

# D-instantons & the non-pert. Complet. of c=1 string theory.

## Motivation

unsatisfying status:

- no robust derivation of duality
- lack of pert. checks,  
addressed in '17 paper

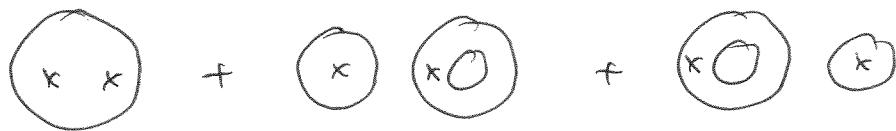
- D-instanton effects & non-pert. string thy.

IB Green-Gutperle  $R^4$

→ fixes normalization via S-duality.

No further 1st principle D-instanton Computation

Fischler - Susskind - Polchinski mechanism



cancel divergence.

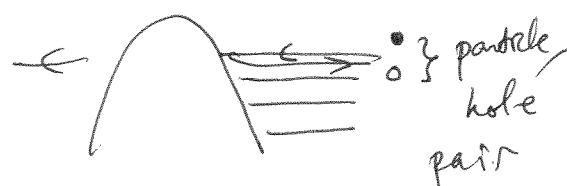
requires closed + open SFT!

- Non-pert duality b/t c=1 string & MQM.

WS - bosonic string.

MQM

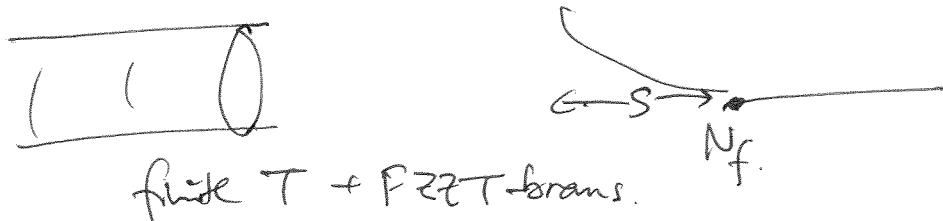
$X^0 + C = 25$  Liouville. + bc.



+ D-instanton effects

- A stringy model of BH's

  
 2D BH as a <sup>closed string</sup> ~~background~~ of  $c=1$  string theory.



$$R < R_c = 2.$$

$$\lambda \approx \frac{g_s N_f e^{-2\pi s R}}{R \sin(\pi R)} \quad N_f, s \rightarrow \infty,$$

$\lambda$  finite

$\downarrow$   
 Dual sine-Gordon definition.

$$\Delta S = \int \lambda \cos(R(x_L - x_R)) \sqrt{\frac{R}{2}}.$$

④  $\hookrightarrow$  thermal bath of long strings?

## The ~~approximate~~ duality

MQM.  $H = \text{Tr} \left( \frac{P^2}{2} + V(x) \right).$

$$X = \Omega' \wedge \Omega. \quad \Psi(x) \rightsquigarrow \Phi(\lambda \langle \Omega' \rangle)$$

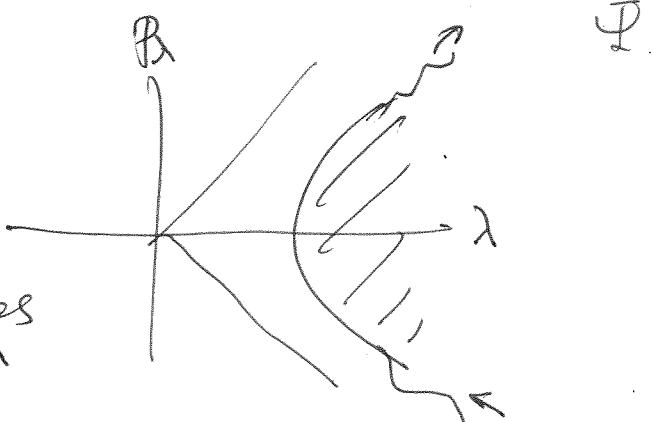
$$H = \Delta^{-1} H' \Delta$$

"  $\Delta^{-1} \Phi'$ "

$$H' = \sum_i \left( -\frac{1}{2} \partial_{x_i}^2 + V(x_i) \right) + \frac{1}{2} \sum_{i \neq j} \frac{R_j R_{ji}}{\lambda_{ij}^2}$$

↑  
free fermi.

