi = \langle\phi|C] \ ,$$

- $\langle\phi|C]$ is invariant under $\phi \to \phi + \nabla_\nu\tilde{\phi}$ and/or $C \to C + \partial\tilde{C}$, where $\nabla_\nu = d - \nu G^{-1}dG$ is twisted differential and $\partial\tilde{C}$ is a boundary (contractable) cycle.
- Therefore, $\langle\cdot|\cdot]$ is defined on the elements of twisted de Rham cohomology and twisted homology. Those are finite-dimensional spaces, therefore we can use basis expansion as IBP.
- Ref. [Cho and Matsumoto, 1995] introduced pairing $\langle\phi_1|\phi_2\rangle$, correctly defined for ∇_ν- and $\nabla_{-\nu}$- de Rham cohomologies.
- Unfortunately, $\langle\phi_1|\phi_2\rangle$ is still very difficult to calculate in general. Perspectives of this approach are quite unclear to me.

Differential equations

Differential equations for master integrals

- Differential equations for master integrals have the form

$$\partial_x \boldsymbol{j} = M(x,\epsilon)\boldsymbol{j}$$

- One can try to simplify the equation by transformation $\boldsymbol{j} = T\tilde{\boldsymbol{j}}$, so that

$$\partial_x \tilde{\boldsymbol{j}} = \widetilde{M}\tilde{\boldsymbol{j}}, \qquad \widetilde{M} = T^{-1}\left[MT - \partial_x T\right]$$

- [Henn, 2013]: there is often a "canonical" basis $\boldsymbol{J} = T^{-1}\boldsymbol{j}$ such that

$$\partial_x \boldsymbol{J} = \epsilon S(x)\boldsymbol{J} \qquad (\epsilon\text{-form})$$

- General solution is easily expanded in ϵ:

$$U = \mathrm{Pexp}\left[\epsilon \int dx S(x)\right] = \sum_n \epsilon^n \iiint\limits_{x>x_n>\ldots>x_0} dx_n \ldots dx_1 S(x_n)\ldots S(x_1)$$

- Algorithm of finding transformation to ϵ-form: [RL, 2015]. Implemented in 3 publicly available codes: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], and recently in `Libra` [RL, 2021].

General structure of reduction algorithm

Algorithm proceeds in three major stages, each involving a sequence of "elementary" transformations.

1. *Fuchsification*: Eliminating higher-order poles

Input: Rational matrix $M(x, \epsilon)$
Output: Rational matrix with only simple poles on the extended complex plane, $M(x, \epsilon) = \sum_k \frac{M_k(\epsilon)}{x - a_k}$.

2. *Normalization*: Normalizing eigenvalues

Input: Matrix from the previous step, $M(x, \epsilon) = \sum_k \frac{M_k(\epsilon)}{x - a_k}$.
Output: Matrix of the same form, but with the eigenvalues of all $M_k(\epsilon)$ being proportional to ϵ.

3. *Factorization*: Factoring out ϵ

Input: Matrix from the previous step.
Output: Matrix in ϵ-form, $M(x, \epsilon) = \epsilon S(x) = \epsilon \sum_k \frac{S_k}{x - a_k}$.

`Libra` program

- `Libra` is a *Mathematica* package useful for treatment of differential systems which appear in multiloop calculations.
- Tools for reduction to ϵ-form
  - Visual interface
  - Algebraic extensions
  - Birkhoff-Grothendieck factorization
- Tools for constructing solution
  - Determining boundary constants.
  - Constructing ϵ-expansion of Pexp.
  - Constructing Frobenius expansion of Pexp.

