

Anomalies

- defects and background useful for symmetry
- backgrounds + partition functions useful for anomalies

$$Z[A + d\lambda] = Z[A] \exp(i \int d^4x \omega(A, \lambda))$$

How do we discuss anomalies via symmetry defects?

background gauge \longleftrightarrow $U_d(\Sigma)$ topological (\longleftrightarrow current conservation)
invariance

Thus anomalies will show up as mild (c-number)

violations of topological property of $U_d(\Sigma)$

Example : Anomalies in QM

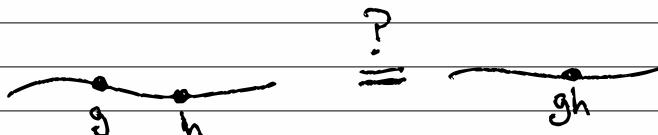


bit of spacetime (1-manifold)

0-form sym defect g inserted

If on S^1 this computes $\text{Tr}_H(g)$ partition function with chemical pot

Examine two such insertions:



naively this
is an allowed
topological move

Isn't this implied by fusion axiom

$$U_g U_h = U_{gh} ?$$

NO! this eqn holds when acting on operators

we are asking whether it holds on states

e.g. $g(h|\Psi\rangle) \stackrel{?}{=} (gh)|\Psi\rangle$

Allow for possible ambiguity :

$$\overbrace{g \circ h} = \exp\left(\frac{i}{2\pi} \gamma(g, h)\right) \overbrace{gh}$$

- overall multiplicative correction so that it cancels when acting on operators ($\Theta \rightarrow g\Theta g^{-1}$)
- phase since symmetry is unitary
- depends only on (g, h) not $|\Psi\rangle$ since arises locally from defect fusion

Now correction must be associative :

$$*\quad \gamma(g, h) + \gamma(gh, k) = \gamma(h, k) + \gamma(g, hk) \bmod \mathbb{Z}$$

Can modify U_g by phase, $U_g \rightarrow \exp\left(\frac{i}{2\pi} \eta(g)\right) U_g$

$$** \quad \gamma(g, h) \rightarrow \gamma(g, h) + \eta(g) + \eta(h) - \eta(gh)$$

Solutions to * modulo ** define a cohomology grp

$$H^2(G, U(1)) \ni [\gamma] \text{ anomaly}$$

sym grp

Non-zero $[g]$ in practical terms :

- operators in faithful representation of G
- states in projective representation of G
(all states have same g)

e.g. if $G \cong SO(3)$ and $[g] \neq 0$ we have

operators \leftrightarrow integer spin, states \leftrightarrow half integer spin

consequence : $[g] \neq 0 \Rightarrow$ all energy levels degenerate!
(no proj reps of don't exist)

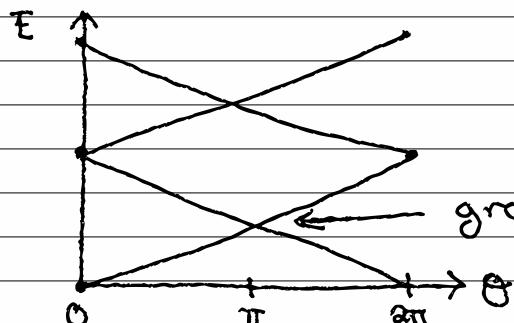
Explicit Example : particle on circle

$$q \sim q + 2\pi \quad L = \frac{1}{2} \dot{q}^2 + \frac{\Theta}{2\pi} \dot{q}, \quad \Theta \in [0, 2\pi)$$

At $\Theta = \pi$ $\mathbb{Z}_2 \times \mathbb{Z}_2$ sym $q \rightarrow -q, q \rightarrow q + \pi$

realized with non-zero anomaly $[g]$.

Energy levels for general Θ $\frac{1}{2}(n - \frac{\Theta}{2\pi})^2 = E_n$



$\Theta = 0$ $n=0$ ground state
 $\pm n$ degenerate

ground state degeneracy. !!

can continue to anomalies in higher dimension ...

eg 2d:

(a kind of crossing eqn!)

?

\equiv

$g_1 g_2 g_3$

Examples with 1-form symmetry

$SU(N) \text{ YM} + \Theta \text{ term: } \left(\frac{\Theta}{8\pi^2} \int \text{Tr}(F_A F^A) \right)$

background B for $\mathbb{Z}_N^{(1)}$ $\leadsto \mathbb{Z}_N[B]$

Q: Can there be an anomaly for $\mathbb{Z}_N^{(1)}$?
(without consideration of other sym.)

A: NO! Gauging $\mathbb{Z}_N^{(1)}$ leads to $SU(N)/\mathbb{Z}_N$ gauge theory which we know exists

\Rightarrow any anomaly for $\mathbb{Z}_N^{(1)}$ involves another sym. (T)

Periodicity of Θ

for all

$$I = \frac{1}{8\pi^2} \int \text{Tr}(F_A F^A) \in \mathbb{Z}$$

$SU(N)$ bndy

$\Rightarrow \Theta \text{ } 2\pi\text{-periodic}$

what about $SU(N)/\mathbb{Z}_N$? Fractional instantons

$$I = \frac{1}{8\pi^2} \int \text{Tr}(F \wedge F) = \frac{1}{2N} \int w_2 \cup w_2 + \text{integer}$$

(spin manifold; $w_2 \cup w_2 \sim w_2 \wedge w_2$; N even)

Simple Explicit Example (abelian AC $SO(3)$)

spacetime $\sim T^4$ $x^i \sim x^{i+1}$ $A = A_i^3 \frac{\epsilon^{ij}}{2} dx^j$

$$A_1 = \pi x^2, A_2 = -\pi x^1, A_3 = \pi x^4, A_4 = -\pi x^3$$

$$F_{12} = 2\pi, F_{34} = 2\pi \quad (\text{all others zero})$$

$$\Rightarrow F = dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$

$$\Rightarrow \int w_2 = \int_{\text{1-2 torus}} w_2 = 1 \quad (\text{so doesn't lift})$$

$\int_{\text{3-4 torus}} w_2$

Compute :

$$I = \frac{1}{8\pi^2} \int \text{Tr}(F \wedge F) = \frac{1}{2}; \quad \frac{1}{4} \int w_2 \cup w_2 = \frac{1}{2} \quad \checkmark$$

Fractional Instantons \Rightarrow Larger Θ Periodicity

$$\text{in } SU(N)/\mathbb{Z}_N \quad \Theta \sim \Theta + 2\pi N$$

Mixed Anomaly at $\Theta = \pi$ (N even)

B field for $\mathbb{Z}_N^{(1)}$ is w_2 of $SU(N)/\mathbb{Z}_N^{(1)}$ bundle

If $B \neq 0$ is theory T invariant?

$$T(S_\Theta) = T(\Theta I) = -\Theta I$$

so if $\Theta = \pi$

usually 1

if $B=0$.

$$\exp(i\pi I) = \exp(-i\pi I) \exp(2\pi i I)$$

And:

$$\exp(2\pi i I) = \exp\left(\frac{2\pi i}{2N} \int u_2 \cup u_2\right) = \exp\left(\frac{i\pi}{N} \int B \cup B\right)$$

non-zero! So with background B , T broken!

\Rightarrow mixed $\mathbb{Z}_N^{(1)} - T$ anomaly

\Rightarrow such YM at $\Theta=\pi$ non-trivial in IR!