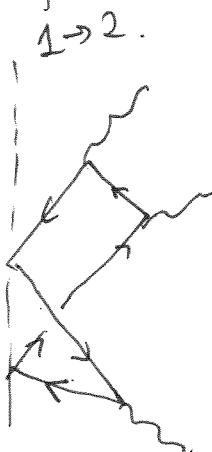
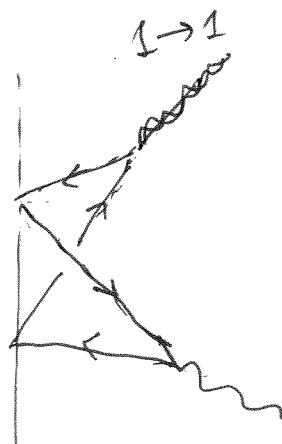
↓  
on  $U(N)$ -int



closed strips

dual to collective modes  
of Fermi surface

↑  
particle/hole pairs.



$$A_{1 \rightarrow k} = \sum (-)^{|S_2| - 1} \int_0^\infty dx.$$

$\oplus S_1 \cup S_2 = \{ \omega_1, \dots, \omega_k \}$

$$R_p(-\mu + \omega - x) R_h(\mu - x).$$

$$R_p(E) = i\mu^{iE} \sqrt{\frac{1}{1+e^{2\pi E}} \cdot \frac{\Gamma(\frac{1}{2}-iE)}{\Gamma(\frac{1}{2}+iE)}}$$

single particle reflection amplitude.

$$R_h = R_p^{-1}$$



Band-resummed

$$A_{1 \rightarrow k} = \sum_{g=0}^{\infty} \cancel{(\frac{1}{\mu})^{k-1+2g}} A_{1 \rightarrow k}^{\text{pert},(g)}$$

$$+ \sum_{n=1}^{\infty} e^{-2\pi n \mu} \sum_{L=0}^{\infty} (\frac{1}{\mu})^L A_{1 \rightarrow k}^{\text{n-inst},(L)}$$

$$\begin{aligned} \sum g^n a_n &= f(g), \\ \rightarrow \sum a_n \frac{t^n}{n!} &= \hat{f}(t). \\ f(g) &= \int_C e^{-\frac{t}{g}} \cdot \hat{f}(t) dt \end{aligned}$$

$$A_{1 \rightarrow 1}^{\text{pert},(0)} = \omega$$

$$A_{1 \rightarrow k}^{\text{I-inst},(0)} = -\frac{2^{k+1}}{4\pi} \prod_{i=1}^k \text{sh}(\pi\omega_i)$$

$$A_{1 \rightarrow 1}^{\text{pert},(1)} = \frac{1}{24} (i\omega^2 + 2i\omega^4 - \omega^5) A_{1 \rightarrow 1}^{\text{I-inst},(1)}(\omega)$$

$$= -\frac{i}{2\pi^2} \omega \left( \frac{\pi\omega}{\tanh(\pi\omega)} - 1 \right)$$

$$A_{1 \rightarrow 2}^{\text{pert},(0)} = i\omega \omega_1 \omega_2$$

$$\sinh^2(\pi\omega)$$

$$A_{1 \rightarrow 3}^{\text{pert},(0)} = i\omega \omega_1 \omega_2 \omega_3 (1+i\omega)$$

$$A_{1 \rightarrow k}^{\text{n-inst},(0)} = \frac{1}{2\pi^{3/2}} \frac{(-)^n}{n} \frac{\Gamma(n+\frac{1}{2})}{n!}$$

$$\times e^{\pi\omega n} {}_2F_1\left(-\frac{1}{2}, -n; -\frac{1}{2}-n; e^{-2\pi\omega}\right)$$

$$\times 2^k \prod_{i=1}^k \text{sh}(n\pi\omega_i)$$

## Worldsheet description

pert. amplitudes.

