Theory of Special Relativity

Romanian High-School Students Internship Program

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Some numbers

Elementary particles are entities so small that even when they have a relatively small energy can reach high and even very high velocities:

- an electron of H atom has velocity of about $2.19 \times 10^6 m/s$ but it has a small energy (about $13.6eV \approx 21.8 \times 10^{-19} J$.

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- a beta decay electron, can reach even $5.515 MeV \approx .8.8 \times 10^{-13} J$, corresponding to a velocity of about $2.987 \times 10^8 m/s \approx 0.99c$, $c = 2.99792458 \times 10^8 m/s$ is the speed of light in vacuum.

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- in 1897 Joseph John Thomson (Nobel Prize 1906) discovered the electron;
- in 1900 Max Planck (Nobel Prize 1918) postulated the idea that the energy emitted by a black body could only take on discrete values (the idea of quanta).

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- is the æther dragged by bodies in motion, or is it not? Is it total or a partial dragg?
- For example, does the Earth in its orbital motion drag with it the æther? In other words, is there a “wind” of æther?

Some answers (theory perspective)

Here are the answers to this question, given by three prominent physicists:

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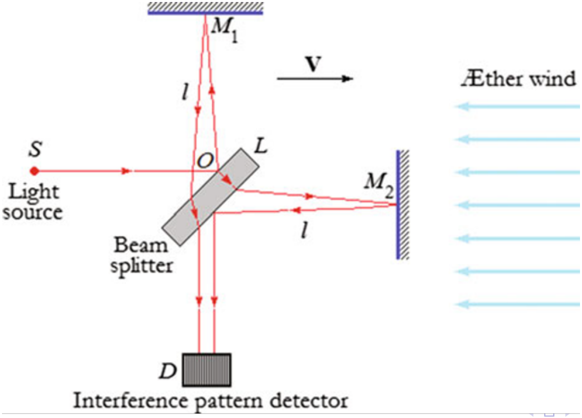
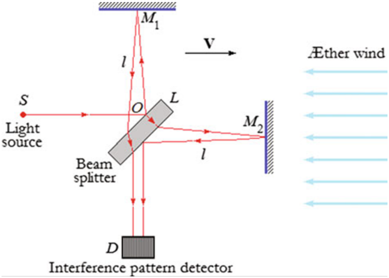
- Hendrik Antoon Lorentz (1853–1928) considered the æther as being immobile;
- George Gabriel Stokes (1819–1903), on the contrary, conceived the æther as completely dragged by media in motion.

Experiment perspective

- 1887 Albert Abraham Michelson (1852–1931) and Edward Williams Morley (1838–1923).
- The device they used in the experiment was an interferometer, which Michelson had invented earlier and which is schematically represented below:



Experimental results



- Even if the precision of the measuring system was one-hundredth of an interfringe, Michelson was surprised to detect no displacement of fringes.
- The experiment was resumed, with higher precision and various improvements, by Morley and Miller in 1902–1904, Dayton Clarence Miller (1866–1941) in 1921–1926, and again Michelson et al. in 1929, etc., but the result was always the same.

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- Michelson was very disappointed about the "negative result" of his experiment, that became what might be called the "most famous failed experiment to date".
- But, as the development of science showed, the experiment "offered the most important negative result in the history of science" (John Desmond Bernal).

Galileo-Newton transforms

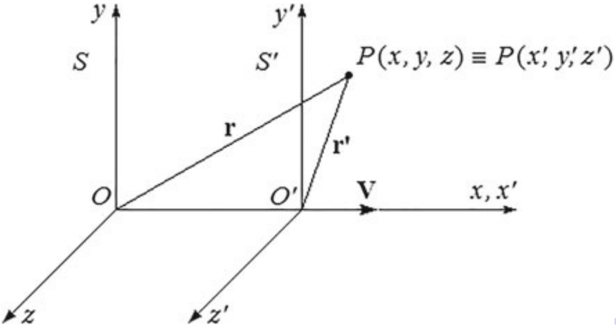
Until Einstein developed TRS (in 1905), space and time had different statuses:

- space was "relative", while
- time was "absolute".