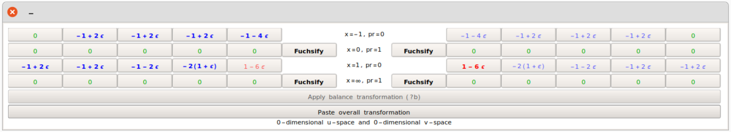
Libra tools for reduction to ϵ-form

- Fuchsification and normalization.
  - Automatic tool (useful for simple cases)

```
In[1]:  t=Rookie[M,x,ϵ];
```

  - Interactive tool (useful for most cases)

```
In[1]:  t=VisTransformation[M,x,ϵ];
```



- Factorization.

```
In[2]:  t=FactorOut[M,x,ϵ,μ];
```

- General solution

```
In[3]:  U=PexpExpansion[{M,6},x];
```

Boundary conditions

Suppose we have found a transformation $T(x) = T(x, \epsilon)$ to ϵ-form, $\mathrm{j} = T\mathrm{J}$. Then we can write

$$\mathrm{J}(x) = U(x, x_0)\mathrm{J}(x_0),$$
$$\mathrm{j}(x) = T(x)U(x, x_0)[T(x_0)]^{-1}\mathrm{j}(x_0)$$

But the point x_0 should be somewhat special to simplify the evaluation of $\mathrm{j}(x_0)$ as compared to $\mathrm{j}(x)$. As a rule, "special" boils down to "singular", i.e., we can expect simplifications for x_0 being a singular point of the differential system. Let it be $x_0 = 0$ for simplicity.

Problem

$U(x, x_0)$ diverges when x_0 tends to zero. Therefore, we have to consider not the values, but the asymptotics of $\mathrm{j}(x_0)$ at $x = 0$.

`Libra` can determine which asymptotic coefficients, c, are sufficient to calculate and find the "adapter" matrix L relating those with the column of boundary constants, $\mathrm{C} = L\mathrm{c}$.

```
In[4]:  {L,cs}=GetLcs[M,T,{x,0}];
```

Algebraic extensions and non-polylogarithmic integrals

- Sometimes, in order to find the transformation to ϵ-form, one has to extend the class of transformations by passing from x to y, such that $x = x(y)$ is some rational function. `Libra` has tool for it:

```
In[1]:  ChangeVar[ds,x→(4 y*y)/(1 - y*y),y];
```

- Moreover, in many cases there is no common rationalizing variable. Thus, `Libra` implements a more powerful way to treat such algebraic extensions, with

```
In[1]:  AddNotation[ds,y → x(1-y*y) - 4 y*y];
```

  One may add as many notations as needed, and `Libra` will take care of them (minimizing their appearance, correctly treating their differentiation).

- Unfortunately, there are cases when the system can not be reduced to ϵ-form even with algebraic extensions. Libra implements Birkhoff-Grothendieck factorization to help to detect such cases (see [RL and Pomeransky, 2017]):

```
In[1]:  {L,T,R}=BirkhoffGrothendieck[t,x];
```

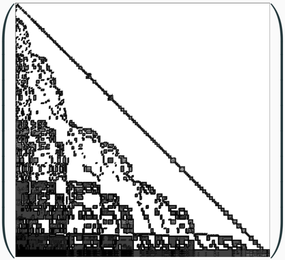
  There is no general approach for such cases in this case. Proper treatment of transcendental extensions is needed?

Example of using `Libra`

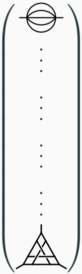
One of many 4-loop massless vertex topologies with two off-shell legs.

- Differential system

$\partial_x j =$



$j, \qquad$ where $j =$



374×374 matrix

- Maximum size of the diagonal blocks is "only" 11×11.
- No global rationalizing variable. Three algebraic extensions are needed for the reduction to ϵ-form:

$$x_1 = \sqrt{x}, \qquad x_2 = \sqrt{x - 1/4}, \qquad x_3 = \sqrt{1/x - 1/4}$$

Summary

- Each step towards increasing the # of loops and/or # of scales requires new methods. Those involve both technological advances (e.g. massive parallelization) and new algorithms coming various fields of mathematics.
- IBP reduction still remains a bottleneck for many calculations. New ideas of IBP reduction appear, whether they will be successful is yet to find out.
- Differential equations method is already in very good shape. Exception: the systems irreducible to ϵ-form.

Summary

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Thank you!

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