$$S_L = \frac{1}{4\pi} \int (\partial\phi)^2 + Q R \phi + \mu e^{2b\phi}$$

$$Q = b + \frac{1}{b}, \quad b \rightarrow 1.$$

$$V_p \sim S(p)^{-\frac{1}{2}} e^{(Q+2ip)\phi} + S(p)^{\frac{1}{2}} e^{(Q-2ip)\phi}$$

$$-\left(\frac{\Gamma(2ip)}{\Gamma(-2ip)}\right)^2 = S(p) \xrightarrow[\text{boundary reflection phase.}]{\text{Lamelle}} \phi \rightarrow -\infty.$$

$$C(P_1, P_2, P_3) = \frac{1}{2\sum_i (1+i(P_1+P_2+P_3))} \left[ \frac{2P_1 \sum_i (1+2ip_i)}{\sum_i (1+i(P_2+P_3-P_i))} \right]$$

Barnes double fn:  
 $\prod_{i=1}^k$  entire fn, zeros at  $\mathbb{Z} \setminus \{0\}$ .  $\times_{\text{perm}}$  ].

$$V_\omega^\pm = g_s e^{\pm i \omega X^0} V_{p=\frac{\omega}{2}}$$

$$\langle \omega | \omega' \rangle = \omega \delta(\omega - \omega')$$

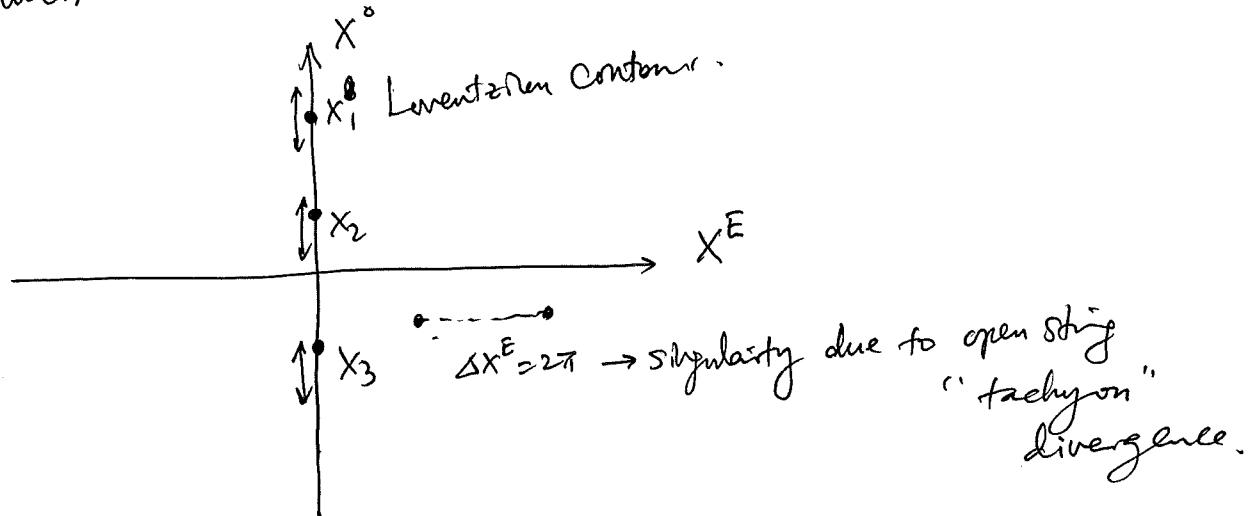
$$= \int d\sigma \text{ (diagram with one leg p)}$$

$$= \int \text{ (diagram with two legs p1, p2)} \quad \checkmark$$

$$= \int \text{ (diagram with two legs p1, p2)}$$

- D-instanton : . ZZ-brane in Lorentz  
 (ZZ-inst) . Dirichlet in  $X^0$ .

multi-instanton



We will also need to take into account higher ZZ-instantons, w/ (n, 1) ZZ-brane config. in Lorentz.