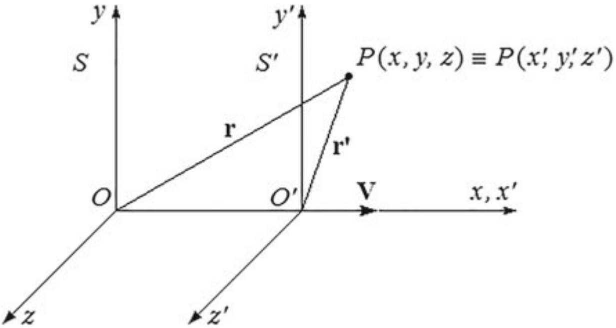
These two properties are expressed by the two relations:

$$\vec{r}' = \vec{r} - \vec{V} \cdot t \tag{1}$$

$$t' = t \tag{2}$$



Lorentz-Einstein Transformations (I)



$$t' = \gamma \left(t - \frac{Vx}{c^2} \right)$$
$$x' = \gamma \left(x - V \cdot t \right)$$
$$y' = y$$
$$z' = z$$

Lorentz-Einstein Transformations (II)

$$t' = \gamma \left(t - \frac{Vx}{c^2} \right) \tag{3}$$

$$x' = \gamma \left(x - V \cdot t \right) \tag{4}$$

$$y' = y \tag{5}$$

$$z' = z \tag{6}$$

where (t, x, y, z) and (t', x', y', z') are the coordinates of an event, determined by the two observers (from S and S' respectively), c is the speed of light in vacuum, and

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}} \tag{7}$$

$$\text{and } \beta = \frac{V}{c} \tag{8}$$

is so-called Lorentz factor.

Lorentz-Einstein Transformation (III)

In the general case, when the relative motion of the two referentials is made in a certain direction, which is not parallel to any of the three coordinate axes, the Lorentz-Einstein relations are written as follows:

$$t' = \gamma \left[t - \frac{V}{c^2} (\vec{r} \cdot \vec{v_0}) \right] \tag{9}$$

$$\vec{r}' = (\gamma - 1) (\vec{r} \times \vec{v_0}) \times \vec{v_0} + \gamma (\vec{r} - \vec{V} t) \tag{10}$$

where $\vec{v_0} = \frac{\vec{V}}{V}$ is the unit vector for $\vec{V}$. The above relations are called Lorentz-Herglotz transformations (some authors called them "generalized Lorentz transforms").

Principles of Special Relativity

In 1905, Albert Einstein published what is now called Special Relativity, by deriving the Lorentz transformations under the assumptions of two postulates:

(i) The laws of physics are the same in all inertial reference frames. This postulate is an extension from the Newtonian principle of relativity, which states that the laws of mechanics are the same for all observers in uniform motion.

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(ii) The speed of light in empty space is the same in all inertial frames. This means that the velocity of light in free space appears the same to all observers, regardless of the motion of the source of light and of the observer.

- In classical mechanics, the simultaneity has an absolute character: two events simultaneous in one inertial reference frame S, are simultaneous in any other inertial frame S'.

Consequences of the Lorentz Transformations

Relativity of Simulataneity

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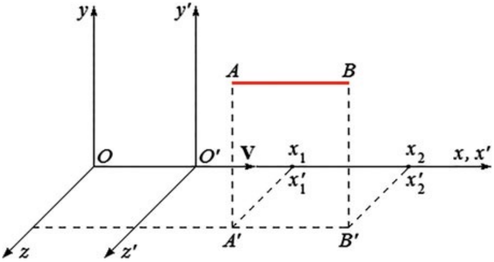
- It should be emphasized that the relativity of simultaneity does not contradict the principle of causality, provided the speed of light is the maximum speed at which signals are transmitted.

Consequences of the Lorentz Transformations

Length Contraction

The measurements are performed exclusively by light signals. Since the speed of light is finite, the observer from S must determine the two ends A and B of the bar at the same time, $t_1 = t_2$ (otherwise, during the measurements the bar would change its place).

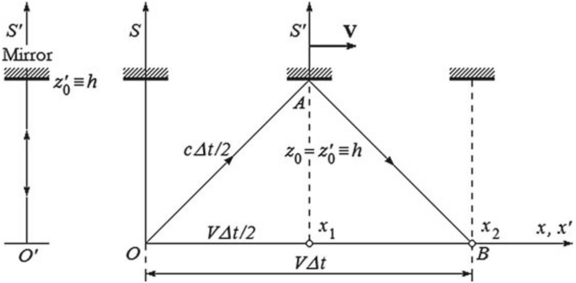
$$l = l_0\sqrt{1 - \frac{V^2}{c^2}} < l_0. \quad (11)$$



Example: Cab contraction

Consequences of the Lorentz Transformations

Time dilation



Gedanken Experiment – to see how durations change when we pass from one frame to another.

$$\Delta t = \frac{2h}{c}\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \gamma \Delta t' > \Delta t'$$

Example: Muon time dilation in own frame of reference

Mass in Special Relativity

The word mass has two meanings in special relativity:

- invariant mass (also called rest mass) is an invariant quantity which is the same for all observers in all reference frames
- relativistic mass is dependent on the velocity of the observer.

