$T\bar{T}$ deformations and holography*

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Abstract

This is a set of introductory lectures to the $T\bar{T}$ deformation and its holographic interpretation. The first two lectures review very basic field-theoretical aspects of the $T\bar{T}$ deformation, such as its definition, its universal effect on the energy levels in finite volume and on the S-matrix. In the third lecture, we review the holographic dictionary for $T\bar{T}$ deformed CFTs, and also explain its relation to the sharp geometric cutoff proposal of McGough *et al.* In the fourth lecture, we sketch the basics of a single-trace analogue of the $T\bar{T}$ deformation, which provides a tractable holographic dual to a non-asymptotically AdS spacetime.

Contents

*These notes are *very* preliminary. They are meant to give a very brief (mostly bullet-point) overview of a selection of developments in this field that the author finds interesting and/or representative. The focus is on simplicity, so many important research directions, generalizations and references are left out.

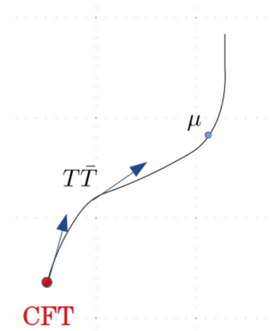
1. Introduction

The subject of these lectures is the $T\bar{T}$ deformation of two-dimensional QFTs and, in particular, CFTs. Unlike operator deformations that are commonly studied, the $T\bar{T}$ deformation is *irrelevant*. Usually, when adding an irrelevant operator to a QFT, the deformed theory does not make sense in the UV on its own. What makes the $T\bar{T}$ deformation special is that the deformed QFT appears to be *well-defined* up to arbitrary scales, is *solvable* in a certain sense, and the effect of the deformation on various physical observables can be computed without much effort for *finite* values of the deformation parameter.

The deformation is defined by incrementally adding to the action the $T\bar{T}$ operator, constructed from the components of the stress tensor, as

$$\frac{\partial S}{\partial \mu} = -2 \int d^2 z (T_{zz} T_{\bar{z}\bar{z}} - T_{z\bar{z}}^2)_\mu \tag{1.1}$$

This definition specifies the operator that is turned on at each point along the flow. Since the deforming operator has dimension four in two dimensions, the coupling μ has dimensions of $(length)^2$.



This deformation has a number of remarkable properties:

- it is *universal* (one can deform an arbitrary local $2d$ QFT/CFT)
- even though the deformed QFT has a length scale, $\sqrt{\mu}$, at which it becomes *non-local*, certain observables (e.g., the S-matrix) appear to be well-defined up to *arbitrarily high energies*
- to the extent that $T\bar{T}$-deformed QFTs are *UV complete*, their UV behaviour is not captured by a local UV fixed point. This new type of *UV* behaviour has been termed *asymptotic fragility*

There are multiple reasons that $T\bar{T}$-deformed QFTs are interesting - here we list just a few of them:

- there are strong indications that the deformed theory is *UV complete*, yet intrinsically non-local. It is thus very interesting to gain a theoretical understanding of this new type of UV behaviour
- they exhibit certain features expected of theories of $2d$ quantum gravity, such as a time delay in scattering proportional to the energy, as well as a minimum length. However, it is not yet clear whether $T\bar{T}$-deformed QFTs entirely lack off-shell observables, as expected in a theory of gravity
- they posses many special properties: universality, integrability, solvability and, possibly, more special structures to be discovered
- they are closely related to the effective string theory approach to study the QCD string and provide a very computationally effective approach for comparison with lattice QCD data
- they have interesting applications to holography, as tractable irrelevant deformations of AdS_3/CFT_2:

- $T\bar{T}$ - deformed CFTs defined as in (1.1) are holographically dual to AdS_3 gravity with mixed boundary conditions for the (non-dynamical) graviton. In typical states, the $T\bar{T}$-deformed observables coincide with those measured by bulk observers at a fixed radial position in AdS_3
- a *single-trace variant* of the $T\bar{T}$ deformation, defined in a particular string-theoretical setting, provides a tractable example of non-AdS holography and shows interesting connections to little string theory. This example also suggests that there exist generalizations of $T\bar{T}$ - deformed QFTs that are less universal, yet share the same type of UV behaviour

- the construction can be generalized to deformations produced by different combinations of conserved currents (e.g., $J\bar{T}$), which share many of the remarkable properties of $T\bar{T}$ deformations (such as solvability, UV completeness), yet have different physical properties (e.g., partially preserved conformal symmetry) and applications (e.g., to the holographic understanding of extremal black holes), which makes them interesting to study in their own right.

The plan for these lectures is as follows:

1. *Definition and basic properties of* $T\bar{T}$

   This lecture is mainly based on [1, 2]. We derive the exact flow equation for the deformed finite-size spectrum of $T\bar{T}$-deformed QFTs, and solve it explicitly in the special case of CFTs.

2. $T\bar{T}$ *- deformed free boson(s)*

   This is the simplest concrete example of a $T\bar{T}$ deformation, yet it exhibits extremely rich physics and an interesting relation to the worldsheet theory of a bosonic string. The theory is integrable and the S-matrix, captured by a simple scattering phase, can be computed exactly and has non-trivial physical effects. Largely based on [3].

3. *Holographic dictionary for* $T\bar{T}$*-deformed CFTs*

   We explain how to derive the holographic dictionary for $T\bar{T}$ - deformed CFTs via a straightforward application of the rules of holography in presence of double-trace deformations. We then clarify the relation between this dictionary and the geometric bulk cutoff proposal of [4]. Based on [5].

4. *A single-trace variant of* $T\bar{T}$ *and non-AdS holography*

   I will start with a very brief review of the NS5-F1 system and its connection to little string theory. Then, I will sketch the main idea behind the single-trace variant of the $T\bar{T}$ deformation proposed in [6] and mention an interesting set of observables that have been studied.

2. Definition and basic properties

2.1 The $T\bar{T}$ operator

Consider a local $2d$ QFT on Euclidean flat space, $\mathbb{R}^2$, with coordinates $x^\alpha = (\sigma, \tau)$. Assuming that the QFT action is translationally invariant, one can define a conserved stress tensor, $T_{\alpha\beta}$, as the Noether current associated with translations along x^β, $\partial^\alpha T_{\alpha\beta} = 0$. Assuming in addition that the QFT is Lorentz invariant, this tensor can also be chosen to be symmetric, $T_{\alpha\beta} = T_{\beta\alpha}$.

In this QFT, consider the following two-point correlation function

$$\mathcal{C}_{vac}(x,y) = -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\langle T_{\alpha\gamma}(x)T_{\beta\delta}(y)\rangle_{vac} \tag{2.1}$$

This correlator has the remarkable property that it is independent of x, y.

Proof: Using the two-dimensional identity $\epsilon_{\alpha\beta}V_\rho + \epsilon_{\beta\rho}V_\alpha + \epsilon_{\rho\alpha}V_\beta = 0$ with $V_\alpha = \partial_\alpha$, we find

$$\partial_{x^\rho}\mathcal{C}_{vac}(x,y) = -\frac{1}{8}\partial_{x^\rho}\epsilon_{\alpha\beta}\epsilon^{\gamma\delta}\langle T^\alpha{}_\gamma(x)T^\beta{}_\delta(y)\rangle = -\frac{1}{8}(\epsilon_{\rho\beta}\partial_{x^\alpha} + \epsilon_{\alpha\rho}\partial_{x^\beta})\epsilon^{\gamma\delta}\langle T^\alpha{}_\gamma(x)T^\beta{}_\delta(y)\rangle \tag{2.2}$$

The first term vanishes by conservation of the stress tensor. We are left with

$$\partial_{x^\rho}\mathcal{C}_{vac}(x,y) = -\frac{1}{8}\epsilon_{\alpha\rho}(\partial_{x^\beta} + \partial_{y^\beta} - \partial_{y^\beta})\langle T^\alpha{}_\gamma(x)T^\beta{}_\delta(y)\rangle = 0 \tag{2.3}$$

where the sum of the first two terms vanishes because of the translational invariance of the vacuum *state*, which implies that $\mathcal{C}_{vac}(x,y) = \mathcal{C}_{vac}(x-y)$, and the last term vanishes by the conservation of the stress tensor.

Thus, we find the remarkable fact that the expectation value, in an arbitrary $2d$ QFT in the vacuum state, of this particular combination of stress tensor components, is entirely independent of the insertion points. Note that by taking $y \to \infty$ and using cluster decomposition, one can show that the correlator (2.1) factorizes

$$\mathcal{C}_{vac}(x,y) = -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\langle T_{\alpha\gamma}(x)\rangle_{vac}\langle T_{\beta\delta}(y)\rangle_{vac} \tag{2.4}$$

Note that due to Lorentz invariance, the vacuum expectation value of all components of the stress tensor but the trace vanishes.

Exercise: Show that for any two conserved currents, $J^{(A)}_\alpha, J^{(B)}_\beta$, the correlator

$$\mathcal{C}^{AB}_{vac}(x,y) = \epsilon_{AB}\epsilon^{\alpha\beta}\langle J^{(A)}_\alpha(x)J^{(B)}_\beta(y)\rangle_{vac} \tag{2.5}$$

is independent of the insertion points (and thus factorizes on $\mathbb{R}^2$).

Notice that the $T\bar{T}$ correlator is just a special case of this, with $J^{(A)}_\alpha = T_\alpha{}^A$ (the generator of translations along x^A) and $J^{(B)}_\beta = T_\beta{}^B$.

Implications for the OPE

The operator product expansion (OPE in short) captures the short-distance behaviour as two operators approach each other

$$\lim_{x\to y}\mathcal{O}_i(x)\mathcal{O}_j(y) \sim \sum_k C_{ijk}(x-y)\mathcal{O}_k(y) \tag{2.6}$$

and is best justified when the UV of the theory is controlled by a CFT fixed point, case in which the coefficient functions C_{ijk} behave as (negative) power laws at short distances.

Consider now the *operator*

$$\hat{\mathcal{C}}(x,y) = -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}T_{\alpha\gamma}(x)T_{\beta\delta}(y) \tag{2.7}$$

whose vacuum expectation value we were computing before. We take $x \neq y$. A series of identical manipulations to the ones above shows that

$$\partial_{x^\rho}\hat{\mathcal{C}}(x,y) = -\frac{1}{8}\epsilon_{\alpha\rho}(\partial_{x^\beta} + \partial_{y^\beta})\,\epsilon^{\gamma\delta}T^\alpha{}_\gamma(x)T^\beta{}_\delta(y) \tag{2.8}$$

Thus, the x^ρ derivative of this operator is a total derivative (the same holds for the y^ρ derivative). Taking $x \to y$ and assuming that the product of stress tensors in (2.7) has an OPE expansion of the type (2.6), we immediately conclude that either the coefficient functions are constant, or they multiply an operator that is a total derivative (this follows just from the translation invariance of the coefficient functions). Given this, we can *define* the $T\bar{T}$ operator as

$$\lim_{x\to y}\hat{\mathcal{C}}(x,y) = \mathcal{O}_{T\bar{T}}(y) + \textit{derivative terms} \tag{2.9}$$

where we have set the constant in front of this operator to one. The reason for the name is that if we work in complex coordinates, $z=\sigma - i\tau$, we have[1]

$$\mathcal{O}_{T\bar{T}}(z,\bar{z}) = \lim_{\epsilon\to 0} T_{zz}(z+\epsilon)T_{\bar{z}\bar{z}}(z) - T_{z\bar{z}}(z+\epsilon)T_{z\bar{z}}(z) \tag{2.10}$$

which indeed corresponds to "$T\bar{T}$" in the CFT limit $T_{z\bar{z}}=0$. However, more generally this operator would rather correspond to a "$\det T$" operator.

Notice that our previous argument concerning the independence of the vacuum expectation value of $\hat{\mathcal{C}}(x,y)$ of x,y extends to any translationally invariant state. In the limit $x\to y$, the expectation value of $\hat{\mathcal{C}}(x,y)$ reduces to that of the $T\bar{T}$ operator. If it is also possible to take the points x,y to have an arbitrarily large separation, then we can use cluster decomposition to argue that the expectation value of the $T\bar{T}$ operator factorizes in such states

$$\langle \mathcal{O}_{T\bar{T}}\rangle = \langle T_{zz}\rangle\langle T_{\bar{z}\bar{z}}\rangle - \langle T_{z\bar{z}}\rangle^2 \tag{2.11}$$

in terms of the one-point functions of the stress tensor components. This relation is valid in *any* 2d QFT.

2.2 Expectation value of $T\bar{T}$ in an energy eigenstate on the cylinder

Let us now place the 2d QFT on an euclidean cylinder of circumference R, with $z=\sigma-i\tau$, $\sigma\sim\sigma+R$. Consider an eigenstate $|n\rangle$ of the energy and momentum

$$\mathcal{H}|n\rangle = E_n|n\rangle\ , \qquad \mathcal{P}|n\rangle = P_n|n\rangle \tag{2.12}$$

Since the state $|n\rangle$ is translationally-invariant, our previous results show that

$$\mathcal{C}_n(x,y) \equiv -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\langle n|T_{\alpha\gamma}(x)T_{\beta\delta}(y)|n\rangle \tag{2.13}$$

is independent of x,y. However, since the coordinate σ is now compact, we can no longer use cluster decomposition to argue for factorization of the expectation value of the $T\bar{T}$ operator in this state.

Factorization does, nevertheless, hold, as was shown in [1]. To see this, insert a complete set of energy-momentum eigenstates in the above correlator, with $x=(\sigma,\tau)$

$$\begin{aligned}\mathcal{C}_n(x,x') &= -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\sum_{n'}\langle n|T_{\alpha\gamma}(\sigma,\tau)|n'\rangle\langle n'|T_{\beta\delta}(\sigma',\tau')|n\rangle \\ &= -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\sum_{n'}e^{-(E_n-E_{n'})(\tau'-\tau)-i(P_n-P_{n'})(\sigma-\sigma')}\langle n|T_{\alpha\gamma}(\sigma,\tau)|n'\rangle\langle n'|T_{\beta\delta}(\sigma,\tau)|n\rangle\end{aligned} \tag{2.14}$$

The only way for the correlator $\mathcal{C}_n$ to be independent of σ',τ' is if the terms in the sum with $E_n\neq E_{n'}$, $P_n\neq P_{n'}$, cancel among each other. Assuming the spectrum to be non-degenerate, this implies that only states with $|n'\rangle = |n\rangle$ contribute to the sum. Consequently, the correlator factorizes

$$\mathcal{C}_n = \langle n|\mathcal{O}_{T\bar{T}}|n\rangle = -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\langle n|T_{\alpha\gamma}|n\rangle\langle n|T_{\beta\delta}|n\rangle \tag{2.15}$$

[1] In our conventions, $\tau = it$.

Thus, we found that the expectation value of the $T\bar{T}$ operator in an energy eigenstate on the cylinder factorizes in terms of the one-point functions of the stress tensor components. This relation can be alternatively written as

$$\langle n|\mathcal{O}_{T\bar{T}}|n\rangle = \frac{1}{8}\left(\langle n|T^{\alpha\beta}|n\rangle\langle n|T_{\alpha\beta}|n\rangle - \langle n|T^{\alpha}{}_{\alpha}|n\rangle^2\right) = -\frac{1}{4}\left(\langle n|T_{\tau\tau}|n\rangle\langle n|T_{\sigma\sigma}|n\rangle - \langle n|T_{\tau\sigma}|n\rangle^2\right) \quad (2.16)$$

In turn, the one-point functions $\langle n|T_{\alpha\beta}|n\rangle$ are related to the global conserved charges of the state as

$$\langle n|T_{\tau\tau}|n\rangle = -\frac{E_n(R)}{R}\,, \qquad \langle n|T_{\tau\sigma}|n\rangle = -\frac{iP_n(R)}{R}\,, \qquad \langle n|T_{\sigma\sigma}|n\rangle = -\frac{\partial E_n(R)}{\partial R} \quad (2.17)$$

which simply follow from the definition of the energy and momentum conserved charges and translational invariance of the one-point functions.

Exercise: Show that $\langle n|T_{\sigma\sigma}|n\rangle = -\partial_R E_n(R)$. (*Hint:* consider the QFT on the euclidean torus and interchange the labeling of space and time.)

To summarize, what we have shown in this subsection is that the expectation value of the $T\bar{T}$ operator in an energy eigenstate on the cylinder factorizes and is determined solely by the global conserved charges of the state, as

$$\langle n|\mathcal{O}_{T\bar{T}}|n\rangle = -\frac{1}{4R}\left(E_n\frac{\partial E_n}{\partial R} + \frac{P_n^2}{R}\right) \quad (2.18)$$

a relation we obtained by combining (2.16) and (2.17). This relation is *universally* valid.

2.3 Deforming two-dimensional QFTs by $T\bar{T}$

We have so far discussed properties of the $T\bar{T}$ operator in an arbitrary $2d$ QFT that was assumed to have usual ultraviolet behaviour (e.g., governed by a UV CFT fixed point). We would now like to do something somewhat different – and possibly less well-defined – which is to *deform* a $2d$ QFT by the $T\bar{T}$ operator. The deformation is defined by incrementally adding to the already deformed QFT action the $T\bar{T}$ operator constructed from the stress tensor of the deformed theory

$$\frac{\partial S_E(\mu)}{\partial \mu} = -2\int d^2z\,(\mathcal{O}_{T\bar{T}})_\mu \quad (2.19)$$

Since the deforming operator is irrelevant, this procedure can (and will) change rather drastically the UV behaviour of the QFT. Usually, an irrelevant deformation gives rise to an effective field theory with a cutoff set by the irrelevant coupling. However, as we will argue in the next lecture, the $T\bar{T}$ deformation is special in that it rather seems to produce a UV complete theory. The latter will become non-local [2] at a scale set by the dimensionful parameter μ, whose units are $[\mu] = (length)^2$. Thus, in order for the above definition to make sense, we should restrict ourselves to length scales much longer than $\sqrt{|\mu|}$, at which the QFT can be treated quasi-locally. This will allow us to associate a conserved Noether current $T_{\alpha\beta}$ to the translational symmetries, which can then be used to construct the $T\bar{T}$ operator.