$$|Z Z_{(m,n)}\rangle = \int_0^\infty \frac{dp}{\pi} \cdot \tilde{\Psi}^{(m,n)}(p) |V_p\rangle.$$



$$\left| \begin{array}{c} / \\ \backslash \\ / \\ \backslash \\ \end{array} \right\rangle_{(m,n)} \quad \mathcal{H} \ni \psi_{m,n}. \quad h = 1 - \frac{(m+n)^2}{4}.$$

$$\tilde{\Psi}^{(m,n)}(p) = \frac{5}{24\sqrt{\pi}} \frac{\sinh(2\pi m p) \sinh(2\pi n p)}{\sinh(2\pi p)}.$$

inv't under  $m \leftrightarrow n$

$$S_{(1,1) \text{ ZZ-mat}} = \frac{1}{g_s}$$

$$S_{(m,n)} = \lim_{P \rightarrow i} \frac{\mathcal{V}^{(m,n)}(P)}{\mathcal{V}^{(1,1)}(P)} S_{(1,1)} = \frac{mn}{g_s}$$

$$\mathcal{V}_w^{(m,n)} = g_s \frac{C_P}{2\pi} \mathcal{V}^{(m,n)}\left(P = \frac{\omega}{2}\right) e^{\pm i\omega x}$$

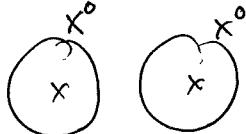
$$= 2e^{\pm i\omega x} \frac{\sinh(m\pi\omega) \sinh(n\pi\omega)}{\sinh(\pi\omega)}$$

closed string

$1 \rightarrow 1$  amplitude  
 $A(\omega \rightarrow \omega')$

order

$$e^{-\frac{1}{g_s}}$$



Note: unitarity rel'n.

$$\sum_{k=1}^{\infty} \int d\omega_i \frac{|A_{i \rightarrow k}|^2}{\omega_i \omega_1 \dots \omega_k} = 1.$$

$\omega = \omega_1 + \dots + \omega_k$

$$\text{String loop} \sim (\leftarrow)^2.$$

$$= N \int dx^\circ e^{-S_{zz}} \langle \mathcal{V}_w^+ \rangle_{zz, x^\circ}^{\Phi^2} \langle \mathcal{V}_{w'}^- \rangle_{zz, x^\circ}^{\Phi^2}$$

$$= 2\pi N e^{-\frac{1}{g_s}} \delta(\omega - \omega') \cdot 4 \sinh^2(\pi\omega).$$

match w/ MQM,  $\leftrightarrow$

$$N (= N_1) = -\frac{1}{8\pi^2}.$$

## Comment

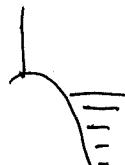
- OB MQM



$$R_h = R_p^*$$

would not agree w/  
WS D-mstanton  
② order  $e^{-\frac{t}{g_s}}$ .

neither is



etc.

- ~~if there is no D-mstanton correction at all,~~  
e.g.  $N=0$ , etc.

~~modified MGF~~ result (Borel-resummed pert.  
series)  
would not admit any MQM/free fund.  
interpretation, as

