

Defects and Backgrounds

- $g\sigma \leftarrow \begin{cases} \sigma & \text{if } \sigma \in U_g \\ \emptyset & \text{otherwise} \end{cases}$
- symmetry via background fields $\sim \Sigma [A]$
- symmetry via defects $U_g (\Sigma)$

In fact: symmetry defects \leftrightarrow background fields

$$\text{defect cuts spacetime into patches}$$

at $P_1 \cap P_2$ transition function g
 $P_1 \quad \begin{array}{c} \nearrow U_g \\ \downarrow \\ P_2 \end{array} \quad \Rightarrow i \int_A \gamma = g$
background field

More formally let G be a discrete group (eg \mathbb{Z}_N)

Background gauge field for G = assignment of holonomy $g \in G$
 for each cycle $\gamma \in$ spacetime

gauge equivalence class $[A] \in H^1(M, G)$ cohomology
 valued in G

• closed $\exp(i \int_\gamma A) = 1$ if $\gamma = \partial \Sigma$

• modulo gauge trans (exact elements)

Defects and backgrounds related by Poincaré duality

$$H^1(M, G) \ni [A] \longleftrightarrow \underbrace{[\Sigma]}_{\text{symmetry defect locus}} \in H_{d-1}(M, G)$$

Some idea works for q -form sym:

$$\int_{\Sigma} U_g(\Sigma) \quad \text{introduce a background } B \text{ } (q+1)\text{-form}$$

$\text{PD}(\Sigma) = \gamma_{q+1} \exp(i \int_{\Sigma} B)$

Wilson surface
analog of holonomy

defects encode:
backgrounds

$$U_g(\Sigma) = \text{PD}([B])$$

$$[B] \in H^{q+1}(M, G^{(q)})$$

q -form
symmetry
group

Caveat

* discussion above correct for discrete symmetry

for continuous symmetry, defects encode flat backgrounds

Examples

- 4d Maxwell theory 1-form sym $U(1)_e^{(1)} \times U(1)_m^{(1)}$

elec current $F_{\mu\nu}$, mag current $(\#F)_{\mu\nu}$

action coupled to backgrounds $S = \int F_{\mu\nu} F^{\mu\nu} + B_e^{\mu\nu} F_{\mu\nu} + B_m^{\mu\nu} (\#F)_{\mu\nu}$

B_e, B_m 2-form gauge fields with gauge trans:

$$B_e \rightarrow B_e + d\lambda_e, \quad B_m \rightarrow B_m + d\lambda_m$$

gauge parameters λ_e, λ_m 1-forms.

- 3d $U(1)_K$ CS theory level 'k

$$S = \frac{k}{4\pi} \int A \wedge dA \quad \text{has } \mathbb{Z}_K^{(1)} \text{ symmetry}$$

Why? Measure charges of Wilson lines

(looks like \mathbb{Z} valued charge naively)

screening effects : monopole operator $\int_{S^2} F \neq 0$ surrounding \mathcal{O}

CS term $\sim k \int A dA \Rightarrow$ monopoles have electric charge k , are dynamical

$$\Rightarrow U(1)^{(1)} \text{ (naive)} \rightarrow \mathbb{Z}_k^{(1)}$$

coupling to background $S = \frac{k}{4\pi} \int A \wedge dA + \frac{1}{2\pi} \int A \wedge B$

Check reduction to \mathbb{Z}_k . Say $B = k b$ with b properly quantized $\sum b \in \mathbb{Z}$. Want to show this is equivalent to no background. Write

b as curvature of gauge field c (locally $b = dc$)

Change variables : $A \rightarrow A - c$ (allowed since b quantized)

$$S \rightarrow \frac{k}{4\pi} \int (A - c) \wedge (A - c) + \frac{k}{2\pi} \int (A - c) dc$$

$$= \underbrace{\frac{k}{4\pi} \int A \wedge dA}_{\text{sourceless } S \text{ CS}} - \underbrace{\frac{k}{4\pi} \int c \wedge dc}_{\text{counterterm}}$$

so $B \mathbb{Z}_k$
value modulo
counterterms
(anomalies!)

- 4d $SU(N)$ YM $\mathbb{Z}_N^{''}$ (related to confinement)

Background field $B \in H^2(M, \mathbb{Z}_N)$

closely related to bundle topology.

simple case $SU(2)$ and $SO(3)$

- $SU(2)$ bundles labelled by instanton number

$$\Sigma \ni I = \frac{1}{8\pi^2} \int \text{Tr}(F_A F)$$

path integral defining $SU(2)$ YM has :

$$\sum_{I \in \mathbb{Z}} \int D A$$

- integral over connections on fixed bundle
- sum over bundle topologies

- $SO(3)$ has more gauge bundles. In addition to instanton density. There is now

"2nd Steifel Whitney class" $= w_2 \in H^2(M, \mathbb{Z}_2)$

\mathbb{Z}_2 - valued magnetic flux $\int_{\Sigma} w_2 = 0 \text{ or } 1$

e.g. write $A = A^\alpha \sigma^\alpha / 2 \quad \alpha = 1, 2, 3$ say only $A^3 \neq 0$

$$\int_{\Sigma} w_2 \equiv \frac{1}{2\pi} \int_{\Sigma} dA^3 \bmod 2$$

evaluate for monopole (divide in two patches)



$$A_N^3 - A_\delta^3 = d\theta$$

where $e^{i\Theta(x)} \sigma^3$ gauge trans relating patches

Note: $SU(2) \rightarrow SO(3)$ double covering so

if in fact the bundle is $SU(2) \Rightarrow \Theta = 2\phi$

$$\Rightarrow w_2 = \int_{S^2} dA^3 = \oint d\phi = 0 \pmod{2}$$

equator

General Result: $SU(2)$ bundles are special

case of $SO(3)$ bundles where $\Theta = w_2 \in H^2(M, \mathbb{Z}_2)$

$$\Rightarrow SO(3) \text{ path integral : } \sum_{w_2} \sum_{I \in \mathbb{Z}} \left\{ \int dA \right\}_{\substack{\text{connections} \\ \text{on fixed} \\ \text{bundle}}} \quad \text{bundle topology}$$

$SU(N)$ story similar :

- $SU(N)$ bundle top labelled by $I \in \mathbb{Z}$
- $SU(N)/\mathbb{Z}_N$ bundle top has additional invariant $w_2 \in H^2(M, \mathbb{Z}_N)$ (\mathbb{Z}_N -valued magnetic flux)
- $SU(N)/\mathbb{Z}_N$ path integral includes sum over w_2

return to backgrounds for $\mathbb{Z}_N^{(1)}$ in $SU(N)$ YM

$$\mathcal{Z}[B] = \sum_{I \in \mathbb{Z}} \int dA e^{-S_{\text{YM}}}$$

\uparrow

$w_2 = B$
fixed

$(SU(N)/\mathbb{Z}_N \text{ bundles})$

Gauging $\mathbb{Z}_N^{(1)}$ in $SU(N)$ YM

- gauging means summing over $B \in H^2(M, \mathbb{Z}_N)$

(making background dynamical)

- reproduces $SU(N)/\mathbb{Z}_N$ path integral

$$\Rightarrow SU(N) \text{ YM} + \text{gauged } \mathbb{Z}_N^{(1)} \cong SU(N)/\mathbb{Z}_N \text{ YM}$$