As mentioned in the introduction, one reason that the $T\bar{T}$ deformation is interesting is that many observables can be computed exactly. In the following, we show how to compute one such observable, which is the exact spectrum of energies, $E_n(\mu, R)$, of the deformed QFT placed on a circle of circumference R, for a finite deformation parameter μ.

[2] This non-locality can be seen very explicitly in the example discussed in the next lecture, where the $T\bar{T}$ deformation is shown to generate an infinite number of higher-derivative interactions.

Spectrum of $T\bar{T}$-deformed QFTs on the cylinder

As beautifully shown in [2] (see also [7]), using the result (2.18) for the expectation value $\langle n|\mathcal{O}_{T\bar{T}}|n\rangle$ of the $T\bar{T}$ operator in an energy eigenstate on the cylinder, one can easily obtain a *universal* equation for how the energy spectrum $E_n(\mu,R)$ depends on μ.

Concretely, as μ is *infinitesimally* changed, the definition of the deformed QFT by the addition of the instantaneous $T\bar{T}$ operator implies that the change in the energy of the n^{th} energy eigenstate is given by usual quantum-mechanical first order perturbation theory

$$\frac{\partial E_n(\mu,R)}{\partial\mu} = -4\langle n|\int_0^R d\sigma\,(\mathcal{O}_{T\bar{T}})_\mu|n\rangle = -4R\langle n|(\mathcal{O}_{T\bar{T}})_\mu|n\rangle \tag{2.20}$$

where the additional factor of 2 with respect to (2.19) comes from the change of measure, $d^2z = 2d\tau d\sigma$. Combining this with (2.18), we find

$$\boxed{\frac{\partial E_n(\mu,R)}{\partial\mu} = E_n(\mu,R)\frac{\partial E_n(\mu,R)}{\partial R} + \frac{P_n^2(R)}{R}} \tag{2.21}$$

This is the universal flow equation that we were looking for. While this derivation has the advantage of being quite intuitive, it is somewhat heuristic, since the deforming "potential" contains an infinite number of derivatives. A more rigorous derivation can be found in e.g. [8], who carefully evaluate the torus partition function of the deformed QFT and extract the energy spectrum from it.

Equation (2.21) can be recognised as the inviscid Burger's equation, with the momentum squared playing the role of a forcing term. Notice that the dependence of the momentum on μ and R is entirely fixed by the fact that $\sigma \sim \sigma + R$, so it must obey the quantization condition

$$P_n = \frac{2\pi k_n}{R}\,, \qquad k_n \in \mathbb{Z} \tag{2.22}$$

In particular, the momentum cannot depend on the continuous parameter μ.

If the finite-size spectrum of the seed QFT at $\mu = 0$ is known (as a function of R), then one can integrate the Burger's equation to find the spectrum at arbitrary finite μ. This is particularly easy to see for those states that have $P_n = 0$, for which the following holds:

Exercise: Show that for states with $P_n = 0$, the function $E_n(\mu,R) = E_n(0, R + \mu E_n(\mu,R))$ solves Burger's equation (2.21) with the correct initial condition.

For general P_n, the solution can be obtained via similar manipulations [7]. In the following, we discuss the explicit solution for the case of $T\bar{T}$-deformed *conformal* field theories, where the R - dependence of the undeformed energy spectrum is particularly simple, being fixed by conformal invariance.

The spectrum of $T\bar{T}$-deformed CFTs

In a CFT, the state-operator correspondence maps the energy and momentum of a state on the cylinder to the conformal dimension Δ and spin s of the corresponding CFT operator on the plane, as

$$E_{\Delta,s}(0,R) = 2\pi\,\frac{\Delta - \frac{c}{12}}{R}\,, \qquad P_{\Delta,s}(R) = \frac{2\pi s}{R} \tag{2.23}$$

where the label n has now been replaced by Δ, s. The shift by $-c/12$ is the usual Casimir energy on the cylinder, where c is the CFT central charge. We would now like to solve the equation (2.21), subject to the above initial condition.

Let us first concentrate on zero-momentum states, $s = 0$. Then, the general result proven in the last exercise implies that

$$E_\Delta(\mu, R) = 2\pi \, \frac{\Delta - \frac{c}{12}}{R + \mu \, E_\Delta(\mu, R)} \tag{2.24}$$

which yields a quadratic equation for the energy, with solution

$$E_\Delta(\mu, R) = -\frac{R}{2\mu} \pm \sqrt{\left(\frac{R}{2\mu}\right)^2 + \frac{2\pi(\Delta - \frac{c}{12})}{\mu}} \tag{2.25}$$

In order to have a smooth limit to the CFT spectrum as $\mu \to 0$, one needs to select the upper sign if $\mu > 0$, and the lower sign if $\mu < 0$.

More generally, for states with non-zero momentum, the solution for the deformed energy is[3]

$$\boxed{E_{\Delta,s}(\mu, R) = \frac{R}{2\mu}\left(-1 + \sqrt{1 + \frac{8\pi\mu(\Delta - \frac{c}{12})}{R^2} + \left(\frac{4\pi\mu s}{R^2}\right)^2}\right)} \tag{2.26}$$

Let us now briefly discuss the salient properties of this solution, focussing for simplicity on the case $s = 0$. If $\mu > 0$, then for those states which have $\Delta < c/12$ (such as the ground state, which has $\Delta = 0$) the deformed energy becomes complex if $\mu/R^2 > 1/(8\pi|\Delta - c/12|)$. For a seed CFT with a discrete spectrum, this represents a finite number of states. In particular, the ground state energy becomes complex for

$$R < R_c \equiv \sqrt{\frac{2\pi\mu c}{3}} \tag{2.27}$$

Since the energy of excited states with $\Delta < c/12$ would acquire an imaginary part at radii smaller than R_c, we conclude that for positive μ, $T\bar{T}$ - deformed CFTs can be placed on a circle, provided its circumference is larger than R_c. Notice that for CFTs with a large central charge, R_c is much larger than the non-locality scale $\sqrt{\mu}$ set by the $T\bar{T}$ coupling.

For $\mu < 0$ on the other hand, the formula for the deformed energy implies that for fixed $|\mu|$ and R, an *infinite* number of energy eigenstates, namely all levels with

$$\Delta > \Delta_{max} \equiv \frac{c}{12} + \frac{R^2}{8\pi|\mu|} \tag{2.28}$$

acquire an imaginary energy. This is a far more worrisome behaviour, as it is present for any finite R, no matter how large it is.

The above maximum value of Δ corresponds to an upper bound on the energy

$$E_{max} = \frac{R}{2|\mu|} \tag{2.29}$$

attained just before the complex energy states set in - see figure 2. This maximum energy is sometimes referred to as the "UV cutoff" of $T\bar{T}$ - deformed CFTs with $\mu < 0$, in the sense that states of the finite volume system whose energies are below E_{max} are to be kept, while states that acquire complex energies are to be discarded. This terminology requires some qualification. Usually, a UV cutoff denotes a high energy or short distance scale beyond which the given description stops making sense, and various pathologies may start appearing. If the $T\bar{T}$ - deformed CFT were an effective field theory with cutoff set by the deformation parameter, one would estimate it to be $E_{max} \sim 1/\sqrt{|\mu|}$.

[3] We have chosen the branch with a smooth $\mu \to 0$ limit. However, for states with large momentum at μ fixed, the other branch may become relevant. See e.g. [5] for details.

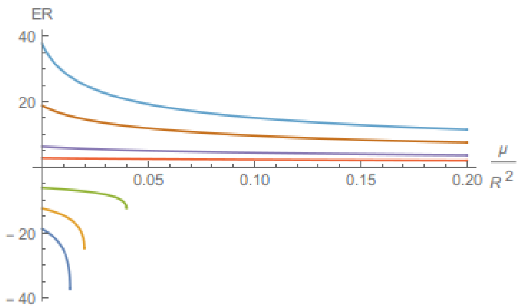


Figure 1: The dimensionless energy ER as a function of the dimensionless coupling μ/R^2 for $\mu > 0$.

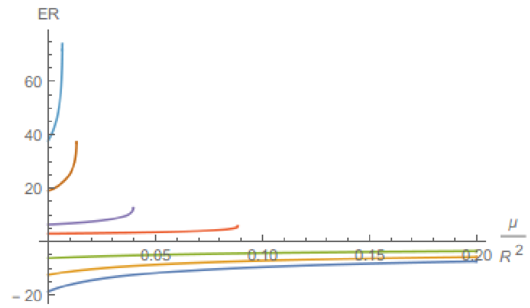


Figure 2: The dimensionless energy ER as a function of the dimensionless coupling μ/R^2 for $\mu < 0$.

The maximum energy (2.29) is quite different from this expectation, in that the IR scale R enters explicitly in the expression for E_{max}, and in particular the complex energies disappear if we take $R \to \infty$ first; indeed, the $T\bar{T}$ deformation has no effect on the spectrum in infinite volume.

An alternative interpretation of the complex energies for $\mu < 0$ was given in [9]. There, it was pointed out that the $\mu < 0$ theory exhibits superluminal propagation around positive energy backgrounds. While in two dimensions, superluminality does not immediately imply the existence of closed timelike curves (CTCs), it does lead to CTCs if the theory is placed in finite volume. In particular, Δ_{max} corresponds to the value of the energy for which the time advance gained in propagating once around the circle is comparable to the circumference of the circle. Since, for any given μ and R, there are always states in the original CFT with $\Delta > \Delta_{max}$, CTCs will always form if R is finite. Thus, it is inconsistent to place the $\mu < 0$ $T\bar{T}$ - deformed CFT in finite volume, and the complex energy states are simply a manifestation of this inconsistency. It is not clear whether manually removing the states around which CTCs appear makes the truncated theory well-defined.

Even if $\mu < 0$ $T\bar{T}$ - deformed CFTs appear to not make sense in finite volume, they may still be well defined in infinite volume[4]. Nonetheless, as discussed in [9], such theories with superluminal propagation are rather peculiar.

The R - dependence of the deformed energies, and in particular of the ground state energy, indicates that $T\bar{T}$ - deformed CFTs have a rather unusual behaviour in the UV; in particular, they are not governed by a local CFT fixed point. Indeed, if that were the case, then in the $R \to 0$ limit the ground state energy should have behaved as $-c/(12R)$, which is not the case for either sign of μ.

In the above, we only discussed the spectrum for $s = 0$. The analysis can be straightforwardly extended to general states, see [9] or [5] for a discussion.

Thermodynamics

When the parameter μ of the irrelevant deformation is small, $T\bar{T}$-deformed CFTs should have a behaviour close to that of the original CFT. It is though interesting to ask how the entropy $S(E)$ behaves for large energy and finite μ. We will concentrate on the case $\mu > 0$, where the finite size spectrum is well-defined at large energies.

In the undeformed CFT, the degeneracy of states at large enough conformal dimension is given by Cardy's formula

$$S(h, \bar{h}) = 2\pi\sqrt{\frac{c}{6}\left(h - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{c}{6}\left(\bar{h} - \frac{c}{24}\right)}\,, \qquad h, \bar{h} = \frac{\Delta \pm s}{2} \tag{2.30}$$

[4]This is precisely what happens for $J\bar{T}$ - deformed CFTs, for which the spectrum on the cylinder also contains an infinite number of complex energy states, but the spectrum of conformal dimensions on the plane, while non-trivially deformed, does not suffer from any obvious inconsistency [10].

Using the relation (2.23) between Δ and E, this leads to $S(E) \sim \sqrt{cER}$ at large energies.

Since, according to the exact formula (2.26), each energy level is smoothly deformed and does not cross other levels as μ is varied, the number of states within a fixed interval $\delta\Delta, \delta s$ centered around a given (Δ, s) does not change with μ, since these are the invariant labels of the state. To compute the entropy $S(E, P)$, i.e. the log of the degeneracy of levels per unit energy (and momentum) interval $\delta E, \delta P$, we simply need to replace Δ in (2.30) by its expression in terms of the deformed energy, obtained by inverting (2.26). Concentrating, for simplicity, on states with $s = 0$, we find[5]

$$S(E) = 2\pi\sqrt{\frac{c}{3}\left(\Delta(E) - \frac{c}{12}\right)} = 2\pi\sqrt{\frac{c}{6\pi}(ER + \mu E^2)} \tag{2.31}$$

The first term is nothing but the CFT Cardy entropy and dominates for $E << R/\mu$. At very high energies, the entropy exhibits Hagedorn behaviour, $S(E) = \beta_H E$, where the Hagedorn temperature is given by

$$\beta_H = T_H^{-1} = \sqrt{\frac{2\pi\mu c}{3}} \tag{2.32}$$

An entropy proportional to the energy means that in the canonical ensemble, the system cannot be in equlibrium at temperatures larger than T_H. Notice that β_H precisely equals the critical radius R_c defined in (2.27): thus, the fact that $T\bar{T}$ - deformed CFTs do not make sense on circles whose size is smaller than R_c maps (via a modular transformation) to the fact that it cannot be brought to temperatures higher than T_H. Hagedorn growth is of course familiar from the behaviour of the partition function of a free string at high temperatures. In fact, as we will see in the next lecture, there is a close connection between $T\bar{T}$ - deformed CFTs and the worldsheet theory on a free string. Note however that while in string theory, the Hagedorn growth is superseded by non-perturbative effects and there is a phase transition before T_H is reached, it is not clear this would be the case in $T\bar{T}$. The behaviour of the heat capacity close to the transition can give us more insights into its behaviour.

Exercise: Compute the heat capacity and show that it is positive and divergent as $T \to T_H$. What happens if one includes the first logarithmic correction to the Cardy entropy formula?

3. $T\bar{T}$-deformed free boson(s)

In this section, we discuss the simplest example of a $T\bar{T}$-deformed CFT: the $T\bar{T}$-deformed free boson(s). Despite its simplicity, this model has surprisingly rich physics, and captures all of the representative physical properties of $T\bar{T}$-deformed QFTs: the preservation of integrability, the universal modification of the S-matrix, the field-dependent coordinate transformation that relates the deformed and undeformed theories. Moreover, this model displays a rather interesting connection to the worldsheet theory of an infinitely long free bosonic string.

In the following, we explicitly derive the *classical* Lagrangian of a $T\bar{T}$-deformed free boson (the generalization to several bosons being straightforward) and relate it to the Nambu-Goto action for a string in a particular gauge. Using the Nambu-Goto perspective, we show that the deformed and the undeformed theories are related via a field-dependent coordinate transformation. In 3.2, we turn to the quantum case and describe the calculation of the deformed S-matrix, which is just a phase, using integrability techniques. Next, in 3.3 we discuss the basic physical manifestations of the scattering phase: a time delay in scattering proportional with the energy and the existence of a minimum length, both of which point towards a (quantum) - gravitational interpretation of $T\bar{T}$ - deformed QFTs. A large part of this section is based on [3]. In 3.4, we very briefly comment on deforming more general QFTs.

[5] The fact that we now measure the number of states in an interval δE, as opposed to the number of states in an interval $\delta\Delta$ does not affect the exponential factor that yields the leading contribution to the entropy.

3.1 $T\bar{T}$-deformed free boson(s): classical analysis

We start with the Lorentzian action for a free boson, given by

$$S_0 = \frac{1}{2}\int dt d\sigma \left[(\partial_t\phi)^2 - (\partial_\sigma\phi)^2\right] \equiv \frac{1}{2}\int dt d\sigma \left(\dot{\phi}^2 - \phi'^2\right) \tag{3.1}$$

Under the $T\bar{T}$ deformation, the Lorentzian action flows as[6]

$$\frac{\partial S_L[\mu]}{\partial\mu} = +\frac{1}{2}\int d^2\sigma\sqrt{\gamma}\,(T^{\alpha\beta}T_{\alpha\beta} - T^2) = \int dt d\sigma\,(T_{tt}T_{\sigma\sigma} - T_{t\sigma}^2) \tag{3.2}$$

In these notes, we follow the conventions of [5]. The stress tensor is defined as

$$T^\alpha{}_\beta = \frac{\partial L}{\partial(\partial_\alpha\phi)}\partial_\beta\phi - L\,\delta^\alpha_\beta \tag{3.3}$$

where L is the Lagrangian density. Since the $T\bar{T}$ deformation preserves Lorentz invariance, the symmetries of the problem and dimensional analysis imply that the deformed action must take the form

$$S_L[\mu] = \frac{1}{\mu}\int dt d\sigma\,\mathcal{F}(\mu\,\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi) \equiv \frac{1}{\mu}\int dt d\sigma\,\mathcal{F}\,(x) \tag{3.4}$$

for some function $\mathcal{F}(x)$, $x \equiv \mu\,(\phi'^2 - \dot{\phi}^2)$, whose expansion around $x = 0$ starts as $\mathcal{F}(x) = -x/2 + \mathcal{O}(x^2)$. The canonical stress tensor computed using this action reads

$$T_{tt} = -T^t{}_t = \mathcal{L} - \dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\mathcal{F} + 2\mu\dot{\phi}^2\mathcal{F}'}{\mu}\,, \qquad T_{\sigma\sigma} = \frac{-\mathcal{F} + 2\mu\phi'^2\mathcal{F}'}{\mu}\,, \qquad T_{\sigma t} = 2\phi'\dot{\phi}\mathcal{F}' \tag{3.5}$$

The flow equation (3.2) reduces to

$$\partial_\mu(\mathcal{F}/\mu) = \frac{x\mathcal{F}' - \mathcal{F}}{\mu^2} = \frac{2x\mathcal{F}\mathcal{F}' - \mathcal{F}^2}{\mu^2} \tag{3.6}$$

with solution $\mathcal{F}(x) = \frac{1}{2}(1 \pm \sqrt{1 + cx})$ for some constant c. We take μ to be positive. The solution with the correct behaviour as $x \to 0$ corresponds to choosing the lower sign and $c = 2$.

