A Reconstruction Conjecture: Deciphering the Structure of the Dark Sector from the Matter Power Spectrum

Brooks Thomas
LAFAyETTE COLLEGE

Based on work done in collaboration with:

This research supported in part by NSF
The Basic Question

- The early-universe dynamics which produces the dark matter gives rise to a particular dark-matter phase-space distribution $f(p)$. This, in turn, affects the shape of the matter power spectrum $P(k)$.

To what extent can we work backwards and reconstruct the properties of $f(p)$ — and the dynamics that gave rise to it — from information encoded in $P(k)$?

While the maps from the underlying physics to $f(p)$, and from $f(p)$ to $P(k)$ are clearly not invertible, it is nevertheless possible to “work backwards” and obtain substantial information about the dark sector from information contained in the matter power spectrum.
Describing the Phase-Space Distribution

• We’re going to describe the phase-space distribution in a slightly atypical way. We’ll begin with some motivation.

\[ x(t) = x(t') \frac{a(t)}{a(t')} \]
\[ p(t) = p(t') \frac{a(t')}{a(t)} \]

\[ \frac{d \log(p)}{dt} = -H(t) \]

• Physical number density:

\[ n(t) = \frac{g_{\text{int}}}{2\pi^2} \int dp p^2 f(p, t) \]

• Comoving number density:

\[ N(t) = n(t) a^3(t) = \frac{g_{\text{int}}}{2\pi^2} \int d \log p p^3 a^3 f(p, t) \]

• This motivates us define the “log-space” DM phase-space distribution:

\[ g(p, t) = a^3(t) p^3 f(p, t) \]

\[ \text{Area under curve: } \mathcal{N} \equiv \frac{2\pi^2}{g_{\text{int}}} N(t) \]
The Cosmological Conveyor Belt

• The reason $g(p, t)$ turns out to be a useful quantity is that it evolves with time in a particularly straightforward manner.

• Indeed, in the absence of sources/sinks, $N(t') = N(t)$ is conserved and $g(p, t)$ quantity evolves with time according to the relation

$$g(p(t'), t') = g(p(t), t)$$

• Thus, as $t$ increases, the $g(p, t)$ distribution retains the same overall profile, which simply redshifts undistorted to lower values of $\log(p)$.

• In other words, $g(p, t)$ is carried along like a cucumber on a **conveyor belt**, moving to lower and lower $\log(p)$ at speed $|d \log p / dt| = H(t)$, but retaining a fixed shape.
Dark-Sector Dynamics and Non-Trivial $g(p)$

- Things become more interesting, of course, when sources and sinks are included, such as those associated with particle decays.

- Consider a dark sector involving three scalars $\phi_0$, $\phi_1$, and $\phi_2$ with similar quantum numbers and masses $m_2 > 2m_1 > 4m_0$.

- For purposes of illustration, let’s assume:

\[
\text{BR}(\phi_2 \rightarrow \phi_1 \phi_0) \approx \text{BR}(\phi_1 \rightarrow \phi_0 \phi_0) \approx 1
\]

- We’ll also work in the instantaneous-decay approximation, wherein each $\phi_i$ decays completely at its lifetime $\tau_i$.

- We’ll also assume that $\phi_2$ is initially the only state populated.
Dark-Sector Dynamics and Non-Trivial $g(p)$

- Things become more interesting, of course, when sources and sinks are included, such as those associated with *particle decays*.

- Consider a dark sector involving three scalars $\phi_0$, $\phi_1$, and $\phi_2$ with similar quantum numbers and masses $m_2 > 2m_1 > 4m_0$.

- For purposes of illustration, let’s assume
  
  $\text{BR}(\phi_2 \rightarrow \phi_1 \phi_0) \approx \text{BR}(\phi_1 \rightarrow \phi_0 \phi_0) \approx 1$

- We’ll also work in the *instantaneous-decay approximation*, wherein each $\phi_i$ decays completely at its lifetime $\tau_i$.

- We’ll also assume that $\phi_2$ is initially the only state populated.
Dark-Sector Dynamics and Non-Trivial $g(p)$

- Things become more interesting, of course, when sources and sinks are included, such as those associated with particle decays.

Consider a dark sector involving three scalars $\phi_0$, $\phi_1$, and $\phi_2$ with similar quantum numbers and masses $m_2 > 2m_1 > 4m_0$.