According to the concept of mass–energy equivalence, invariant mass is equivalent to rest energy, while relativistic mass is equivalent to relativistic energy (also called total energy).

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- In contrast, "invariant mass" is usually preferred over rest energy. The measurable inertia and the warping of spacetime by a body in a given frame of reference is determined by its relativistic mass, not merely its invariant mass.
- For example, photons have zero rest mass but contribute to the inertia (and weight in a gravitational field) of any system containing them.

Relativistic mass I

The relativistic mass is the sum total quantity of energy in a body or system (divided by c^2). Thus, the mass in the formula

$$E = m_{\text{rel}} c^2$$

is the relativistic mass. For a particle of finite rest mass m moving at a speed v relative to the observer, one finds

$$m_{\text{rel}} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- In the center of momentum frame, $v = 0$ and the relativistic mass equals the rest mass.
- In other frames, the relativistic mass (of a body or system of bodies) includes a contribution from the "net" kinetic energy of the body.

Relativistic mass II

However, for given single frames of reference and for isolated systems, the relativistic mass is also a conserved quantity. The relativistic mass is also the proportionality factor between velocity and momentum,

$$\vec{p} = m_{\text{rel}} \vec{v}.$$

Newton's second law remains valid in the form

$$\vec{F} = \frac{d(m_{\text{rel}} \vec{v})}{dt}.$$

When a body emits light of frequency ν and wavelength λ as a photon of energy $E = h\nu = hc/\lambda$, the mass of the body decreases by $E/c^2 = h/\lambda c$, which some authors interpret as the relativistic mass of the emitted photon since it also fulfills $p = m_{\text{rel}} c = h/\lambda$.

Relativistic energy-momentum equation I

The relativistic expressions for E and p obey the relativistic energy–momentum relation:

$$E^2 - (pc)^2 = \left(mc^2\right)^2$$

where the m is the rest mass, or the invariant mass for systems, and E is the total energy.
The equation is also valid for photons, which have $m = 0$:

$$E^2 - (pc)^2 = 0$$

and therefore

$$E = pc$$

A photon's momentum is a function of its energy, but it is not proportional to the velocity, which is always c.

Relativistic energy-momentum equation II

For an object at rest, the momentum $p = 0$, therefore $E_0 = mc^2$ [true only for particles or systems with momentum $= 0$] The rest mass is only proportional to the total energy in the rest frame of the object.
When the object is moving, the total energy is given by

$$E = \sqrt{(mc^2)^2 + (pc)^2}$$

To find the form of the momentum and energy as a function of velocity, it can be noted that the four-velocity, which is proportional to $(c, \vec{v})$, is the only four-vector associated with the particle's motion, so that if there is a conserved four-momentum $(E, \vec{p}c)$, it must be proportional to this vector.

Relativistic energy-momentum equation III

This allows expressing the ratio of energy to momentum as

$$pc = E\frac{v}{c},$$

resulting in a relation between E and v:

$$E^2 = \left(mc^2\right)^2 + E^2\frac{v^2}{c^2},$$

This results in

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma mc^2$$

and momentum

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma mv.$$

Relativistic energy-momentum equation IV - final

- When working in units where c = 1, known as the natural unit system, all the relativistic equations are simplified and the quantities energy, momentum, and mass have the same natural dimension:

$$m^2 = E^2 - p^2.$$

- The equation is often written this way because the difference

$$E^2 - p^2$$

is the relativistic length of the energy momentum four-vector, a length which is associated with rest mass or invariant mass in systems.

- Where $m > 0$ and $p = 0$, this equation again expresses the mass–energy equivalence $E = m$.

Instead of conclusions

Invariants

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- Note that the invariant mass of a system of particles may be more than the sum of the particles' rest masses, since kinetic energy in the system center-of- mass frame and potential energy from forces between the particles contribute to the invariant mass. As an example, two particles with four-momenta $(5GeV/c, 4GeV/c, 0, 0)$ and $(5GeV/c, -4GeV/c, 0, 0)$ each have (rest) mass $3GeV/c^2$ separately, but their total mass (the system mass) is $10GeV/c^2$. If these particles were to collide and stick, the mass of the composite object would be $10GeV/c^2$.