Thus, the *classical* $T\bar{T}$-deformed free boson action is

$$S_L[\mu] = \frac{1}{2\mu}\int dt d\sigma \left(1 - \sqrt{1 + 2\mu\,(\phi'^2 - \dot{\phi}^2)}\right) \tag{3.7}$$

When expanding this action in μ, it contains an infinite number of higher derivative terms. However, these higher derivative terms turn out to have a highly constrained structure, since (3.7) precisely coincides with the Nambu-Goto action

$$S_{NG} = -\frac{1}{\ell_s^2}\int d^2\sigma\sqrt{-\det\gamma_{\alpha\beta}}\,, \qquad \gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\,, \qquad \ell_s^2 = 2\pi\alpha' \tag{3.8}$$

in three Minkowski target space dimensions in *static gauge*, i.e.

$$X^0 = t\,, \quad X^1 = \sigma\,, \quad X^2 = \sqrt{2\mu}\,\phi \tag{3.9}$$

Indeed, in this gauge the induced metric on the string worldsheet is

[6] Notice the sign difference with respect to the euclidean definition (2.19) in the previous section. It follows from the fact that $\tau_E = it$.

$$\gamma_{\alpha\beta} = \begin{pmatrix} -1 + 2\mu\,\dot{\phi}^2 & 2\mu\,\dot{\phi}\,\phi' \\ 2\mu\,\dot{\phi}\,\phi' & 1 + 2\mu\,\phi'^2 \end{pmatrix} \tag{3.10}$$

and the square root of the metric determinant agrees with (3.7) (up to a constant shift), provided we identify $\mu = \ell_s^2/2$.

Thus, at classical level, the action of a $T\bar{T}$-deformed free boson is nothing but the Nambu-Goto action for an infinitely long string embedded in three-dimensional Minkowski space in static gauge. It is not hard to show that the connection between the $T\bar{T}$ deformation and the Nambu-Goto action persists if we deform several free bosons

Exercise: Show that the $T\bar{T}$-deformed action for n free bosons is the Nambu-Goto action for a string in $D = n + 2$ Minkowski target space dimensions in static gauge.

The Nambu-Goto action is invariant under $ISO(D-1,1)$, the Poincaré symmetries of the target space, of which the Lorentz boosts are non-linearly realised. Whether these non-linearly realised boost symmetries are preserved at quantum level is an interesting question addressed in [11] that we briefly comment on in the next subsection. In any case, notice that in the particular case $D = 26$, we recover the action for a free infinitely long bosonic string in the critical dimension, which is known to preserve the target space Lorentz symmetry at the full quantum level.

As a simple check of this equivalence, one can show that the target space energy of a critical bosonic string with N left-moving and $\tilde{N}$ right-moving oscillators turned on and with winding one around $X^1 \sim X^1 + R$ (which in static gauge is the same as the worldsheet energy)

$$E(N, \tilde{N}, R) = \sqrt{\left(\frac{R}{\ell_s^2}\right)^2 + \frac{4\pi}{\ell_s^2}\left(N + \tilde{N} - \frac{D-2}{12}\right) + \left(\frac{2\pi(N - \tilde{N})}{R}\right)^2} \tag{3.11}$$

precisely coincides[7] with the $T\bar{T}$ deformed energy (2.26) of 24 free bosons, for which $\Delta = N + \tilde{N}$, $s = N - \tilde{N}$, $c = D - 2 = 24$ and, as before, $\mu = \ell_s^2/2$. The connection with the bosonic string provides a physical interpretation for the complex energy of the ground state of the $T\bar{T}$ - deformed free bosons on a circle of size smaller than R_c in (2.27) in terms of the well-known tachyonic instability of the bosonic string: when the radius is large with respect to ℓ_s, the winding energy of the string dominates over the tachyonic contribution and the system is stable. At small radii though, the tachyon wins and the system becomes unstable.

The bosonic string picture, explored at length in [3], gives important insights into the physics of the $T\bar{T}$ deformation.

Exercise: Starting from the Nambu-Goto action with Minkowski target space

$$S_{NG} = -\ell_s^{-2} \int dt d\sigma \sqrt{(\dot{X}^\mu X'_\mu)^2 - (\dot{X}^\mu \dot{X}_\mu)(X'_\mu X'^\mu)} \tag{3.12}$$

show that the canonical variables satisfy the following primary constraints

$$\Pi^\mu X'_\mu = 0\,, \qquad \Pi^2 + X'^2 = 0\,, \qquad \Pi^\mu \equiv \frac{\partial L}{\partial \dot{X}^\mu} \tag{3.13}$$

Write down the conditions for conformal gauge and show they are compatible with the constraints. Do these gauge conditions completely fix the reparametrization symmetry? Show that in conformal gauge, X^μ satisfy the free wave equation.

[7] Away from the critical dimension, (3.11) can be identified with the target space energy of a string computed using lightcone gauge, which in general will not have target space Lorentz symmetry.

As is well known, in conformal gauge ($\gamma_{\alpha\beta} \propto \eta_{\alpha\beta}$), the string dynamics reduces to that of 24 free bosons, which parametrize fluctuations in the transverse directions to the string. Thus, from the point of view of the $T\bar{T}$ deformation, the undeformed CFT (describing 24 free bosons) corresponds to the Nambu-Goto action in conformal gauge, while the deformed CFT corresponds to the Nambu-Goto action in static gauge. We conclude that there should be a change of coordinates on the string worldsheet that takes the deformed CFT to the undeformed one, which we will now exemplify for a string in three target space dimensions (single boson).

Let $U = \sigma + t = X^1 + X^0$, $V = \sigma - t = X^1 - X^0$ be the worldsheet coordinates in static gauge (which are identified with two null coordinates in the target space), and u, v be the corresponding null worldsheet coordinates in conformal gauge. In static gauge (3.9), the induced worldsheet line element takes the form

$$ds^2 = dUdV + 2\mu(\partial_U\phi\, dU + \partial_V\phi\, dV)^2 \tag{3.14}$$

Let us now change the coordinates on the worldsheet to u, v

$$\begin{aligned} ds^2 &= (\partial_u U\, du + \partial_v U\, dv)\,(\partial_u V\, du + \partial_v V\, dv) + 2\mu(\partial_u\phi\, du + \partial_v\phi\, dv)^2 \qquad (3.15) \\ &= \left(\partial_u U\partial_u V + 2\mu(\partial_u\phi)^2\right) du^2 + \left(\partial_v U\partial_v V + 2\mu(\partial_v\phi)^2\right) dv^2 + (\partial_u U\partial_v V + \partial_v U\partial_u V + 4\mu\partial_u\phi\partial_v\phi)\, du dv \end{aligned}$$

and set $\gamma_{uu} = \gamma_{vv} = 0$, as required by the conformal gauge condition. These equations are not sufficient to fix the map between U, V and u, v, so we additionally require that $\partial_u U = \partial_v V = 1$, a condition that is compatible with both the conformal gauge equations of motion for U, V (which, as shown in the exercise, are $\partial_u\partial_v U = \partial_u\partial_v V = 0$) and the requirement that as $\mu \to 0$, we have $U = u$ and $V = v$. We thus find

$$U = u - 2\mu \int^v T_{vv} dv\ , \qquad V = v - 2\mu \int^u T_{uu} du \tag{3.16}$$

where $T_{uu} = (\partial_u\phi)^2$ and $T_{vv} = (\partial_v\phi)^2$ are the non-zero components of the stress tensor of the free boson. Written in the form above, the transformation (3.16) will generalize to any $T\bar{T}$ - deformed CFT.

Exercise: Show that the solutions $\phi^{[\mu]}(U, V)$ of the deformed CFT (3.7) equal those of the undeformed one at the field-dependent values (3.16) of the coordinates: $\phi^{[\mu]}(U, V) = \phi^{[0]}(u(U, V), v(U, V))$.

Thus, the $T\bar{T}$ - deformed free boson is related to the undeformed free boson via the above field-dependent coordinate transformation[8], which involves the integral of the stress tensor. As we will see, much of the physics of the $T\bar{T}$ deformation is encoded in this field-dependent coordinate transformation. A signature effect that is visible already at the classical level is a universal time delay proportional to the energy.

To see this, consider for simplicity a purely left-moving classical background $\phi(U) = \phi(u)$[9]. The relation between the $T\bar{T}$ coordinates U, V and the conformal gauge ones is, in this background

$$U = u\ , \qquad V = v - 2\mu \int T_{uu}\, du \tag{3.17}$$

Consider now the propagation of left/right-moving waves on the worldsheet immersed in this background, which by definition propagate on lines of $u = const$, $v \in (\infty, -\infty)$ and respecively $v = const$,

[8]The field-dependent coordinate transformation maps the equations of motion to each other, though not the action itself.

[9]This can be seen to solve the $T\bar{T}$-deformed equations of motion, using the result of the above exercise and the fact that for a purely left-moving background, we have $U = u$.

$u \in (-\infty, \infty)$. It is clear that the presence of the background does not at all affect the left-moving excitations. As for the right-moving ones, they acquire a shift

$$\Delta V = -2\mu \int_{-\infty}^{\infty} T_{uu} du = -2\mu \Delta E_L \tag{3.18}$$

where ΔE_L is the total left-moving energy of the background. If we wait for the wave to arrive at the same position σ near infinity, $\Delta V = -\Delta t$, where

$$\Delta t = 2\mu \Delta E_L \tag{3.19}$$

is the *time delay* (for $\mu > 0$) for the wave to arrive, as compared to if no background had been present. A time delay proportional to the energy is typical of gravitational scattering, reason for which it has been proposed that the $T\bar{T}$-deformation turns the orginal QFT into a gravitational theory. The time delay is present if $\mu > 0$; for $\mu < 0$ one obtains instead a time advance, which is related to the fact discussed at the end of the first lecture, that $T\bar{T}$-deformed CFTs with $\mu < 0$ exhibit superluminal propagation [9]. While this behaviour is unusual, it is not immediately inconsistent, provided one stays in infinite volume.

3.2 The S-matrix of the $T\bar{T}$-deformed free bosons CFT

In this section, we move on to the quantum theory and consider the scattering of worldsheet excitations around the long string background.

There are several ways in which the S-matrix for this scattering process has been computed: via brute force perturbative calculations [11, 13]; by promoting the field-dependent coordinate transformation (3.16) to an operator equation and evaluating its effect on the S-matrix [15]; via integrability techniques [3]. The latter two methods produce the full answer for the S-matrix, whereas the direct perturbative calculation has been so far been carried out up to two loops. In this lecture, we follow the integrability - based approach, which is both very powerful and connects nicely with the results of the first lecture.

The free boson theory is a trivial example of an integrable theory - in particular, it has an infinite number of conserved charges, roughly given by $\int d\sigma (\partial\phi)^n$. Since, as we have seen above, the $T\bar{T}$-deformed free boson is related to the undeformed free boson by a simple change of gauge, the deformed theory is also integrable[10]. This can be also seen explicitly from the exact spectrum (3.11): since states with a definite particle number, $N, \tilde{N}$ are exact energy eigenstates, one concludes there should be no particle production in scattering.

Consider now the $2 \to 2$ scattering of the worldsheet bosons, taken to be identical for simplicity. In the free boson CFT, the S-matrix is simply one, and can be defined despite the absence of a mass gap because the particles do not interact. Since the deformed CFT is integrable, the $2 \to 2$ S-matrix will be given by a phase, $S = e^{i\delta(p_i)}$, where p_i are the particles' momenta.

Now, in an integrable theory, it is possible to relate the scattering phase $\delta(p_i)$ to the ground state energy in finite-size, $E_0(R)$, via the so-called *Thermodynamic Bethe Ansatz* (TBA) equations. We will not use the full TBA equations herein but, following [3], we will use a simpler argument, to find the effect of the deformation on the S-matrix.

Remember that the energy $E(N, \tilde{N}, R)$ of N left-moving and $\tilde{N}$ right-moving $T\bar{T}$-deformed free bosons on a cylinder of circumference R is given by (3.11). Consider a two-particle eigenstate of the Hamiltonian on the cylinder (e.g., $a^\dagger_{-N}\tilde{a}^\dagger_{-\tilde{N}}|0\rangle$) with zero total momentum, so $\tilde{N} = N$. In the Schrödinger picture, the time evolution of this state is

$$|N, N, t\rangle = e^{-iE(N,N,R)\, t}|N, N, 0\rangle \tag{3.20}$$

[10]It is in fact possible to show, on very general grounds, that the $T\bar{T}$ deformation always preserves integrability [2].

The same state can be thought of as describing two massless particles, one left-moving and one right-moving, which circle around the cylinder in opposite directions and interact every $\Delta t = R/2$. At each interaction, the two-particle wavefunction picks up a phase shift $e^{i\delta(p_i)}$, which can be taken to be the flat space scattering phase if R is very large. It will be convenient to work in terms of the Mandelstam variable

$$s = (p_1 + p_2)^2 = E_{cm}^2 = \frac{16\pi^2 N^2}{R^2} \tag{3.21}$$

The scattering phase only depends on the external momenta p_i though s due to the special $2d$ kinematics. From the point of view of the two particles, the state at time $t >> R$ can be written as

$$|N, N, t\rangle = e^{-i(\Delta E(N,0,R)+\Delta E(0,N,R)+E(0,0,R))t+i\delta(s)2t/R}|N, N, 0\rangle \tag{3.22}$$

where $\Delta E(N, \tilde{N}, R)$ is the energy of the respective state with respect to the vacuum, whose energy is $E(0, 0, R)$. Equating the two exponents, we find that

$$\delta(s) = \lim_{R\to\infty} \frac{R}{2}(\Delta E(N, 0, R) + \Delta E(0, N, R) + E(0, 0, R) - E(N, N, R)) \tag{3.23}$$

i.e., the scattering phase is nothing but the binding energy of the two-particle state. As we send $R \to \infty$, we would like to keep the particles' momenta, $2\pi N/R$ fixed, which yields in the following result for $\delta(s)$

$$\delta(s) = \frac{8\pi^2 N^2 \mu}{R^2} = \frac{\mu s}{2} \tag{3.24}$$

Thus, the exact S-matrix takes the extremely simple form

$$\boxed{S = e^{i\mu s/2}} \tag{3.25}$$

This expression can also be derived using the TBA equations [3].

Despite its unusual form, the above S-matrix is perfectly consistent with the S-matrix axioms (unitarity, analyticity, crossing). It takes the form of a CDD factor - a meromorphic function (in this case, a phase factor polynomial in s, which is generally set to zero in order to avoid exponential behaviour of the S-matrix as $s \to \infty$), by which one can multiply the S-matrix while still satisfying the axioms. Its peculiar, non-polynomial behaviour as $s \to \infty$ indicates that the S-matrix does not correspond to the S-matrix of a local QFT. This will be clearly seen in the next section, where we show the theory exhibits a minimum length.

Did we not have access to the string theory description, the static gauge Nambu-Goto action would naively appear to correspond to a non-renormalizable theory, due to the presence of an infinite number of higher derivative terms. However, given that the associated S-matrix for scattering of worlsheet excitations can be computed exactly for *arbitrarily large s* and appears to be well-defined at all energies, we learn that static gauge Nambu-Goto rather corresponds to a UV complete theory. At least for $D = 26$, this is of course obvious from the string picture.

Comments on the brute force calculation of the S-matrix

The integrability-based approach allows one to compute the exact scattering phase via the magic of the TBA equations, which relate it to the known finite-size spectrum. One can alternatively perform the calculation of the S-matrix the hard way, i.e. using perturbation theory in μ, which is the approach initially undertaken in [11]. The usefulness of this exercise is that it renders more transparent the various assumptions at play, the structure of divergences one encounters and the possible choices of counterterms needed to subtract them, the role played by integrability and what is special about the critical dimension.

We will take D to be arbitrary. The static gauge Nambu-Goto action has manifest $SO(D-2)$ flavour symmetry, which requires the amplitude for scattering particles i, j into k, l to take the form

$$\mathcal{M}_{ij,kl} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk} \ , \qquad i,j,k,l \in \{1,\ldots,D-2\} \tag{3.26}$$

where A is the amplitude for annihilation, B for transmission, and C for reflection. They are functions of the Mandelstam variables s, t, u and the string coupling ℓ_s^2 and are related by crossing. Due to a peculiarity of $2d$ kinematics, one has either $t = 0$ with $u = -s$, or $u = 0$ with $t = -s$.

As we noted earlier in this section, the classical $T\bar{T}$ - deformed free boson action posesses non-linearly realised target space Poincaré symmetry for *any* D. An interesting question, adressed in [11], is whether this symmetry survives in the full quantum theory.