- For purposes of illustration, let’s assume

$$\text{BR}(\phi_2 \rightarrow \phi_1 \phi_0) \approx \text{BR}(\phi_1 \rightarrow \phi_0 \phi_0) \approx 1$$

- We’ll also work in the instantaneous-decay approximation, wherein each $\phi_i$ decays completely at its lifetime $\tau_i$.

- We’ll also assume that $\phi_2$ is initially the only state populated.
Dark-Sector Dynamics and Non-Trivial $g(p)$

- Things become more interesting, of course, when sources and sinks are included, such as those associated with particle decays.

- Consider a dark sector involving three scalars $\phi_0$, $\phi_1$, and $\phi_2$ with similar quantum numbers and masses $m_2 > 2m_1 > 4m_0$.

- For purposes of illustration, let’s assume

$$\text{BR}(\phi_2 \to \phi_1 \phi_0) \approx \text{BR}(\phi_1 \to \phi_0 \phi_0) \approx 1$$

- We’ll also work in the instantaneous-decay approximation, wherein each $\phi_i$ decays completely at its lifetime $\tau_i$.

- We’ll also assume that $\phi_2$ is initially the only state populated.
Dark-Sector Dynamics and Non-Trivial $g(p)$

Things become more interesting, of course, when sources and sinks are included, such as those associated with particle decays.

Consider a dark sector involving three scalars $\phi_0$, $\phi_1$, and $\phi_2$ with similar quantum numbers and masses $m_2 > 2m_1 > 4m_0$.

For purposes of illustration, let’s assume

\[ \text{BR}(\phi_2 \rightarrow \phi_1 \phi_0) \approx \text{BR}(\phi_1 \rightarrow \phi_0 \phi_0) \approx 1 \]

We’ll also work in the instantaneous-decay approximation, wherein each $\phi_i$ decays completely at its lifetime $\tau_i$.

We’ll also assume that $\phi_2$ is initially the only state populated.
Dark-Sector Dynamics and Non-Trivial $g(p)$

- Things become more interesting, of course, when sources and sinks are included, such as those associated with particle decays.

- Consider a dark sector involving three scalars $\phi_0$, $\phi_1$, and $\phi_2$ with similar quantum numbers and masses $m_2 > 2m_1 > 4m_0$.

- For purposes of illustration, let’s assume

\[
\text{BR}(\phi_2 \rightarrow \phi_1 \phi_0) \approx \text{BR}(\phi_1 \rightarrow \phi_0 \phi_0) \approx 1
\]

- We’ll also work in the instantaneous-decay approximation, wherein each $\phi_i$ decays completely at its lifetime $\tau_i$.

- We’ll also assume that $\phi_2$ is initially the only state populated.
Dark-Sector Dynamics and Non-Trivial $g(p)$

- Things become more interesting, of course, when sources and sinks are included, such as those associated with *particle decays*.

- Consider a dark sector involving three scalars $\phi_0$, $\phi_1$, and $\phi_2$ with similar quantum numbers and masses $m_2 > 2m_1 > 4m_0$.

- For purposes of illustration, let’s assume $\text{BR}(\phi_2 \rightarrow \phi_1\phi_0) \approx \text{BR}(\phi_1 \rightarrow \phi_0\phi_0) \approx 1$.

- We’ll also work in the *instantaneous-decay approximation*, wherein each $\phi_i$ decays completely at its lifetime $\tau_i$.

- We’ll also assume that $\phi_2$ is initially the only state populated.
Dark-Sector Dynamics and Non-Trivial $g(p)$

• Generally speaking, we expect multi-modal phase-space distributions to arise in dark-sector decay scenarios wherein...

  - **Multiple decay pathways** to the lightest (stable) state exist.
  - Decay pathways with substantial overall branching fractions have **significantly different timescales**.
  - Scattering rates in the dark sector are sufficiently low that states **don’t have time to thermalize**.

• These conclusions remain robust even when we include all relevant physical effects (exponential decay, time-dilation, etc.).

• Let’s therefore investigate how we might obtain evidence of such multi-modality in the matter power spectrum.