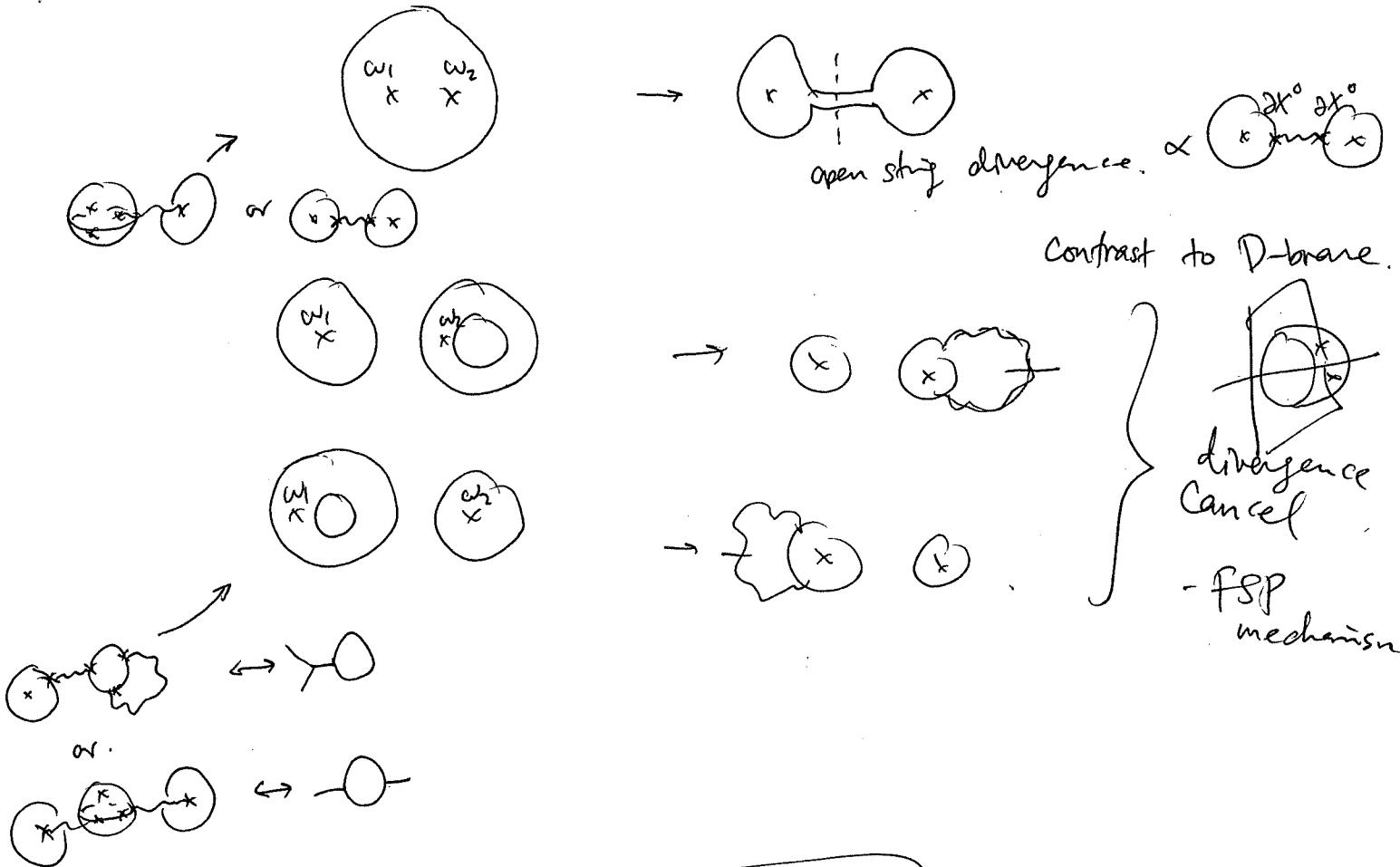
$$R^{\text{pert}}(E) = i \mu^E \sqrt{\frac{\Gamma(\frac{1}{2} - iE)}{\Gamma(\frac{1}{2} + iE)}}$$

would have branch cuts on  
lower cplx E-plane.  
rather than poles  
as in  $R(E)$ .

$$\sim \sqrt{\int \frac{1}{1 + e^{2\pi E}} \frac{\Gamma(\frac{1}{2} - iE)}{\Gamma(\frac{1}{2} + iE)}}$$

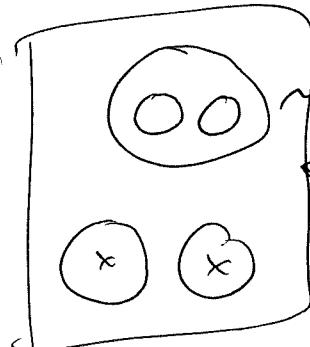
Next order

$$e^{-\frac{1}{g_s}} \cdot g_s$$



2 ambiguities.