The answer is generally no, except if $D = 3, 26$. The rough argument is that the formula (3.11), which represents the exact spectrum of $D-2$ $T\bar{T}$ deformed free bosons, indicates that states with the same level, but different $SO(D-2)$ quantum numbers are exactly degenerate, from which one concludes that the annihilation part, A, of the amplitude should be zero. In [11], the coefficients A, B, C above were computed up to one loop, using the derivative expansion of the Nambu-Goto actionand regulating divergences using dimensional regularization. It was found that

$$A_{tree} = 0 \ , \qquad B_{tree} = \frac{l_s^2}{2}s^2 \ , \qquad A_{1-loop} = -\frac{D-26}{192\pi}l_s^4 s^3 \ , \qquad B_{1-loop} = i\frac{l_s^4}{16}s^3 \tag{3.27}$$

As advertised, in $D = 26$ the annihilation part of the amplitude vanishes and the Nambu-Goto action plus counterterms can (and does) equal the $T\bar{T}$ - deformed bosons. The case $D = 3$ is special, as the index i only takes one value, so it only makes sense to talk about the contribution to the full amplitude $A + B + C$, which does vanish. One can easily check that the transmission part of the amplitude, B, precisely equals the expansion of (3.25) with $\mu = \ell_s^2/2$ up to this order.

In all other dimensions, in order to have $A = 0$, one needs to add to the action a so-called Polchinski-Strominger (PS) term [12], which to this order in static gauge takes the form

$$S_{PS} = -\frac{D-26}{192\pi}\int d^2\sigma\, \partial_\alpha\partial_\beta X^i \partial^\alpha\partial^\beta X^i\, \partial_\gamma X^j \partial^\gamma X^j \tag{3.28}$$

This term explicitly breaks the non-linearly realised Lorentz symmetry. Thus, we conclude that the non-linearly realised Poincaré symmetry of the classical Nambu-Goto action is not preserved at the quantum level, except for $D = 3, 26$. In all other dimensions, the $T\bar{T}$ deformation of $D-2$ free bosons is an integrable theory, whose exact S-matrix is given by (3.25) and is associated, via integrability, to the exact spectrum (3.11). The action perturbatively reproducing this amplitude is $S_{NG} + S_{ct} + S_{PS}$, which also contains the higher order analogues of the PS term required to cancel the Nambu-Goto contribution to the annihilation amplitude up to the desired order. As for the counterterms, one needs ever new ones as the loop level is increased (an infinite number of them), and the only rule that appears to fix their finite part is to require integrability, i.e. match to (3.25).

Requiring integrability for any D at the expense of the target space Lorentz symmetry is natural from the $T\bar{T}$ point of view. However, as mentioned in the introduction, one is sometimes interested in the Nambu-Goto action as a departure point for an effective string description of the QCD string. In that case, one would instead like to preserve the non-linearly realized Lorentz symmetry at the expense of integrability, and therefore does not add the PS term to the action. The (low-order) S-matrix is still quite constrained by approximate integrability, as e.g. non-polynomial contributions to the amplitude only start appearing at three loops [13].

3.3 Physical manifestations of the scattering phase

In this subsection, we discuss two important physical effects of the scattering phase, still in the context of the $T\bar{T}$-deformed free bosons: time delay and the presence of a minimum length.

To see them, we consider the scattering of two Gaussian wavepackets, one left-moving and one right-moving, with profile functions $f_L(p_L)$ and $f_R(p_R)$ given by[11]

$$f_L(p_L) = \frac{1}{\sqrt{\Delta p_L \sqrt{\pi}}} \exp\left(-\frac{(p_L - \bar{p}_L)^2}{2(\Delta p_L)^2}\right) \tag{3.29}$$

and similarly for f_R. The *in* state, prepared at $t \to -\infty$, takes the form

$$|in\rangle = \int_0^\infty dp_L dp_R f_L(p_L) f_R(p_R)\, a^\dagger(p_L)\tilde{a}^\dagger(p_R)|0\rangle \tag{3.30}$$

where $a^\dagger$ and $\tilde{a}^\dagger$ are the creation operators for left/right-moving modes. As $t \to \infty$, we have

$$|out\rangle = \int_0^\infty dp_L dp_R f_L(p_L) f_R(p_R)\, e^{2i\mu\, p_L p_R}\, a^\dagger(p_L)\tilde{a}^\dagger(p_R)|0\rangle \tag{3.31}$$

since $s = 2\,\eta^{\alpha\beta} p_{L,\alpha} p_{R,\beta} = 4\, p_L p_R$. The momentum-space reduced density matrix for the left-movers, obtained after tracing out the right-moving modes, is given by

$$\begin{aligned}\rho(p_L, p_L') &= Tr_{RM}|out\rangle\langle out| = f(p_L) f_L^\star(p_L') \int_0^\infty dp_R |f_R(p_R)|^2 e^{2i\mu(p_L - p_L')p_R} \\ &= f(p_L) f_L^\star(p_L')\, e^{2i\mu(p_L - p_L')\bar{p}_R - \mu^2 (p_L - p_L')^2 \Delta p_R^2}\end{aligned} \tag{3.32}$$

The probability density to find the left-moving wavepacket (whichat some x_L (large and negative) for t large, is given by the Fourier transform of the above expression

$$\rho(x_L, x_L, t) = \int dp_L e^{-ip_L(t+x_L)} \int dp_L' e^{-ip_L'(t+x_L)} \rho(p_L, p_L') \approx e^{-\frac{(t+x_L-2\mu\bar{p}_R)^2}{(\Delta x_L)^2}} \tag{3.33}$$

where Δx_L is given by

$$(\Delta x_L)^2 = \mu^2 (\Delta p_R)^2 + \frac{1}{4(\Delta p_L)^2} \tag{3.34}$$

The answer above has several rather interesting features. The first feature is the time delay: if in absence of the deformation, the probability would have peaked at $t \approx -x_L$, in presence of the deformation it peaks *later* by an amount $2\mu\bar{p}_R$ proportional to the energy of the right-moving wavepacket. This is the quantum version of the effect we have seen earlier at classical level, which was induced by the field-dependent coordinate transformation.

The second very interesting effect is the existence of a minimum length. As we see from above, the width of the wavepacket after the scattering is given by Δx_L, and there will be a similar expression for the post scattering width Δx_R of the right-moving wavepacket. The two satisfy the inequality

$$\Delta x_L \Delta x_R \geq \mu \tag{3.35}$$

which was nicknamed a "stringy uncertainty principle" in [3] and indicates that it is impossible to resolve lengths smaller than $\sqrt{\mu}$. A third interesting effect one can already notice from the momentum-space expression (3.32) is that the off-diagonal matrix elements of the density matrix are highly suppressed, implying that the outgoing wavepackets are highly entangled with each other after the scattering. In [3] this has been likened with a very toy version of black hole evaporation.

Thus, even though the S-matrix is an extremely simple phase, it encodes some surprisingly rich physics, which shows that the $T\bar{T}$ deformation produces a non-local theory with certain gravitational properties.

[11] For $\bar{p} >> \Delta p$, we can replace the integration range to be $(-\infty, \infty)$, instead of $(0, \infty)$.

3.4 Comments on general $T\bar{T}$-deformed QFTs

The definition (1.1) of the $T\bar{T}$ deformation discussed in the first lecture has the advantage of being based on an action, and allows one to compute the exact spectrum of the deformed QFT; however, this definition only makes sense at distances much longer than the non-locality scale $\sqrt{\mu}$, where the QFT can be treated quasi-locally and one is able to associate a stress tensor to the translational symmetries.

The fact that the deformed QFT could actually be UV complete was instead seen, at least for the case of the $T\bar{T}$ - deformed free bosons, from the exact expression (3.25) for the S-matrix, which is well-defined up to arbitrarily high energies. Given this, one may be lead [16] to provide an alternative definition of the $T\bar{T}$ deformation, based on its effect on the S-matrix:

$$S_\mu(p_i) = e^{i\mu \sum_{i<j} \epsilon^{ab} p_{ia} p_{jb}} S_{QFT}(p_i) \tag{3.36}$$

where the particles are naturally ordered according to their rapidities. This expression is the natural generalization of the scattering phase to massive particles. This S-matrix-based definition makes the UV completeness manifest; it nonetheless has the drawback of not being based on an action principle.

In the following, we briefly sketch how these two approaches could be unified. First, for the special case of integrable QFTs, one can explicitly check the equivalence of these two definitions by using the TBA equations. Then, we mention a general, path-integral definition of the $T\bar{T}$ deformation of *arbitrary* QFTs on flat space, which is able to reproduce both the deformed spectrum and the deformed S-matrix.

Integrable QFTs

In integrable QFTs, the finite-size spectrum is related to the scattering phase via the Thermodynamic Bethe Ansatz (TBA) equations. Since, as shown in [2], the $T\bar{T}$ deformation preserves integrability if initially present, we can hope to relate the universal modification of the spectrum to a universal modification of the scattering phase. Below, we briefly sketch the argument.

We consider the scattering of massive particles with mass m in an integrable QFT, so the S-matrix is well-defined. It is convenient to write the particles' momenta in terms of the rapidities, β

$$p_i^0 = m \cosh \beta_i \ , \qquad p_i^1 = m \sinh \beta_i \tag{3.37}$$

In the integrable case, the $2 \to 2$ S-matrix is a pure phase, $S = e^{i\delta(\beta_i)}$, with

$$\delta(\beta_i) = \delta_{QFT}(\beta_i) + \delta_\mu(\beta_i) \ , \qquad \delta_\mu(\beta_1, \beta_2) = \mu\, \epsilon_{\alpha\beta} p_1^\alpha p_2^\beta = \mu m^2 \sinh(\beta_1 - \beta_2) \tag{3.38}$$

where δ_{QFT} is the scattering phase of the undeformed QFT and δ_μ is the additional shift due to the $T\bar{T}$ deformation.

Let us now very briefly review the TBA approach. As beautifully explained in [14], the derivation of these equations proceeds in two steps. The first step is to consider the partition function of the $2d$ QFT on a torus of size (L, R), where $L >> R$. Depending on the choice of euclidean time direction, this partition function can be approximated as either the ground state energy in finite volume, or the finite-temperature free energy in infinite volume

$$Z(L, R) \approx e^{-LE_0(R)} \approx e^{-Rf(R)L} \tag{3.39}$$

In the second step, the free energy in infinite volume is estimated by doing statistics over the particles. The scattering phase $\delta(p_i, p_j)$ enters in the quantization condition for their momenta as

$$mL \sinh \beta_i + \sum_{j \neq i} \delta(\beta_i, \beta_j) = 2\pi n_i \ , \qquad n_i \in \mathbb{Z} \tag{3.40}$$

In the limit of a large number of particles, it is convenient to introduce the particle and level densities, which are constrained by the above equation. The free energy is computed using usual thermodynamic considerations and then minimized. Specializing to the "bosonic" case, at the minimum one finds

$$E_0(R) = Rf(R) = \frac{m}{2\pi}\int d\beta \cosh\beta \ln(1 - e^{-\varepsilon(\beta)}) \tag{3.41}$$

where the "pseudoenergy" ε is constrained to obey

$$\varepsilon(\beta) = mR\cosh\beta + \frac{1}{2\pi}\int d\beta' \frac{\partial\delta(\beta', \beta)}{\partial\beta'} \ln(1 - e^{-\varepsilon(\beta')}) \tag{3.42}$$

We assume that the momentum $P = m\int d\beta \sinh\beta \ln(1 - e^{-\varepsilon(\beta)}) = 0$.

Let us now turn to the $T\bar{T}$ deformation. Using the expression (3.38) for the scattering phase, it is easy to see that the solution for $\varepsilon(\beta)$ in the deformed QFT is related to $\varepsilon(\beta)$ in the undeformed one by the formal replacement $R \to R + \mu E$. Remember from section 2.3 that the finite-size energies of the deformed theory (with momentum set to zero, for simplicity), could be obtained from the finite-size energies of the undeformed QFT by replacing $R \to R + \mu E(\mu, R)$. This establishes the link between the scattering phase (3.38) and the deformed spectrum. A generalization to non-zero momentum and to excited states is also possible [7].

$T\bar{T}$ as coupling to topological gravity

It is interesting to ask whether the action-based definition of $T\bar{T}$ valid at large scales and the S-matrix-based non-perturbative definition (3.36) can be unified also for non-integrable QFTs. This was achieved in [17], who proposed a path integral definition of the $T\bar{T}$ deformation

$$Z_{T\bar{T}}[\mu] = \int \mathcal{D}X^a \mathcal{D}e^a_\alpha \exp\left[-\frac{1}{2\mu}\int d^2x\, e\, \epsilon_{ab}\, \epsilon^{\alpha\beta}(\partial_\alpha X^a - e^a_\alpha)(\partial_\beta X^b - e^b_\beta) + S_{QFT}(e^a_\alpha)\right] \tag{3.43}$$

which can be thought of as coupling the original quantum field theory to a topological theory of gravity.

This formula can be justified by treating the $T\bar{T}$ deformation using the Hubbard-Stratonovich method, which in this case amounts to coupling the original QFT to a dynamical metric. However, as shown in [18], the conservation of the stress tensor implies that the path integral reduces to one only over flat metrics, at least infinitesimally. When passing from metrics to vielbeine, a simple way to enforce this constraint is to introduce the auxiliary fields X^a as above, whose equations of motion impose the flatness condition $\partial_\alpha e^a_\beta = 0$. The vielbein equations of motion impose

$$\partial_\alpha X^a = e^a_\alpha + \mu\, \epsilon^{ab}\epsilon_{\alpha\beta}T^\beta_b \tag{3.44}$$

which is nothing but the generalization of the field-dependent coordinate transformation from static ($\partial_\alpha X^a = \delta^a_\alpha$) to conformal gauge ($e^a_\alpha = \delta^a_\alpha$) that we have seen before in the Nambu-Goto action. This equation points to interpreting the $T\bar{T}$ deformation as providing a set of dynamical coordinates, X^a, through which the underlying QFT dynamics is seen [19].

By computing the torus partition function using the above definition, [17] were able to derive a flow equation for the partition function, which precisely reproduces the flow equation (2.21) on the energy levels. Concommitently, in [15] it was shown that the effect of coupling to the non-dynamical vielbein precisely reproduces the dressing (3.36) on the S-matrix. Thus, the path integral above is indeed able to unify the two previous definitions of the $T\bar{T}$ deformation.

4. The holographic dictionary for $T\bar{T}$-deformed CFTs

So far, we have been focussing on basic quantum field theoretical properties of the $T\bar{T}$ deformation: its definition, the deformed spectrum, the S-matrix. The remarkable lesson that we have learned is that these observables are exactly computable at finite μ, despite the deformation being irrelevant.

In this lecture, we will discuss the $T\bar{T}$ deformation in the holographic context. Consequently, we will restrict our attention to seed CFTs with a large central charge $c >> 1$ and a large gap in the spectrum of conformal dimensions, known as a holographic CFTs (see e.g. [20] for a nice introduction). From the perspecive of a low-energy observer, the holographic dual of such a CFT consists of Einstein gravity in AdS_3 coupled to some light matter fields. The question that we would like to answer in this lecture is:

What is the holographic dual of the $T\bar{T}$ deformation of a holographic CFT?

As we will see, the answer is extremely simple and predictable. $T\bar{T}$ is a double-trace deformation in holographic parlance, and double-trace deformations have long been known to coorrespond to mixed boundary conditions for the dual bulk fields. As we will show, the holographic dictionary for the $T\bar{T}$ deformation can be derived at *precision* level[12], and follows from a straightforward application of the rules of holography in presence of double-trace deformations.

The plan of this lecture is as follows. We start by reviewing double-trace deformations in AdS/CFT and how the holographic dictionary for them is derived. In 4.2, we apply this procedure to the $T\bar{T}$ deformation and use it to *derive* the holographic dictionary. We then exemplify how this holographic dictionary reproduces the finite-size energy spectrum (2.26) in the $T\bar{T}$ - deformed CFT. In subsection 4.3, we explain the relation between the holographic dictionary and an earlier proposal that links the $T\bar{T}$ deformation to AdS_3 gravity in presence of a sharp radial cutoff.

4.1 Double-trace deformations in holography

Brief review of usual AdS/CFT

The AdS/CFT holographic dictionary states that the partition function of a d-dimensional CFT in presence of sources, J, for its single-trace operators, $\mathcal{O}$, equals the partition function of the dual $d+1$-dimensional gravitational theory with prescribed (usually Dirichlet) boundary conditions on the dual bulk fields Φ. Taking Φ to be a free scalar (which is usually a good approximation near the boundary), its asymptotic equation of motion fixes the radial dependence to be of the Fefferman-Graham form

$$\Phi(z, x^\mu) = \phi^{(0)}(x^\mu)\, z^{d-\Delta} + \ldots + \phi^{(\Delta)}(x^\mu) z^\Delta + \ldots \tag{4.1}$$

where z is the Poincaré radial coordinate, with the conformal boundary of AdS lying at $z = 0$. The mode proportional to $\phi^{(0)}$ is non-normalizable, and is identified with the CFT source $J(x^\mu)$ via the AdS/CFT dictionary

$$Z_{CFT}[J] = \int \mathcal{D}\psi\, e^{-S_{CFT}[\psi] - \int J\mathcal{O}[\psi]} = Z_{grav}[\phi^{(0)} = J] \tag{4.2}$$

Because it is non-normalizable, the coefficient $\phi^{(0)}$ is to be kept fixed and provides the boundary conditions that the gravitational path integral obeys.

The one-point function of $\mathcal{O}$ in the CFT is given by

$$\langle \mathcal{O}(x^\mu) \rangle = \frac{\delta W[J]}{\delta J(x^\mu)} \tag{4.3}$$

[12]The restriction to large c, large gap is mostly in order to have a manageable holographic dual; the effect of $T\bar{T}$ can likely be followed through exactly.

where $W[J] = -\ln Z_{CFT}[J]$ is the generating functional of connected CFT correlators. In the classical approximation, the gravitational path integral with fixed boundary conditions $\phi^{(0)}$ is approximated by the exponential of the holographically renormalized on-shell action $S^{ren}_{on-shell}[\phi^{(0)}]$, which is therefore equated with $W[J]$. Taking the functional derivative, one finds that the mode $\phi^{(\Delta)}(x^\mu)$, which is normalizable and thus allowed to fluctuate, is identified[13] with the expectation value of the dual operator in the particular state that the CFT finds itself in

$$\frac{\delta S^{ren}_{on-shell}[\phi^{(0)}]}{\delta\phi^{(0)}(x^\mu)} = \phi^{(\Delta)}(x^\mu) = \langle \mathcal{O}(x^\mu)\rangle \tag{4.4}$$

Note that since our discussion is rather schematic, we have neglected all normalization factors.

Review of double-trace deformations in AdS/CFT

According to the above discussion, if we want to add a source J for a single-trace CFT operator, all we need to do in the holographic dual is to impose the boundary condition $\phi^{(0)} = J$ when computing the gravitational path integral. What happens if we simultaneously add a source for the double-trace operator $\mathcal{O}^2$ to the CFT action?

The generating functional of connected correlators in the deformed CFT now reads

$$e^{-W_\mu[J^{[\mu]}]} = \int \mathcal{D}\psi\, e^{-S_{CFT}[\psi] - \frac{\mu}{2}\int \mathcal{O}[\psi]^2 - \int J^{[\mu]}\mathcal{O}[\psi]} \tag{4.5}$$

where $J^{[\mu]}$ is the (tunable) source for the single-trace operator $\mathcal{O}$ in the CFT deformed by the double-trace coupling $\mu\,\mathcal{O}^2/2$, where μ is kept fixed[14].

The standard way to treat these deformations is to use the Hubbard-Stratonovich method, which consists of first inserting a resolution of the identity $1 = (\det\mu)^{-1/2}\int d\tilde\sigma\, e^{\frac{1}{2\mu}\int\tilde\sigma^2}$ into the path integral, and then shifting the integration variable as $\tilde\sigma = \sigma' - \mu\,\mathcal{O}$, with the result

$$\begin{aligned} e^{-W_\mu[J^{[\mu]}]} &= (\det\mu)^{-1/2}\int \mathcal{D}\psi\,\mathcal{D}\sigma'\, e^{-S_{CFT}[\psi] - \int (J^{[\mu]}+\sigma')\,\mathcal{O}[\psi] + \frac{1}{2\mu}\int\sigma'^2} \\ &= (\det\mu)^{-1/2}\int \mathcal{D}\sigma\, e^{\frac{1}{2\mu}\int(\sigma - J^{[\mu]})^2}\, e^{-W_0[\sigma]} \approx e^{-W_0[J^{[\mu]}+\mu\langle\mathcal{O}\rangle] + \frac{\mu}{2}\int\langle\mathcal{O}\rangle^2} \end{aligned} \tag{4.6}$$

where the last step involves a saddle point approximation at large N, which yields the saddle-point value $\sigma_* = J^{[\mu]} + \mu\langle\mathcal{O}\rangle$. Thus, we find that at large N, the source in the deformed theory is shifted with respect to the one in the undeformed CFT by the expectation value of the operator

$$J^{[\mu]} = J^{[0]} - \mu\langle\mathcal{O}\rangle\ , \qquad W_\mu = W_0 - \frac{\mu}{2}\int\langle\mathcal{O}\rangle^2 \tag{4.7}$$

and the generating function is shifted by *minus* the double trace. The expectation value $\langle\mathcal{O}\rangle$ stays the same. Notice that (4.7) follows from a purely field-theoretical argument. The large N approximation is only used for the evaluation of the saddle point, but exact results are in principle possible.

The implications of the above shifts for the holographic dictionary in presence of the double-trace deformation are straightforward. Since all we did was to add the boundary term (4.7), the bulk theory is the same, all that can change are the boundary conditions. Before the deformation, $J^{[0]}$ was identified with the non-normalizable mode $\phi^{(0)}$ of the bulk field, while $\langle\mathcal{O}\rangle$ was identified with the normalizable mode $\phi^{(\Delta)}$. Since the new source $J^{[\mu]}$ is a linear combination of the old source $J^{[0]}$ and the expectation value, we find that the deformed CFT corresponds to the same gravitational theory as before, but with *mixed* boundary conditions on the bulk fields, namely. the linear combination

[13] This is true for free scalars; more generally, $\langle\mathcal{O}\rangle$ will be a function of both $\Phi^{(\Delta)}$ and $\phi^{(0)}$.

[14] From now on, an underscript or a superscript in square brackets will indicate whether the corresponding quantity belongs to the deformed or undeformed CFT.

$\phi^{[\mu]} \equiv \phi^{(0)} - \mu\, \phi^{(\Delta)}$ of the normalizable and the non-normalizable modes is held fixed. The holographic dictionary reads

$$Z_{\mu,QFT}\left[J^{[\mu]}\right] = Z_{grav}\left[\phi^{(0)} - \mu\, \phi^{(\Delta)} = J^{[\mu]}\right] \tag{4.8}$$

The functional derivative of Z_{grav} with respect to the new source $\phi^{[\mu]}$ will produce the gravity dual of the expectation value of $\mathcal{O}$ in the deformed CFT which, in the simple example of the scalar field, will equal $\phi^{(\Delta)}$.

To summarize, the holographic dictionary in presence of double-trace deformations proceeds in two steps:

1. a *purely field-theoretical* step, in which one uses the Hubbard-Stratonovich method and large N to derive the relation between sources and expectation values before and after the deformation

2. the interpretation of the new sources and expectation values in terms of the coefficients in the asymptotic expansion of the dual bulk field, which consists of simply plugging in the *undeformed* holographic dictionary into the results of step 1

Variational principle approach

There exists an alternate - and, in practice, much simpler - approach for finding the relation between the deformed and undeformed CFT data, which I will call the 'variational principle' approach. It is nicely explained in [21] and it works at the level of the *classical* AdS/CFT dictionary, involving only (super)gravity fields.

By definition, the variation of the undeformed/deformed generating functional with respect to the source is

$$\delta W_0 = \int \langle\mathcal{O}\rangle_0 \delta J^{[0]}\ , \qquad \delta W_\mu = \int \langle\mathcal{O}\rangle_\mu \delta J^{[\mu]} \tag{4.9}$$

From (4.7) we have that $W_\mu = W_0 - S_{d.tr.}$. Taking the variation, we have

$$\delta W^{[\mu]} = \delta W^{[0]} - \frac{\mu}{2}\delta \int \mathcal{O}^2 = \int \mathcal{O}\, \delta(J^{[0]} - \mu\mathcal{O}) \tag{4.10}$$

Equating this with δW_μ in (4.9) and separately matching the terms inside the variation and their coefficients, we can effectively read off the same holographic dictionary as above. Clearly, this is a much simpler way to derive the dictionary, and it is particularly useful as we consider more complicated operators, e.g. those carrying spin, and expectation values that shift under the deformation.

This exercise is very naturally phrased in bulk language. From the point of view of the canonical formulation of gravity in AdS, in the radial Hamiltonian formalism, where z plays the role of time, $\phi^{(0)}$ should be viewed as a generalized coordinate on the initial surface $z = 0$, while $\phi^{(\Delta)}$ represents the (holographically renormalized) canonically conjugate momentum. As before, $W[J]$ is identified with the *classical* on-shell renormalized gravitational action $S^{ren}_{on-shell}[\phi^{(0)}]$. The addition of the double trace boundary term induces a canonical transformation of the above phase space variables, which can be read off from the variational principle as above, under the identifications $W \to S, J \to \phi^{(0)}$ and $\mathcal{O} \to \phi^{(\Delta)}$.

The main message is that, rather than performing the steps of the Hubbard-Stratonovich procedure, followed by the saddle point approximation to find the relation between the new sources and expectation values and the undeformed ones, at large N, when $W[J]$ is well approximated by the classical on-shell action $S^{ren}_{on-shell}[\phi^{(0)}]$, we can derive the same data from the (much simpler to handle) variational principle approach. We emphasize that the two procedures are *equivalent* at large N.

4.2 The holographic dictionary for $T\bar{T}$-deformed CFTs

We are now ready to derive the classical holographic dictionary for $T\bar{T}$-deformed holographic CFTs. This simply amounts to applying the recipe described in the previous section to the double-trace $T\bar{T}$ deformation[15].

As explained in the previous subsection, the first step of the recipe is purely field-theoretical and uses the Hubbard-Stratonovich method to find the deformed sources and expectation values in terms of the original ones. Since, as we argued, for the purpose of deriving the classical holographic dictionary, the Hubbard-Stratonovich method is equivalent with the much simpler variational principle approach, in the following we will use the latter to find the desired field-theoretical relation.

Step 1: the relation between the deformed and undeformed sources and vevs

The definition of the $T\bar{T}$ deformation was to incrementally add to the QFT action the $T\bar{T}$ operator of the deformed theory

$$\partial_\mu S^{[\mu]}_{QFT} = -\frac{1}{2}\int d^2x\sqrt{\gamma}(T^{\alpha\beta}T_{\alpha\beta} - T^2)_\mu \tag{4.11}$$

Here the source coupling to the stress tensor is the background metric $\gamma^{\alpha\beta}$ and, by definition, we have $\delta S = \frac{1}{2}\int d^2x\sqrt{\gamma}\,T_{\alpha\beta}\delta\gamma^{\alpha\beta}$. These definitions make sense in the classical, large N limit; indeed, outside this limit it is not clear if the operator can be defined on a non-flat background metric. We are working in euclidean signature and follow the conventions of [5]. The variations of the generating functionals in two nearby $T\bar{T}$ - deformed CFTs related by infinitesimally changing $\mu \to \mu + \Delta\mu$ satisfy

$$\begin{aligned}\delta W^{[\mu+\Delta\mu]} &= \delta W^{[\mu]} - \Delta\mu\,\delta S_{(T\bar{T})_\mu} = \frac{1}{2}\int d^2x\sqrt{\gamma}\,T^{[\mu]}_{\alpha\beta}\delta\gamma^{\alpha\beta}_{[\mu]} - \Delta\mu\,\delta S_{(T\bar{T})_\mu} \\ &\equiv \frac{1}{2}\int d^2x\left(\sqrt{\gamma}\,T_{\alpha\beta}\delta\gamma^{\alpha\beta}\right)_{[\mu+\Delta\mu]}\end{aligned} \tag{4.12}$$

In the limit $\Delta\mu \to 0$, this equation can be rewritten as

$$\partial_\mu\left(\frac{1}{2}\int d^2x\sqrt{\gamma}\,T^{[\mu]}_{\alpha\beta}\delta\gamma^{\alpha\beta}_{[\mu]}\right) = \delta\left(\frac{1}{2}\int d^2x\sqrt{\gamma}(T^{\alpha\beta}T_{\alpha\beta} - T^2)_\mu\right) \tag{4.13}$$

We need to solve this for an arbitrary variation of the sources $\gamma^{[\mu]}$. Separately equating the terms under the variations and their coefficients, one obtains the following flow equations for the source and the expectation value of the stress tensor (see [5] for details)

$$\partial_\mu\gamma_{\alpha\beta} = -2(T_{\alpha\beta} - \gamma_{\alpha\beta}T) \equiv -2\,\hat{T}_{\alpha\beta}\,, \qquad \partial_\mu\hat{T}_{\alpha\beta} = -\hat{T}_{\alpha\gamma}\hat{T}_\beta{}^\gamma\,, \qquad \partial_\mu(\hat{T}_{\alpha\gamma}\hat{T}_\beta{}^\gamma) = 0 \tag{4.14}$$

This set of equations is trivial to integrate, and the solution is

$$\boxed{\begin{aligned}\gamma^{[\mu]}_{\alpha\beta} &= \gamma^{[0]}_{\alpha\beta} - 2\mu\,\hat{T}^{[0]}_{\alpha\beta} + \mu^2\hat{T}^{[0]}_{\alpha\rho}\,\hat{T}^{[0]}_{\sigma\beta}\,\gamma^{[0]\rho\sigma} \\ \hat{T}^{[\mu]}_{\alpha\beta} &= \hat{T}^{[0]}_{\alpha\beta} - \mu\,\hat{T}^{[0]}_{\alpha\rho}\,\hat{T}^{[0]}_{\sigma\beta}\,\gamma^{[0]\rho\sigma}\end{aligned}} \tag{4.15}$$

[15] The story is a bit more complicated than this. In the usual case of scalar deformations, one normally requires that $\mathcal{O}^2$ be a relevant or marginal operator, so that its effect is tractable. This implies that the dimension of $\mathcal{O}$ is $\leq d/2$. Such an operator corresponds to a bulk field quantized with Neumann boundary conditions, or alternate quantization. The case of $T\bar{T}$ is in a certain sense simpler, in that one deforms the theory in the usual quantization. The reason that the effect of this *irrelevant* double-trace operator is still tractable from a holographic point of view is that the dual bulk field is the $3d$ metric, which is not dynamical. In particular, its non-normalizable part is pure gauge and thus does not backreact on the local geometry, even at full non-linear level.

which represent the relations between the sources and expectation values in the $T\bar{T}$ - deformed CFT and the undeformed one. The expectation value of the deformed stress tensor is determined by that of $\hat{T}$ via

$$T^{[\mu]}_{\alpha\beta} = \hat{T}^{[\mu]}_{\alpha\beta} - \gamma^{[\mu]}_{\alpha\beta}\,\hat{T}^{[\mu]} \tag{4.16}$$

Notice these relations are rather non-linear and the change in the expectation value of the stress tensor is non-trivial. We emphasize these relations follow from the definition of the $T\bar{T}$ deformation via a purely (large N) field theoretical derivation. The flow equations (4.14) can be additionally used to show that $\partial_\mu(R\sqrt{\gamma}) = 0$, that the deforming operator does not flow, $\partial_\mu(\sqrt{\gamma}\,\mathcal{O}_{T\bar{T}}) = 0$, and the trace relation $T^{[\mu]} = c/24\pi R^{[\mu]} - \mu\,\mathcal{O}^{[\mu]}_{T\bar{T}}$. See [5] for more details.

One can in principle add sources for "matter" fields, here taken for simplicity to be scalars. Since the double-trace deformation only involves the stress tensor, the variation with respect to the matter operator sources just goes along for the ride - at infinitsimal level at least - as can be seen from the variational principle

$$\int (\sqrt{\gamma}\,\mathcal{O}\delta J)_{[0]} = \int (\sqrt{\gamma}\,\mathcal{O}\delta J)_{[\mu]} \tag{4.17}$$

This implies that the sources for matter operators are unaffected by the deformation, while the expectation values are related via $\mathcal{O}^{[\mu]} = \mathcal{O}^{[0]}\sqrt{\gamma^{[0]}/\gamma^{[\mu]}}$.

Step 2: the holographic dictionary

Having found the relation between the deformed and undeformed stress tensor data at large N, the second step of the holographic dictionary consists in interpreting these data in terms of the asymptotic values of the bulk fields in the dual asymptotically AdS_3 spacetime. For this, we need to briefly review the holographic dictionary for the stress tensor in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$.

As explained at the beginning of this section, the holographic dual to a large c, large gap CFT_2 is $3d$ Einstein gravity coupled to various light matter fields. We will concentrate on the gravitational sector, which is the one that captures the dynamics of the stress tensor in the dual CFT. The asymptotic solution for the three-dimensional metric is simplest in the so-called Fefferman-Graham gauge ($g_{\rho\rho} = \ell^2/4\rho^2$, $g_{\rho\alpha} = 0$, where ρ is the radial coordinate and ℓ is the AdS_3 length) and is given by the following expansion

$$ds^2 = \ell^2\frac{d\rho^2}{4\rho^2} + \left(\frac{g^{(0)}_{\alpha\beta}(x^\alpha)}{\rho} + g^{(2)}_{\alpha\beta}(x^\alpha) + \ldots\right) dx^\alpha dx^\beta \tag{4.18}$$

This expansion holds at *non-linear level* provided the matter fields satisfy reasonable boundary conditions (i.e., the non-normalizable mode $\phi^{(0)}$ is set to zero beyond linearized level; however, arbitrary normalizable modes of the matter fields are allowed at full non-linear level). The two leading terms written above are universal and correspond to the source and expectation value of the dual CFT_2 stress tensor. More precisely,

$$g^{(0)}_{\alpha\beta} = \gamma^{[0]}_{\alpha\beta}\,, \qquad g^{(2)}_{\alpha\beta} = 8\pi G\ell\,\hat{T}^{[0]}_{\alpha\beta} \tag{4.19}$$

The asymptotic Einstein equations fix the trace and divergence of $g^{(2)}$ in terms of $g^{(0)}$; these correspond to the holographic Ward identities of the CFT stress tensor - see e.g. [22] for details. The dots correspond to terms that are subleading in the ρ expansion. They are non-universal and depend on the particular matter field *expectation values* that have been turned on.

We now have all the ingredients to describe the holographic dictionary for $T\bar{T}$ - deformed CFTs. The coefficients $g^{(0,2)}$ above encode the source and the expectation value of the stress tensor in the *undeformed* CFT, so they should be identified with $\gamma^{[0]}$ and respectively $\hat{T}^{[0]}$ in (4.15). The source for the deformed stress tensor (i.e., the background metric) in the $T\bar{T}$ - deformed CFT is $\gamma^{[\mu]}$, given in (4.15). Using the undeformed holographic dictionary (4.19), we find the following expression for $\gamma^{[\mu]}$ in terms of the coefficients appearing in the asymptotic metric expansion

$$\gamma^{[\mu]}_{\alpha\beta} = g^{(0)}_{\alpha\beta} - \frac{\mu}{4\pi G\ell} g^{(2)}_{\alpha\beta} + \frac{\mu^2}{(8\pi G\ell)^2}\, g^{(2)}_{\alpha\gamma} g^{(0)\,\gamma\delta} g^{(2)}_{\delta\beta} \tag{4.20}$$

Since we are supposed to keep the source $\gamma^{[\mu]}$ in the deformed theory fixed, we find that the fluctuations of the dual bulk metric satisfy a mixed and rather non-linear boundary condition, given by the right-hand side of the above equation. Note that both the normalizable and non-normalizable (boundary metric) mode of the metric are allowed to fluctuate, as long as the above combination is held fixed.