(1)



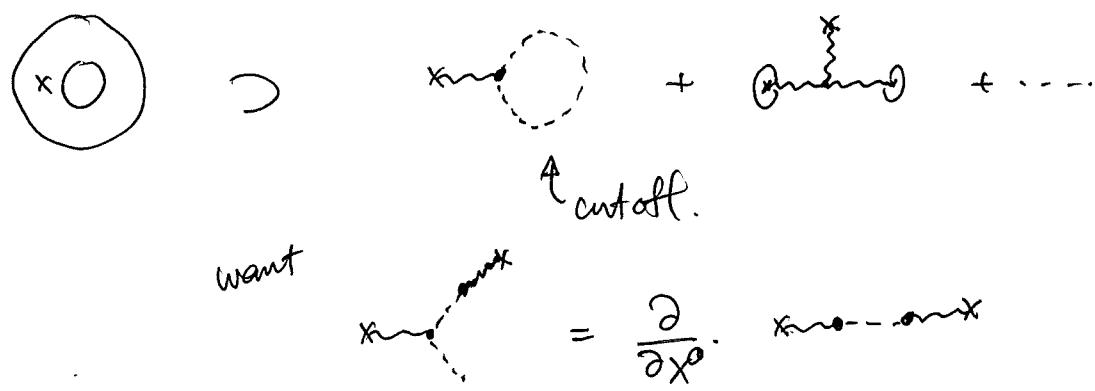
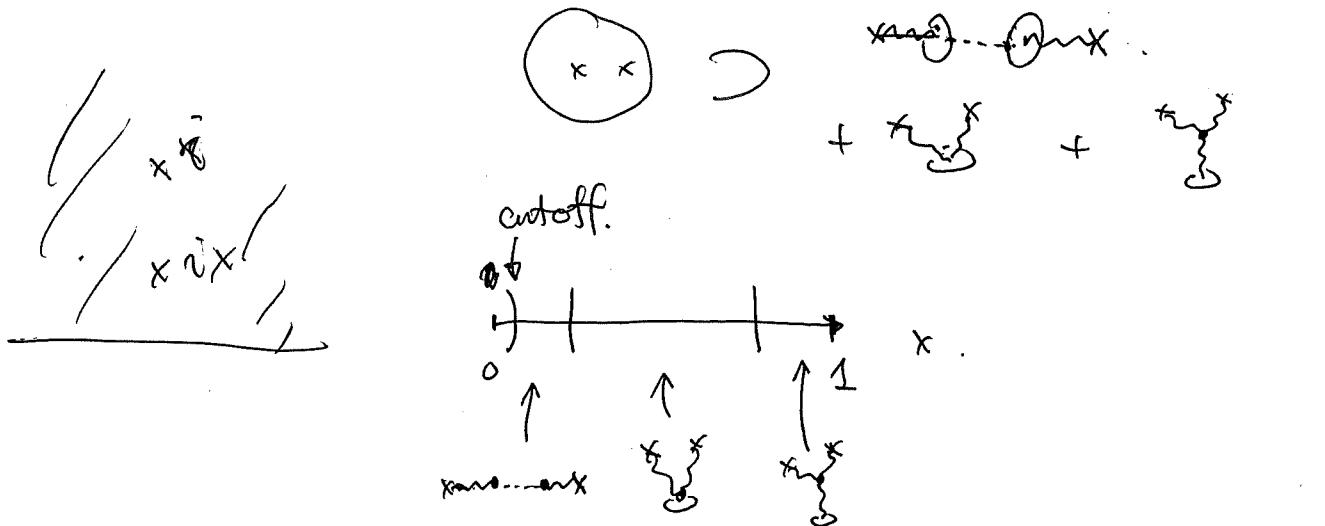
$\mathcal{O}(g_s)$  corrections to  $S_{22}$ .  
need regularization.

$$\propto e^{-\frac{1}{g_s}} g_s \cdot S_{22}^{(1)}$$

↑  
unknown  
const

(2) finite part after canceling log div. via FSP mechanism

## Idea of SFT



A diagram of a complex plane with points  $z_1$ ,  $z_2$ ,  $w_1$ , and  $w_2$ . A curly arrow connects  $z_1$  and  $z_2$ , and another connects  $w_1$  and  $w_2$ . A curved arrow labeled  $w_1 = i\lambda \frac{z_1 - z_2}{z_1 + z_2}$  is shown. To its right, it says "defines  $x$ ".

A diagram of a complex plane with points  $z_1$ ,  $z_2$ ,  $w_1$ , and  $w_2$ . Curly arrows connect  $z_1$  to  $z_2$  and  $w_1$  to  $w_2$ . A curved arrow labeled  $w_2 = i\lambda \frac{z_2 - z_1}{z_2 + z_1}$  is shown. To its right, it says "defines  $x$ ". Below this, it says "compatibility requires  $\tilde{\lambda} = \lambda$ ".