To summarize, the holographic dictionary for $T\bar{T}$-deformed CFTs is

$$Z_{T\bar{T}\text{-}def.\,CFT}\left[J^{[\mu]}, \gamma^{[\mu]}\right] = Z_{grav}\left[\phi^{(0)} = J^{[\mu]}, g^{(0)} - \mu\, g^{(2)} + \frac{\mu^2}{4} g^{(2)} g^{-1}_{(0)} g^{(2)} = \gamma^{[\mu]}\right] \tag{4.21}$$

where, to ease the notation, we have measured μ in units of $4\pi G\ell$. The first argument indicates that all matter fields have same boundary conditions as before the deformation.

Using (4.15) and (4.19), we find that the expectation value of the stress tensor in the deformed theory is given by (4.16), with $\gamma^{[\mu]}$ given in (4.20) and

$$\hat{T}^{[\mu]}_{\alpha\beta} = \frac{1}{8\pi G\ell}\, g^{(2)}_{\alpha\beta} - \frac{\mu}{(8\pi G\ell)^2}\, g^{(2)}_{\alpha\gamma} g^{(0)\,\gamma\delta} g^{(2)}_{\delta\beta} \tag{4.22}$$

A few comments are in place:

i) just like in the undeformed case, the expectation value of the stress tensor *only* involves the *universal* asymptotic metric coefficients $g^{(0)}$ and $g^{(2)}$ that encode the energy and momentum density of the initial state in the undeformed CFT.This will ultimately be responsible for the universality of the deformed energy formula, as we will show.

ii) unlike in the undeformed CFT case, $\hat{T}_{\alpha\beta}$ bears a rather non-linear relation to the asymptotic data, and is computed by a different formula than in AdS_3 with Dirichlet boundary conditions

iii) the boundary conditions for matter field are unaffected, as follows from (4.17)

Building the gravitational phase space

Having established what the sources and expectation values in the $T\bar{T}$ - deformed CFT correspond to in terms of the asymptotic behaviour of the metric components, the next natural question is to understand the phase space of the bulk theory, i.e. what are the most general allowed metric fluctuations for a given, fixed metric $\gamma^{[\mu]}$ in the $T\bar{T}$ - deformed CFT. Knowing the answer for arbitrary $\gamma^{[\mu]}$ is useful, e.g. for computing correlation functions of the stress tensor by repeated functional differentiation of the holographic one-point function.

For $\mu = 0$, the answer to this question is well-known: one allows for all tensors $g^{(2)}$ that are compatible with the holographic Ward identities, which fix its trace and divergence in terms of the boundary metric $g^{(0)}$, which is held fixed. For $\mu \neq 0$, the problem is significantly more complicated, since one needs to find the most general solution for $g^{(0,2)}$, subject to the holographic Ward identities and the non-linear boundary condition (4.20). Solving this set of non-linear algebraic and differential equations for general $\gamma^{[\mu]}$ appears rather cumbersome. We will therefore concentrate on the much simpler problem of finding the most general $g^{(0,2)}$ for which $\gamma^{[\mu]}_{\alpha\beta} = \eta_{\alpha\beta}$. The price to pay is that since we restrict the $T\bar{T}$ metric to be flat, we will not have access to arbitrary correlation functions of the stress tensor, but only to the one-point functions (4.22).

In the particular case $\gamma^{[\mu]}_{\alpha\beta} = \eta_{\alpha\beta}$, i.e. when the $T\bar{T}$-deformed CFT lives on flat space, there is a trick to solve the general equations, which we will now explain. Using the fact that the combination $R\sqrt{\gamma}$ is constant along the flow and that $R[\gamma^{[\mu]}] = 0$, we conclude that $R[g^{(0)}] = 0$, so $g^{(0)}$ and $\gamma^{[\mu]}$

are diffeomorphic to each other (this is not generally the case). Since the Ricci scalar of the boundary metric $g^{(0)}$ vanishes, this implies that there exists a coordinate system in which the metric $\gamma^{(0)}$ equals the two-dimensional Minkowski metric η.

Let U, V be the coordinates of the $T\bar{T}$-deformed CFT, in terms of which $\gamma^{[\mu]} = \eta$, and u, v be the auxiliary set of coordinates, in terms of which $g^{(0)} = \eta$. The asymptotic solution for the bulk metric in the u, v coordinate system is extremely simple, as the holographic Ward identities can be solved explicitly in terms of two arbitrary functions $\mathcal{L}(u)$ and $\bar{\mathcal{L}}(v)$

$$ds^2 = \frac{\ell^2}{4}\frac{d\rho^2}{\rho^2} + \frac{dudv}{\rho} + \mathcal{L}(u)\, du^2 + \bar{\mathcal{L}}(v)\, dv^2 + \ldots \tag{4.23}$$

where the $\ldots$ are $\mathcal{O}(\rho)$ and are non-universal. The functions $\mathcal{L}(u)$ and $\bar{\mathcal{L}}(v)$ are proportional to the expectation values of the holomorphic and, respectively, antiholomorphic stress tensor components T_{uu} and T_{vv}. Plugging in the coefficients $g^{(0,2)}$ read off from above into (4.20), we obtain the following expression for $\gamma^{[\mu]}$ in the u, v coordinate system

$$\gamma^{[\mu]}_{\alpha\beta} dx^\alpha dx^\beta = dudv + \rho_c(\mathcal{L}(u)du^2 + \bar{\mathcal{L}}(v)dv^2) + \rho_c^2 \mathcal{L}(u)\bar{\mathcal{L}}(v)dudv = \big(du + \rho_c\bar{\mathcal{L}}(v)dv\big)\ (dv + \rho_c\mathcal{L}(u)du) \tag{4.24}$$

where we have introduced the shortcut notation

$$\rho_c \equiv -\frac{\mu}{4\pi G\ell} \tag{4.25}$$

The line element (4.24) must equal the Minkowski line element $dUdV$ in terms of the $T\bar{T}$ coordinates U, V. This yields the following relation between the two sets of coordinates

$$U = u + \rho_c \int^v \bar{\mathcal{L}}(v)dv\ , \qquad V = v + \rho_c \int^u \mathcal{L}(u)du \tag{4.26}$$

which is precisely the field-dependent coordinate transformation (3.16), now rederived from holography.

Thus, the most general $g^{(0,2)}$ satisfying the holographic Ward identities and the boundary condition $\gamma^{[\mu]} = \eta$ are given by the asymptotic metric coefficients encoded in (4.23), translated back to the U, V coordinate system. Explicitly, we have

$$g^{(0)}_{\alpha\beta}\, dx^\alpha dx^\beta = dudv = \frac{(dU - \rho_c\bar{\mathcal{L}}(v)dV)(dV - \rho_c\mathcal{L}(u)dU)}{(1-\rho_c^2\mathcal{L}(u)\bar{\mathcal{L}}(v))^2} \tag{4.27}$$

$$g^{(2)}_{\alpha\beta}\, dx^\alpha dx^\beta = \mathcal{L}(u)du^2 + \bar{\mathcal{L}}(v)dv^2 = \frac{\big(1+\rho_c^2\mathcal{L}(u)\bar{\mathcal{L}}(v)\big)\,(\mathcal{L}(u)\,dU^2 + \bar{\mathcal{L}}(v)\,dV^2) - 4\rho_c\mathcal{L}(u)\bar{\mathcal{L}}(v)\,dUdV}{\big(1-\rho_c^2\mathcal{L}(u)\bar{\mathcal{L}}(v)\big)^2}$$

The expectation value of the stress tensor can be read off from (4.22), and is given by

$$T^{[\mu]}_{\alpha\beta} dx^\alpha dx^\beta = (\hat{T}^{[\mu]}_{\alpha\beta} - \eta_{\alpha\beta}\,\eta^{\gamma\delta}\hat{T}^{[\mu]}_{\gamma\delta})\, dx^\alpha dx^\beta = \frac{\mathcal{L}(u)dU^2 + \bar{\mathcal{L}}(v)dV^2 + 2\rho_c\mathcal{L}(u)\bar{\mathcal{L}}(v)dUdV}{8\pi G\ell(1-\rho_c^2\mathcal{L}(u)\bar{\mathcal{L}}(v))} \tag{4.28}$$

in the U, V coordinate system.

Thus, we find that, just like in the case of AdS_3 with Dirichlet boundary conditions, the space of bulk solutions satisfying the mixed boundary conditions with $\gamma^{[\mu]} = \eta$ is parametrized by two arbitrary functions, $\mathcal{L}(u)$ and $\bar{\mathcal{L}}(v)$, where now the coordinates u, v are field-dependent and are determined via (4.26). In addition, one can have arbitrary matter expectation values turned on, which appear at subleading order. The solutions are still asymptotically locally AdS_3, since the non-normalizable mode of the metric (4.27) is pure gauge.

Holographic calculation of the deformed energies

In the first section, we derived the exact formula (2.26) for the $T\bar{T}$-deformed energy spectrum on a cylinder, which is a function of μ, R and only the initial energy and momentum of the state. As a basic check of the proposed holographic dictionary, we would like to reproduce this formula from a holographic calculation.

Since we need to compute the energies in finite volume, we take the AdS_3 boundary to be a cylinder, with $U, V = \sigma \pm t$ where $\sigma \sim \sigma + R$, and consider an energy-momentum eigenstate, characterized by $\mathcal{L}(u) = \mathcal{L}_\mu$ and $\bar{\mathcal{L}}(v) = \bar{\mathcal{L}}_\mu$, both constant. The deformed energy and angular momentum are given by

$$E_\mu = \int_0^R d\sigma\, T_{tt}^{[\mu]} = \frac{R}{8\pi G\ell} \frac{\mathcal{L}_\mu + \bar{\mathcal{L}}_\mu - 2\rho_c \mathcal{L}_\mu \bar{\mathcal{L}}_\mu}{1 - \rho_c^2 \mathcal{L}_\mu \bar{\mathcal{L}}_\mu} , \qquad J_\mu = \int_0^R d\sigma\, T_{t\sigma}^{[\mu]} = \frac{R}{8\pi G\ell} \frac{\mathcal{L}_\mu - \bar{\mathcal{L}}_\mu}{1 - \rho_c^2 \mathcal{L}_\mu \bar{\mathcal{L}}_\mu} \tag{4.29}$$

where we simply plugged in the expression (4.28) for the stress tensor components.

The formula (2.26) expresses the enegy of an energy-momentum eigenstate in the deformed CFT in terms of the energy of the "same" eigenstate in the undeformed one, where by "same" we mean that the two states are connected by adiabatic evolution as μ is increased from zero to its final value. Let the corresponding state in the undeformed CFT be characterized by the left/right-moving energies $\mathcal{L}_0, \bar{\mathcal{L}}_0$. In order to reproduce (2.26), we need to find a way to relate $\mathcal{L}_0, \bar{\mathcal{L}}_0$ to $\mathcal{L}_\mu, \bar{\mathcal{L}}_\mu$.

This question is not as simple as it may seem, since we are comparing states in different theories, i.e. belonging to different gravitational phase spaces, where the asymptotic metric satisfies *different* boundary conditions. A good strategy is to look for quantities that are invariant along the $T\bar{T}$ flow. For example, the momentum of the state is constant along the flow, being quantized, which gives one relation: $J_\mu = J_0 = R(\mathcal{L}_0 - \bar{\mathcal{L}}_0)/(8\pi G\ell)$.

Another quantity that is invariant along the flow is the degeneracy of states around a state labeled by the CFT conformal dimensions $h, \bar{h}$; this was discussed in the first lecture. For $h, \bar{h} >> c/24$, this degeneracy is computed by the horizon area of the black hole characterized by $h, \bar{h}$. Note that this black hole need *not* be dual to the CFT state whose energy we are computing (which can be atypical, represented e.g. by a matter field configuration in the bulk); we are simply using it as an auxiliary tool to estimate the degeneracy of states of similar energy to the CFT state of interest.

To calculate the area of the black hole horizon, we need the full bulk solution. This is fixed by the fact that black holes are solutions of pure $3d$ gravity, for which the Fefferman-Graham expansion happens to truncate at $\mathcal{O}(\rho)$ [23]

$$ds^2 = \frac{\ell^2}{4}\frac{d\rho^2}{\rho^2} + \frac{1}{\rho}\left(g^{(0)}_{\alpha\beta} + \rho\, g^{(2)}_{\alpha\beta} + \frac{\rho^2}{4} g^{(2)}_{\alpha\gamma} g^{(0)\,\gamma\delta} g^{(2)}_{\delta\beta}\right) dx^\alpha dx^\beta \tag{4.30}$$

For the deformed black holes, $g^{(0)}$ and $g^{(2)}$ are given in (4.27). The horizon is located at $\rho_h = (\mathcal{L}_\mu \bar{\mathcal{L}}_\mu)^{-1/2}$ and its area is given by

$$\mathcal{A}_\mu = R \frac{\sqrt{\mathcal{L}_\mu} + \sqrt{\bar{\mathcal{L}}_\mu}}{1 + \rho_c \sqrt{\mathcal{L}_\mu \bar{\mathcal{L}}_\mu}} \tag{4.31}$$

Equating this to the original area $\mathcal{A}_0 = R(\sqrt{\mathcal{L}_0} + \sqrt{\bar{\mathcal{L}}_0})$ yields an second relation between $\mathcal{L}_\mu, \bar{\mathcal{L}}_\mu$ and $\mathcal{L}_0, \bar{\mathcal{L}}_0$, which allows one to solve for the first in terms of the latter[16]. Plugging this into (4.29), one finds precisely the QFT answer (2.26). The details of the calculationcan be found in [5].

Thus, we find that the proposed holographic dictionary perfectly reproduces the QFT answer for the deformed energies. This match works for both signs of μ. Remember from our discussion in the first lecture that for $\mu > 0$, the vacuum and the states close to it can acquire complex energy, whereas

[16] The exact, if unilluminating expression is: $\mathcal{L}_\mu = \frac{\mp(1+(\mathcal{L}_0 - \bar{\mathcal{L}}_0)\rho_c)\sqrt{\rho_c^2(\mathcal{L}_0 - \bar{\mathcal{L}}_0)^2 - 2\rho_c(\mathcal{L}_0 + \bar{\mathcal{L}}_0) + 1} + \rho_c^2(\mathcal{L}_0 - \bar{\mathcal{L}}_0)^2 - 2\bar{\mathcal{L}}_0\rho_c + 1}{2\mathcal{L}_0\rho_c^2}$.

for $\mu < 0$ all states above a certain energy (2.29) do so. From the point of view of our calculation, what accounts for this behaviour is that in these parameter ranges, there is no real, CTC-free bulk metric that satisfies the mixed boundary conditions. This is simply a bulk manifestation of the fact that for $\mu > 0$, the deformed theory cannot be put on a circle of radius smaller than R_c, whereas for $\mu < 0$, it cannot be put in finite volume at all.

It is important to emphasize that the deformed energy only depends on the *asymptotic value* of the metric, encoded by $\mathcal{L}_\mu$, $\bar{\mathcal{L}}_\mu$, and so this derivation works also if matter fields are turned on. This makes perfect sense from the point of view of the deformed energy, as a *universal* energy formula in the QFT can only depend on the *universal* asymptotic data in the bulk.

Asymptotic symmetries

In gravity, on a manifold with a (conformal) boundary, the conserved charges are given by boundary integrals over a fixed time slice. Given a set of boundary conditions on the metric and other fields, it is natural to ask what are the associated *asymptotic symmetries* - the set of diffeomorphisms that are allowed by the boundary conditions and carry non-trivial conserved charges - as well as the Dirac bracket algebra of these charges, known as the *asymptotic symmetry group* (ASG). For example, in AdS_3 with usual Dirichlet boundary conditions, the ASG is famously known to be infinite-dimensional, and consists of two commuting copies of the Virasoro algebra, with central charge $c = 3\ell/2G$.

We can equally well perform the ASG analysis for the mixed boundary conditions associated to $T\bar{T}$-deformed CFTs. As we saw, the most general bulk solution is parametrized by two arbitrary functions $\mathcal{L}(u), \bar{\mathcal{L}}(v)$ of the auxiliary coordinates u, v defined in (4.26). The diffeomorphisms that are allowed yet non-trivial act on the coordinates U, V of the $T\bar{T}$ - deformed CFT by shifts depending on arbitrary functions of these state-dependent coordinates, i.e.

$$U \to U + \epsilon f(u) \, , \qquad V \to V + \epsilon \bar{f}(v) \tag{4.32}$$

The conserved charges are computed using the usual formulae

$$Q_f = \int_0^R d\sigma \, T^{[\mu]}_{t\alpha} \, \xi^\alpha_f \, , \qquad \bar{Q}_{\bar{f}} = \int_0^R d\sigma \, T^{[\mu]}_{t\alpha} \, \xi^\alpha_{\bar{f}} \tag{4.33}$$

After some rather tedious calculations, it can be shown that the charge algebra for the Fourier modes of $f, \bar{f}$ consists of two commuting copies of the Virasoro algebra, with the same central extension as in the undeformed CFT. Since in the holographic context, the ASG of the bulk gravitational theory is identified with the symmetries of its boundary QFT dual, this calculation strongly suggests that at least in the classical large N limit, $T\bar{T}$-deformed CFTs possess full Virasoro $\times$ Virasoro symmetry. These symmetries are however unusual, in that they depend on the field configuration. If the existence of these symmetries is confirmed at non-perturbative level, it could lead to an interpretation of $T\bar{T}$ - deformed CFTs as non-local generalizations of a usual CFT.

4.3 Demystifying the finite bulk cutoff proposal

In the above, we presented a first principles *derivation* of the large N holographic dictionary for $T\bar{T}$-deformed CFTs, for both signs of μ, by simply applying the rules of AdS/CFT to this particular case. We would now like to comment on the relation between this dictionary and an earlier proposal by [4], according to which $T\bar{T}$ - deformed CFTs with $\mu < 0$ (in our conventions) are dual to AdS_3 gravity with a sharp radial cutoff.

For the sake of clarity, it will be useful to split the proposal and results of [4] into two logically distinct steps:

i) the (highly non-trivial) observation that various $T\bar{T}$ observables (such as the deformed energy spectrum, the speed of sound, the thermodynamic relations) match the measurements of a bulk observer sitting at a fixed radial position $r_c \sim 1/\sqrt{|\mu|}$ in the background of a BTZ black hole

ii) the proposal that the bulk degrees of freedom outside the r_c surface be removed: this is the geometric cutoff proposal proper, which is motivated by the presence of complex energies for $\mu < 0$.

   Remember from the discussion in the first lecture that there is a maximum energy, E_{max} (2.29), above which all states acquire complex energies. The authors of [4] made the interesting observation that for a black hole whose mass is at the threshhold E_{max}, its Schwarzschild radius reaches the r_c surface. Thus, by excluding black holes supported on radii larger than r_c, it appears that one excludes the complex energy states from the spectrum. The energy E_{max} was interpreted as UV cutoff in the deformed CFT, and the finite number of states below this cutoff would correspond to the finite number of states of bulk quantum gravity in a finite region.

The relation between a radial cutoff in the bulk and a UV cutoff in the boundary has a long history in terms of the holographic renormalization group [24, 25, 26] : integrating out degrees of freedom above some UV scale in the boundary would correspond to integrating out the fluctuations of the bulk fields outside a given radius. However, the relation between the field-theory cutoff and the bulk radius was never made precise.

Notice that for the case at hand, the reduction in the number of degrees of freedom implicit in holographic RG would be in conflict with the integrability and likely UV completeness of $T\bar{T}$ - deformed *two-dimensional* CFTs we have been arguing for. Therefore, from this point of view, the proposal that the dual bulk theory should have bulk degrees of freedom removed is at least counterintuitive.

In this section, we will discuss the proposal of [4] from the perspective of the holographic dictionary we have described. Concretely:

i) we show that when $\mu < 0$ and we concentrate on *on-shell* solutions of *pure gravity*, the mixed boundary conditions (4.20) effectively reduce to Dirichet boundary conditions at a specific bulk radius, independent of the energy. Also, all $T\bar{T}$ observables coincide with the measurements of an observer at this fixed radius.

   Thus, all the checks performed in step i) of [4]'s proposal are simply checks of this effective Dirichlet boundary condition, which follows from the mixed ones under the particular circumstances listed above. From the point of view of the mixed boundary conditions, the match to the observations at finite bulk radius is a pure coincidence - albeit a fascinating one - having to do with the particular way the asymptotic solution is extended into the bulk. This coincidence *no longer happens* once matter field profiles are turned on.

ii) we argue that the current evidence for a sharp geometric cutoff in the bulk is uncompelling:

   – from a field-theoretical perspective, the geometric cutoff proposal is in tension with the integrability and UV completeness of $T\bar{T}$ - deformed CFTs
   – it is not clear that E_{max} in (2.29) should be interpreted as a UV cutoff, due to the explicit appearance of the IR scale.
   – we show that once matter fields are turned on, a holographic dictionary of the form suggested by [4]
   $$Z_{T\bar{T}}[\mu] = Z_{grav}[r < r_c] \tag{4.34}$$
   cannot hold: the energies do *not* match and moreover, the onset of the complex energy states can be shown to have nothing to do with a bulk distance or the presence of a horizon
   – to the extent that the central charge of the ASG we discussed is a measure of the number of degrees of freedom in the $T\bar{T}$ - deformed CFT, the fact that it is a μ - independent constant that equals the original CFT central charge is nicely consistent with integrability, but not clearly consistent with a UV cutoff interpretation

We will now proceed to explaining these points in turn. For details of the calculations, see [5].

Relation to an observer at a fixed radial distance in the bulk

The analysis in section 4.2 was completely general: in particular, arbitrary (normalizable) matter field profiles could be turned on. We will now concentrate on pure gravity solutions, i.e. those satisfying $R_{\mu\nu} + 2\ell^{-2} g_{\mu\nu} = 0$ everywhere, and not only asymptotically. Because pure $3d$ gravity has no propagating degrees of freedom, such solutions are locally diffeomorphic to AdS_3.

For pure gravity solutions, the Fefferman-Graham expansion for the metric truncates, and the most general form of the metric is given by (4.30). It is then easy to notice that for $\mu < 0$, the $T\bar{T}$ metric $\gamma^{[\mu]}$ in (4.20) precisely *happens to coincide* with the induced metric on a surface of constant $\rho = \rho_c$, with

$$\rho_c = -\frac{\mu}{4\pi G\ell} \tag{4.35}$$

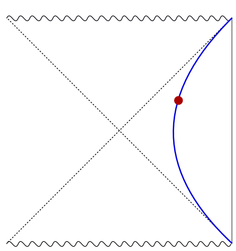
Thus, for pure on-shell gravity solutions with $\mu < 0$ *only*, the mixed boundary conditions at infinity that we have derived coincide with Dirichlet boundary conditions at $\rho = \rho_c$. The coincidence further extends to the stress tensor, as it can be shown that under the same conditions, the expression (4.16), (4.22) for the stress tensor coincides with the Brown-York stress tensor computed on the $\rho = \rho_c$ surface, with a particular counterterm added

$$T_{\alpha\beta} = -\frac{1}{8\pi G}(K_{\alpha\beta} - g_{\alpha\beta}K + \ell^{-1} g_{\alpha\beta}) \tag{4.36}$$

where $K_{\alpha\beta}$ is the extrinsic curvature of the $\rho = \rho_c$ surface and $g_{\alpha\beta} = \gamma^{[\mu]}_{\alpha\beta}/\rho_c$ is the induced metric.

Exercise: By explicitly computing the extrinsic curvature of the $\rho = \rho_c$ surface in the metric (4.30), show that the right-hand side of (4.36) precisely coincides with (4.16), (4.22).

This coincidence explains why the deformed energy computed with this stress tensor agrees with the energy measured by an accelerated bulk observer on the $\rho = \rho_c$ surface. Notice that this interpretation is only possible for CFT states dual to black holes in the bulk. However, since we expect that high-energy typical states are modelled by black holes, we see that the interpretation of [4] that $T\bar{T}$ "moves the CFT into the bulk" does hold for the vast majority of CFT high-energy states, as indicated in the figure below.



In typical high energy states, the $T\bar{T}$-deformed CFT still has the interpretation of describing the experience of an accelerated observer's laboratory in the bulk, located at fixed radial coordinate r_c.

It is worth pointing out that even for the case of pure gravity, the details of the energy calculation in [4] and [5] are not the same. In the calculation described in section 4.2, the black holes at different values of μ belonged to *different* phase spaces, with *different* asymptotic boundary conditions given by (4.27). The surface where $\gamma^{[\mu]} = \eta$ corresponded to $\rho_c \propto |\mu|$ in *Fefferman-Graham* coordinates for these metrics.

By contrast, in [4] the energies that are matched are measured on a surface of fixed *Schwarzschild* coordinate $r_c \propto |\mu|^{-1/2}$, in the background of the usual BTZ black hole (for which $g^{(0)} = \eta$) *for all* μ. Since the relation between the Fefferman-Graham and the Schwarzschild coordinate, as well as that between (4.27) and BTZ, depends on the parameters of the black hole, it is not clear whether the ρ_c and r_c surfaces are the same, and thus why the energy calculation matches. In [5], the relation between the Schwarzschild and the Fefferman-Graham radial coordinate was worked out, which is indeed field-dependent, and reads

$$r^2(\rho) = \frac{\left(1+\mathcal{L}_\mu(\rho-\rho_c)-\bar{\mathcal{L}}_\mu\mathcal{L}_\mu\rho\rho_c\right)\left(1+\bar{\mathcal{L}}_\mu(\rho-\rho_c)-\bar{\mathcal{L}}_\mu\mathcal{L}_\mu\rho\rho_c\right)}{\rho\left(1-\rho_c^2\mathcal{L}_\mu\bar{\mathcal{L}}_\mu\right)^2} \tag{4.37}$$

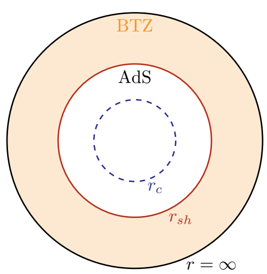
Notice that precisely for $\rho = \rho_c$, the dependence on the black hole parameters drops out and the surface corresponds to r_c, explaining why the two energy calculations agree. This looks like another remarkable coincidence.

Including matter

It should be clear from our discussion that the coincidence between the $T\bar{T}$ observables and the measurements at the finite radius in the bulk no longer happens if matter field profiles are turned on. [5] analysed an extremely simple setup to exemplify this point. It consists of a thin shell of matter with mass M and radius r_{sh}, taken to be much larger than the associated Schwarzschild radius, r_{Schw}. The geometry is BTZ for $r > r_{sh}$, and vacuum AdS_3 inside the shell.

We now consider the state dual to this geometry in the $T\bar{T}$ - deformed CFT. Since the calculation (explained in section 4.2) of the deformed energy using the mixed boundary conditions only takes as input the two leading coefficients in the asymptotic metric expansion, which for the state at $\mu = 0$ are given by their values in BTZ, it is clear that for any μ (positive or negative), the result will reproduce the universal energy formula (2.26) in the deformed CFT.

It is also easy to see that for $\mu < 0$, Dirichlet boundary conditions on the $r_c \sim 1/\sqrt{|\mu|}$ surface *do not* correctly reproduce the boundary QFT prediction for the deformed energy of this state. Indeed, if $|\mu|$ is large enough so that $r_c < r_{sh}$, the geometry inside the shell is just vacuum AdS_3, which does not have the correct energy. Thus, the coincidence between the predictions of the mixed boundary conditions and the effective Dirichlet one at r_c only happens for states described by purely gravitational configurations, where the asymptotic solution is extended into the bulk in the particular way (4.30).



A constant-time slice through the thin shell geometry, which equals BTZ in the shaded outer region and vacuum AdS inside. When the r_c surface is inside the shell, the induced stress tensor is just that of vacuum AdS. The energy on this surface does not agree with the field theory answer for the energy of the deformed theory.

Comments on the finite bulk cutoff proposal

As explained, the authors of [4] went on further to propose that the bulk geometry outside the r_c surface be removed. One reason that one may be tempted to identify the bulk $r_c \sim 1/\sqrt{|\mu|}$ surface with a sharp radial cutoff is that the mass of a black hole whose horizon reaches this surface coincides precisely with the maximum energy (2.29) before the onset of the complex energies. Thus, by removing the geometry outside this radius, one can remove the complex energy states.

It should be clear that this argument only works for black hole states, for which there is a fixed relation between their size (the horizon radius) and their mass. For more general configurations containing matter fields, their spatial support is generally much larger than their associated Schwarzschild radius, and this intuitive picture does not apply, as we now exemplify using the thin shell toy model.

Consider again the thin matter shell configuration, with μ chosen so that $r_c < r_{sh}$, and then start increasing the mass of the shell. The complex energy states should set in when the Schwarzschild radius of the shell reaches r_c; however, nothing special happens to the geometry at this value of the mass, and in particular there is no horizon[17].

However, the QFT formula (2.26) predicts that the energy will become complex. The way that the mixed boundary conditions reproduce this is that they require the solution to belong to the deformed phase space, satisfying the deformed boundary condition (4.20). When the mass is large enough so that $r_{Schw} > r_c$, there is simply no real solution for the metric satisfying this boundary condition. Thus, the complex energies are related to a breakdown that happens near the AdS boundary, which only depends on the energy and is unrelated to the presence of a horizon deep inside the bulk.

Note that in order to reach the complex energy states without the shell turning into a black hole, we needed to take $r_c < r_{sh}$. One may argue that since in the proposal of [4], the geometry outside r_c is to be removed, then this configuration is not the correct one to consider. We only see two other options, both of which are problematic. If one decides to keep the shell states with $r_{Schw} < r_c$ but simply remove the region outside r_c in the bulk dual, then the cut off bulk geometry is just vacuum AdS, whose energy does not match that in the dual QFT. One can alternatively discard these states altogether, though there is no particular reason to do so from the boundary QFT point of view, given that their energy is real and lower than E_{max}. While the thin shell example is somewhat unrealistic, these conclusions would hold whenever the bulk contains non-trivial configurations of matter fields.

Thus, we conclude that the relation between the complex energy states and a geometric cutoff at r_c can only hold in pure gravity, for black hole states. Since most known holographic CFTs do contain operators dual to matter fields, this significantly constrains the theories to which the proposal of [4] could apply.

Let us also remark that the terminology "UV cutoff" may be employed with different meanings in the holographic context. In the usual holographic RG literature, it denotes a scale beyond which the bulk/boundary degrees of freedom are integrated out. It is this interpretation of the cutoff that is in tension with the integrability of the $T\bar{T}$ deformation. The second meaning appears to be specific to the example of $T\bar{T}$, and represents the maximum *finite-size* energy before it becomes complex. Of course, this notion of "cutoff" is consistent with the properties of the $T\bar{T}$ deformation. However, as explained in the first lecture, a more reasonable interpretation is that the $\mu < 0$ theory does not make sense in finite volume, and E_{max} is simply the energy beyond which the time advance characteristic of it leads to the formation of CTCs. The match of energies for $E < E_{max}$ does not necessarily imply that the theory with the complex energy states removed is consistent.

Higher dimensions

Following the bulk cutoff proposal for two-dimensional $T\bar{T}$ - deformed CFTs, a number of higher-dimensional generalizations were proposed for large N CFT$_d$'s [27, 28]. Since an operator with the special properties of $T\bar{T}$ likely does not exist in higher dimensions, these references chose a generalization inspired by the radial flow equation in the bulk, and used large N factorization to obtain an analogue of the energy flow equation (2.21). The resulting energies, which also become complex at large level, were shown to precisely match the energies of bulk black holes as measured by an observer at a fixed radial distance $r_c \sim |\mu|^{-1/d}$. It was subsequently proposed that this higher-dimensional generalization of the $T\bar{T}$ deformation should match bulk gravity with a sharp radial cutoff at r_c.

An important difference with two dimensions is that the higher-dimensional $T\bar{T}$ operator is only defined perturbatively in $1/N$, using the dual bulk description; there is no independent QFT definition for it. Since the deformation is chosen so that it matches the predictions of gravity with a sharp radial cutoff, this leads to a different treatment of matter fields. Specifically, while in two dimensions the fact that the deforming operator was exactly $T\bar{T}$ implied that matter fields would continue to have Dirichlet

[17] Following the prescription of [4], we are working with the undeformed, asymptotically BTZ geometry (with $g^{(0)} = \eta$) even though we are studying the deformed CFT.

boundary conditions at infinity, in the higher-dimensional case it is natural to require that they have Dirichlet boundary conditions at the cutoff surface, which results in deforming the boundary QFT by additional multitrace operators associated to matter. Adding these irrelevant multitrace operators to the CFT action likely results in an EFT with a cutoff related to the irrelevant couplings. This cutoff is in principle different from the maximum energy attained before the complex energy states set in, which only exists in finite volume and depends explicitly on the size of the box.

Since the only available definition of the T^2 operator and the rest of the boundary QFT action is through the dual gravitational theory in presence of a sharp radial cutoff, it is not clear whether the above-mentioned match of the energy, as well as subsequent matches of the entanglement entropy, represent checks of an interesting holographic duality, or simply a rewriting of a number of bulk manipulations in a suggestive field-theoretical form. Besides the QFT not having an independent definition, it is also not clear whether the gravitational side of the would-be duality is defined beyong leading order in $1/N$, due to difficulties in imposing hard Dirichlet boundary conditions in presence of gravity. It is certainly interesting though to think of a more encompassing definition of both sides.

5. A single-trace analogue of $T\bar{T}$ and non-AdS holography

In the previous lecture, we discussed the usual, *double-trace* $T\bar{T}$ deformation, whose main effect was to change the asymptotic boundary conditions for the non-dynamical metric. The new boundary data are simply a reinterpretation of the usual AdS/CFT ones, which explains the universality of the deformed spectrum from a holographic perspective. The price to pay for this universality is that the holographic dictionary is a bit boring, at least at the level of (super)gravity, as the geometry stays locally AdS_3.

In this lecture, we will discuss a *single-trace* variant of the $T\bar{T}$ deformation [6]. Unlike the usual $T\bar{T}$ deformation, which can be performed universally on any CFT, this single-trace analogue is defined in a specific string-theoretical setting, in which the boundary CFT_2 takes the form of a symmetric product orbifold; this structure is essential to be able to define the deformation.

The construction of [6] is very interesting because: i) it provides a rare example of *tractable non-AdS* holography; ii) it shows that there exist deformations with very similar properties, particularly in what concerns the UV behaviour, to $T\bar{T}$, which are less universal and thus more interesting for gaining insight into *general* asymptotically fragile theories; iii) it provides a concrete description of a two-dimensional compactification of little string theory.