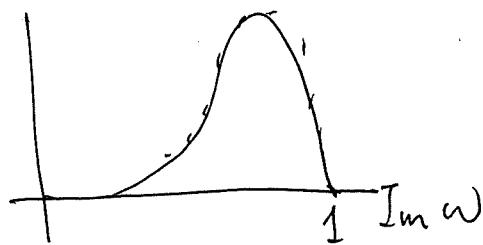
$$A_{1 \rightarrow 1}^{1\text{-Mst}, (1)} (\omega \rightarrow \omega')$$

$$= e^{-\frac{1}{g_s}} g_s \delta(\omega - \omega) \cdot N \cdot \left\{ \begin{array}{l} \cancel{\text{---}} \text{---} \text{---} \\ + \# S_{zz}^{(1)} \sinh^2(\pi\omega) \\ + c' \omega^3 \sinh^2(\pi\omega) \end{array} \right\}$$

fit  $S_{zz}^{(1)}$ ,  $c'$  against MQM answer.

$$\cancel{\text{---}} \text{---} \text{---}$$

$$\omega \left( \frac{\pi\omega}{\tanh \pi\omega} - 1 \right) \sinh^2(\pi\omega)$$



$$\cancel{\text{---}} \text{---} \text{---}$$

$$S_{zz}^{(1)} = 0.496 \pm 0.001$$

$$c' = -1.399 \pm 0.002$$

$\sim 1\%$  error in computing  $\cancel{\text{---}}$

Sol:  $c' = -\ln 4 \approx -1.386$  ✓

Multi-inst. • Lorentzian content in  $x_i^0$ .

•  $(n, 1)$  ZZ-brane  $\rightarrow \mathcal{N}_n$

we find by matching  
w/ MM

$$\mathcal{N}_n = \frac{(-)^n}{4\pi^2 n} \frac{(2n-1)!!}{(2n)!!}$$

e.g. 2-instanton. at order  $e^{-\frac{2}{g_s}}$ .

$$\mathcal{N}_1^2 \cdot \frac{1}{2} \int dx_1 dx_2 \left[ \left( \text{diagram with } \begin{array}{c} 1 \\ \text{x} \\ \omega \end{array} \text{ and } \begin{array}{c} 1 \\ \text{x} \\ \omega' \end{array} \right) + \left( \begin{array}{c} 2 \\ \text{x}\omega \\ \omega \end{array} + \begin{array}{c} 2 \\ \text{x}\omega \\ \omega' \end{array} \right) + \text{diagram with } \begin{array}{c} 2 \\ \text{x}\omega \\ \omega' \end{array} \right. \\ \times \left. e^{-\frac{2}{g_s}} \right] \\ + 2 \left( \begin{array}{c} 1 \\ \omega \\ \omega' \end{array} \right) \times \left( e^{-\frac{2}{g_s}} - 1 \right).$$

$$+ \mathcal{N}_2 \cdot \int dx \quad \begin{array}{c} (2,1) \\ \text{x}\omega \\ \omega' \end{array}$$

$$\textcircled{z} = \int_0^\infty \frac{dt}{2t} \frac{(e^{2\pi t} - 1)}{\eta(-t)} \cdot \frac{e^{-t(\Delta x^E)^2}}{\eta(it)} \cdot \eta(it)^2$$

Lomille       $\underbrace{x^0}_{\text{b.c.}}$

$$= \frac{1}{2} \ln \frac{(\Delta x^E)^2}{(\Delta x^E)^2 - (2\pi)^2}$$

$$e^2 \textcircled{z} = \frac{(\Delta x^E)^2}{(\Delta x^E)^2 - (2\pi)^2}$$

$$\rightarrow \frac{(\Delta x)^2}{(\Delta x)^2 + (2\pi)^2} . \quad \begin{matrix} \text{Lorentz} \\ \Delta x \end{matrix}$$

After combination magic, match w/ MM

at order  $e^{-n/2}$  for all  $n$  !!

also fixes  $N_n$ .