In this lecture, we start by reviewing the relevant string theory setup, which is the NS5-F1 system. Then, we briefly sketch the construction of the single-trace analogue of the $T\bar{T}$ operator and list a number of checks and predictions.

5.1 The NS5-F1 system

The main player in this string-theoretical story is the NS5-F1 system. I will start with a short review of NS5-branes and then add the F1 strings.

NS5 branes are solitonic objects in string theory, which are magnetically charged under the B-field. In type IIB, they are related to D5-branes via S-duality, while in type IIA they are related to M5-branes via uplift. It is interesting to ask whether there exists a limit in which the modes on the NS5 branes decouple from gravitational physics. Unlike for D-branes, where this limit is a low energy limit ($\alpha' \to 0$), for NS5 branes the limit is, rather

$$g_s \to 0 \,, \qquad \alpha' \;\; fixed \tag{5.1}$$

The worldbrane theory obtained in this limit, called little string theory (LST), is non-trivial and its properties depend strongly on whether we are in type IIA or type IIB, see [29] for a review. Some of its properties are: it is non-local, and in particular it exhibits T-duality (since NS5-branes are left invariant by T-duality along their worldvolume, and $g_s = 0$ is a fixed point of T-duality), it is not gravitational (no masslesss spin two excitation is present in the spectrum), it exhibits a Hagedorn growth of states at high energies. The latter property is best seen from holographic dual.

The background that is holographically dual to LST is obtained from the backreacted solution for N_5 NS5-branes, which in string frame reads

$$ds^2 = dx^\mu dx_\mu + \left(1 + \frac{N_5\alpha'}{r^2}\right)(dr^2 + r^2 d\Omega_3^2)\ , \qquad e^{2\Phi} = g_s^2\left(1 + \frac{N_5\alpha'}{r^2}\right)\ , \qquad H = \star_4 d\Phi \tag{5.2}$$

where $x^\mu \in \mathbb{R}^{1,5}$ span the directions along the brane and r is the radial coordinate in the transverse $\mathbb{R}^4$. In the decoupling limit, the value of the dilaton at infinity, $g_s \to 0$, with $\tilde{r} \equiv r/g_s$ fixed. The spacetime becomes

$$\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times S^3\ , \qquad \phi \equiv \sqrt{N_5\alpha'}\ \ln \tilde{r} \tag{5.3}$$

with a linear dilaton, $\Phi = -\phi/\sqrt{N_5\alpha'}$. In the asymptotic region, the string coupling is small , so the bulk can be analysed in string perturbation theory. The worldsheet CFT is exactly solvable in this region. However, as we approach $r \to 0$, the dilaton grows without bound and the string description breaks down.

As mentioned above, it is easy to exhibit the Hagedorn growth of states using the holographic dual. If the NS5 branes are made near-extremal, the metric becomes[18]

$$ds^2 = -f_{\mathcal{E}}dt^2 + ds^2_{\mathbb{R}^5} + N_5\alpha'\left(\frac{dr^2}{r^2 f_{\mathcal{E}}} + d\Omega_3^2\right)\ , \qquad f_{\mathcal{E}} = 1 - \frac{r_0^2}{r^2} \tag{5.4}$$

where r_0 is the location of the horizon, which is assumed to be large enough so that the string coupling, $N_5\alpha'/r_0^2$, is still small there.

Exercise: By analytically continuing to euclidean time, show that the temperature associated to this black brane is independent of r_0 and is given by $T_H^{-1} = 2\pi\sqrt{N_5\alpha'}$.

A temperature that is independent of the energy implies that $S \propto E$, i.e. Hagedorn behaviour.

So far for the little strings. Let us now turn to the setup of [6]. The first step is to get rid of the strong coupling region at small r. One way to achieve this is to add N_1 F1 strings to the system, which extend along one of the NS5 brane directions, denoted x_1 with $x_1 \sim x_1 + R$, and compactify the remaining directions of the NS5 branes on a T^4. The metric and dilaton then become

$$ds^2 = \frac{-dt^2 + dx_1^2}{f_1} + N_5\alpha'\left(\frac{dr^2}{r^2} + d\Omega_3^2\right) + ds^2_{T^4}\ , \qquad e^{2\Phi} = \frac{N_5\alpha'}{r^2 f_1}\ , \qquad f_1 = 1 + \frac{r_1^2}{r^2}\ , \qquad r_1^2 = \frac{N_1\alpha'}{v} \tag{5.5}$$

where v is the volume of the T^4 in units of $(2\pi)^4\alpha'^2$. The solution is also supported by a B-field. The string coupling at $r = 0$ is inversely proportional to N_1 , so for a very large number of F1 strings, the coupling is small everywhere and we can trust string perturbation theory.

The background (5.5) interpolates between $\mathrm{AdS}_3 \times S^3 \times T^4$ in the IR ($r \to 0$) and $\mathbb{R}_\phi \times \mathbb{R}_t \times S^1_{x_1} \times S^3 \times T^4$ in the UV ($r \to \infty$). The worldsheet sigma model for a string propagating in this background is known exactly. The IR geometry is described by an $SL(2,\mathbb{R}) \times SU(2) \times U(1)^4$ WZW sigma model (plus fermions). The full background can be shown to correspond to an exactly marginal $J^-\bar{J}^-$ deformation of this worldsheet CFT, where J^- is a null component of the $SL(2,\mathbb{R})$ current. The coefficient of this deformation is in principle tunable and will be denoted as λ.

[18] From now on, r is rescaled by a factor of g_s with respect to its value at asymptotically flat infinity.

It is also interesting to note that the full background (5.5) can be obtained via a TsT transformation of the near-horizon F1-NS5 solution.

Exercise: Show that the full background (5.5) can be obtained via a TsT transformation of the IR AdS_3 solution, where TsT stands for: T-duality along x_1, a shift s: $t + \lambda\tilde{x}_1$, where $\tilde{x}_1$ is the T-dual coordinate to x_1, followed by a T-duality back. Determine the value of λ.

5.2 Holographic description of the NS5-F1 system

We would now like to understand the holographic description of the above background. The IR AdS_3 region is described by a dual CFT_2, as expected. The departure away from AdS_3 is by a non-normalizable mode, so it corresponds to an irrelevant deformation of the dual CFT. The deep UV corresponds to a $2d$ compactification of LST. The main progress achieved in [6] was to propose a holographic dual the full background (5.5), thus providing a concrete description of compactified LST.

The holographic dual to the AdS_3 region

Let us first concentrate on the $r \to 0$ region, where the geometry becomes AdS_3[19] supported by purely NS-NS three-form flux. For N_1 very large, the dual CFT is conjectured to be a free symmetric product orbifold CFT, of the form $(\mathcal{M}_{6N_5})^{N_1}/S_{N_1}$, where $\mathcal{M}_{6N_5}$ is a CFT with central charge $6N_5$.

Since the background (5.5) is weakly coupled, it can be studied using peturbative worldsheet techniques. The holographic dictionary provides in this case a map between *single-trace* operators in the boundary CFT and vertex operators in the worldsheet WZW model

$$\mathcal{O}_{bnd}(x) = \int d^2z\, \mathcal{O}_{w-sheet}(x,z) \tag{5.6}$$

In particular, the stress tensor of the boundary CFT is given by the following expression

$$T(x) = \frac{1}{2N_5}\int d^2z (\partial_x J(x,z)\partial_x \Phi_1(x,z) + 2\partial_x^2 J(x,z)\Phi_1(x,z))\bar{J}(\bar{x},\bar{z}) \tag{5.7}$$

where $J(x,z)$ is a convenient packaging of the worldsheet $SL(2,\mathbb{R})$ WZW currents, with spacetime scaling dimension $(-1,0)$

$$J(x,z) = e^{-xJ_0^-}J^+(z)e^{xJ_0^-} = x^2J^- - 2xJ^3 + J^+ \tag{5.8}$$

and $\Phi_h(x,z)$ is an operator with spacetime scaling dimension (h,h), which is a primary both on the worldsheet and in the boundary CFT. It is then easy to check that the stress tensor has the correct dimension $(2,0)$ from the boundary point of view.

Since the boundary CFT is a symmetric product orbifold, the total stress tensor takes the form

$$T(x) = \sum_{i=1}^{N_1} T_i(x) \tag{5.9}$$

where T_i is the stress tensor in a single copy of the $\mathcal{M}_{6N_5}$ CFT. A similar expression holds for the antiholomorphic stress tensor. Note that the (double-trace) $T\bar{T}$ operator is, in this boundary CFT

$$T\bar{T} = \sum_i T_i \sum_j \bar{T}_j \tag{5.10}$$

Using the dictionary (5.6), this operator would correspond to a double worldsheet integral.

[19] More precisely, massless BTZ.

A single-trace analogue of $T\bar{T}$

Having described the CFT dual to the IR AdS_3 region, one would now like to understand the holographic dual to the full exact string background (5.5). From the point of view of the IR CFT, one turns on an irrelevant deformation by an operator of dimension $(2, 2)$.

Exercise: By expanding e.g. the dilaton in (5.5) around the IR AdS_3 background, show that infinitesimally away from AdS, the deformation corresponds to turning on an operator of dimension $(2, 2)$.

It was proposed in [6] that the deforming operator is

$$D(x) = \int d^2z(\partial_x J \partial_x \Phi_1 + 2\partial_x^2 J \Phi_1)(\partial_{\bar{x}} \bar{J} \partial_{\bar{x}} \Phi_1 + 2\partial_{\bar{x}}^2 \bar{J} \Phi_1) \tag{5.11}$$

which has the correct spacetime scaling dimension, $(2, 2)$, and upon integration satisfies $\int d^2x D(x) = \int d^2z\, J^-\bar{J}^-$, as required. This operator looks like T on the left and like $\bar{T}$ on the right; however, it is single-trace because it involves a single worldsheet integral. Given these properties, [6] naturally conjectured that in the dual CFT, this operator corresponds to

$$D(x) \propto \sum_{i=1}^{N_1} T_i \bar{T}_i \tag{5.12}$$

It is useful to let the parameter of the deformation be a free parameter, λ, even though in (5.5) it has a fixed value. The identification of the leading deforming operator performed above holds for infinitesimal λ. Remarkably, the results of this analysis can be meaningfully extended to finite λ. On the worldsheet side, the reason for this is that the $J^-\bar{J}^-$ deformation is exactly marginal and thus can be turned on a finite amount. On the dual CFT side, the deformation at finite λ can be defined as the sum of $T\bar{T}$ deformations, one in each copy of $\mathcal{M}_{6N_5}$, by the same amount λ. This leads to the conjecture (supported a posteriori by checks) that the holographic dual to the background (5.5) is the symmetric product orbifold of $T\bar{T}$ - deformed CFTs

$$Z_{string}[\text{F1-NS5}] = Z\left[(T\bar{T}\text{-}def.\ \mathcal{M}_{6N_5})^{N_1}/S_{N_1}\right] \tag{5.13}$$

The regime of validity of this correspondence is meant to be the same as that of the original $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality describing the IR region. The free orbifold structure is quite important, both for being able to define the deformation, and for performing calculations in the boundary QFT.

5.3 Checks and predictions

Black hole entropy

An important check of the proposed duality was to show that the symmetric product orbifold of $T\bar{T}$ - deformed CFTs correctly reproduces the entropy of a black hole in the bulk.

On the gravity side, we consider the non-extremal NS5-F1 solution . The string frame metric reads

$$ds^2 = -\frac{f_\mathcal{E}}{f_1}dt^2 + \frac{dx_1^2}{f_1} + \frac{N_5\alpha'}{f_\mathcal{E}}\frac{dr^2}{r^2}\ , \qquad e^{2\Phi} = g_s^2\frac{N_5\alpha'}{r^2 f_1} \tag{5.14}$$

where $f_1, f_\mathcal{E}$ are defined in (5.4) and (5.5) and r_0^2 is proportional to the mass of the black hole. The Bekenstein-Hawking entropy of the black hole is

$$S_{BH} = \frac{\mathcal{A}_H}{4G_3} = \frac{R\sqrt{f_1}}{4G_3 f_5} = \frac{R\sqrt{r_0^2 r_1^2 + r_0^4}}{4G_3 N_5 \alpha'} \tag{5.15}$$

In the dual theory, the maximal entropy is obtained for equal repartition of the energy between the N_1 copies of the $T\bar{T}$ - deformed $\mathcal{M}_{6N_5}$ CFT, each of which carries energy E/N_1. Using (2.31) for each copy, we find

$$S = N_1 S_0 = 2\pi N_1 \sqrt{\frac{N_5}{\pi}\left(\frac{ER}{N_1} + \lambda \frac{E^2}{N_1^2}\right)} = 2\pi \sqrt{\frac{N_5}{\pi}(ERN_1 + \lambda E^2)} \tag{5.16}$$

Exercise: Express r_0^2 in terms of the mass of the black hole and use it to show that the boundary entropy (5.16) agrees with the bulk Bekenstein-Hawing entropy (5.15) for the appropriate value of λ. Check that this value agrees with that used to match the Hagedorn temperature, computed in a previous exercise.

Energy spectrum

As a cross-check of the proposed duality, one can show that the string theory spectrum, as computed using worldsheet techniques, agrees with that of the boundary symmetric product orbifold QFT. This was verified in [30] using a null coset construction of the worldsheet CFT that corresponds to the background (5.5), and in [31] by using the link to the TsT transformation, which is known to only change the boundary conditions on the worldsheet fields.

Correlation functions

One can use the holographic map between boundary and worldsheet operators to study correlation functions in the deformed boundary CFT using the well-controlled worldsheet techniques. For a large class of boundary operators, this map takes the form

$$\mathcal{O}(x) = \int d^2z\, \Phi_h(x,z)\mathcal{V}(z) \tag{5.17}$$

where $\mathcal{V}$ is a vertex operator associated with the part of the worldsheet CFT that describes the internal space. The above operator satisfies the mass-shell condition

$$-\frac{h(h-1)}{N_5} + \Delta_{\mathcal{V}} = 1/2 \tag{5.18}$$

This method was used in [32, 33] to study the two-point function of operators in the deformed theory. Since the deformation is irrelevant, the correlator becomes non-local. It is thus best described in momentum space, using the Fourier-transformed version of the map (5.17). From a worldsheet perspective, the deformation only modifies the mass shell condition (5.18), and amounts to replacing

$$h(h-1) \to h(h-1) + \frac{\lambda N_5}{2} p^2 \tag{5.19}$$

where p is the momentum of the Fourier-transformed boundary operator, $\mathcal{O}(p)$. The answer for the two-point function is then given by the same formula as the Fourier transform of the CFT two-point function ($\propto (p^2)^{2h-1}$ times an h-dependent prefactor), but now with $h \to h(p)$ as read off from above. This two-point function is well-defined and smooth for real Euclidean momenta; however, in Lorentzian signature it has a branch cut for timelike momenta, whose interpretation remains to be understood.

Entanglement entropy

The entanglement entropy between an interval of length L in the boundary QFT and its complement was computed holographically in [34], using the Ryu-Takayanagi formula. Their results nicely show non-local features, as expected of the non-local boundary theory.

Just like in AdS, the area of the extremal surface is divergent, and needs to be regulated by a short-distance cutoff ϵ. The non-locality manifests itself in that the RT surface stops existing when the length of the boundary interval is smaller than a minimum length L_{min}, with

$$L_{min} = \frac{\pi}{2}\sqrt{N_5 \alpha'} \tag{5.20}$$

The expression for the vacuum entanglement entropy when the length of the boundary interval, L, is large is given by

$$S_{EE} = \frac{c\,\beta_H^2}{24\pi^2\epsilon^2} + \frac{c}{3}\ln\frac{L}{\epsilon} + \mathcal{O}(\beta_H^2/L^2) \tag{5.21}$$

where β_H is the Hagedorn inverse temperature $2\pi\sqrt{N_5\alpha'}$. For $L \approx L_{min}$, [34] find

$$S_{EE} = \frac{c\,\beta_H^2}{24\pi^2\epsilon^2} + \frac{c}{6}\ln\frac{\beta_H(L - L_{min})}{\epsilon^2} + \mathcal{O}(L - L_{min}) \tag{5.22}$$

The second term in the large L expansion is similar to the vacuum entanglement entropy in a CFT_2. The first term - which does not depend on the length of the interval, and would thus drop out from the renormalized entanglement entropy - could be due to contact terms.

6. Conclusions

In this lecture notes, I have reviewed several basic properties of the $T\bar{T}$ deformation: its universal effect on the energy spectrum and on the S-matrix, the holographic dictionary. I have also included a brief sketch of the single-trace analogue of the $T\bar{T}$ deformation proposed by [6], which is interesting from the point of view of developing holography for non-AdS spacetimes, as well as understanding less universal generalizations of the $T\bar{T}$ deformation. Many interesting results and research directions were left out, both for $T\bar{T}$ and other similar irrelevant current-current deformations.

Of the many possible future directions in this field, one that is particularly interesting is to understand whether $T\bar{T}$ - deformed QFTs should be thought of as theories of quantum gravity or, rather, as non-local UV-complete QFTs. The universal time delay proportional to the energy that appears in scattering processes suggests the former interpretation, whereas the Virasoro symmetry found in the classical ASG analysis, *if* it survives at non-perturbative level, suggests the latter. Related questions are whether a stress tensor can be defined at length scales comparable to $\sqrt{\mu}$, as well as understanding the precise connection to LST. One way to distinguish between the two options above would be to determine whether off-shell observables, such as correlation functions, can be non-perturbatively defined in this theory. The first perturbative steps in this interesting direction were taken in [35].

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