Full numerical results for three-field scenarios with:

\[
\frac{m_2}{7} = \frac{m_1}{3} = m_0
\]

\[
\Gamma_{m_\ell}^{\ell} \equiv \Gamma(\phi_\ell \rightarrow \phi_m \phi_n)
\]
Deciphering the Matter Power Spectrum

- The **matter power spectrum** provides an observational handle on the velocity distribution of DM particles.
- Dark matter particles with sufficient speed can escape from gravitational potential wells as they form, leading to a *suppression of structure* on small scales.

**Linear Matter Power Spectrum**

- Departures from a purely CDM cosmology can be expressed in terms of the **transfer function** $T(k)$, where

$$T^2(k) = \frac{P(k)}{P_{CDM}(k)}$$

Particles moving at sufficient speeds can escape potential wells of size $\sim k^{-1}$
Our Approach

• The free-streaming horizon for a particle of mass $m$ and present-day momentum $p$ in an expanding universe is

$$k_{\text{hor}}(p) \equiv \xi \left[ \int_{t_{\text{prod}}}^{t} v(p, t) \frac{dt}{a(t)} \right]^{-1} = \xi \left[ \int_{a_{\text{prod}}}^{1} \frac{da}{Ha^2} \frac{p}{\sqrt{p^2 + m^2a^2}} \right]^{-1}$$

$\mathcal{O}(1)$ constant

• The usual approach (e.g., for warm DM) is to define a single “free-streaming-horizon” scale $k_{\text{FSH}}$ using the average DM velocity $\langle v(t) \rangle$:

$$k_{\text{FSH}} \sim \left[ \int_{t_{\text{prod}}}^{t} \langle v(t) \rangle \frac{dt}{a(t)} \right]^{-1}$$

• By contrast, we shall consider a somewhat unorthodox procedure in which we regard $k_{\text{hor}}(p)$ as a functional map between $p$ and $k$.

• We can use this map to define a phase-space distribution in $k$-space which correspond to $g(p)$ in momentum space.

$$\tilde{g}(k) \equiv g(k_{\text{hor}}^{-1}(k)) \left| \frac{d \log p}{d \log k} \right|$$

Inverse of $k_{\text{hor}}(p)$

Jacobian
Relating $g(p)$ to $T^2(k)$

- Let's first consider the case of a simple $g(p, t_{\text{now}})$ which consists of a single log-normal peak with average momentum $\langle p \rangle$ and width $\sigma$.
- We'll begin simply by fixing $\langle p \rangle$ and $\sigma$ and varying the normalization of the peak, assuming that the rest of $\Omega_{\text{DM}}$ is made up by cold DM.
- Increasing the abundance $\Omega$ associated with the peak, we find...

DM **acoustic oscillations** (the “wiggles”) become more pronounced. The **slope** of $T^2(k)$ at $k$ above the peak in $\tilde{g}(k,t_{\text{now}})$ increases.
Relating $g(p)$ to $T^2(k)$

- Now let’s hold $\Omega$ and $\langle p \rangle$ fixed and **vary the width** $\sigma$ of the peak.
- Different values of $\sigma$ lead to different amounts of suppression in $T^2(k)$ for $k$ above the peak in $\tilde{g}(k,t_{\text{now}})$, but essentially **identical slopes**!

The abundance associated with a peak in $\tilde{g}(k,t_{\text{now}})$ correlates not with suppression in $T^2(k)$, but rather the slope.
Relating \( g(p) \) to \( P(k) \)

- In order to test this conjecture, let’s consider a distribution consisting of two log-normal peaks with the same \( \sigma \), but different values of \( \langle p \rangle \).

- Here, \( \Omega_{DM} \) is partitioned entirely between the two peaks (no extra cold DM component). We vary their relative normalizations.

- Indeed, once again, we observe that the abundance associated with a peak in \( \tilde{g}(k,t_{\text{now}}) \) is correlated with the change in the slope of \( T^2(k) \).
The Hot-Fraction Function

• The slope of the transfer function at a given value of $k$ seems to correlate with the total number density of particles which can free-stream at that value of $k$ – particles with momenta $p > k_{\text{hor}}^{-1}(k)$.

• Motivated by these empirical findings, let us define the "hot-fraction function" $F(k)$ as follows:

$$F(k) = \frac{\int_{-\infty}^{\log k} \widetilde{g}(k) \, d\log k'}{\int_{-\infty}^{\infty} \widetilde{g}(k) \, d\log k'}$$

Fraction of DM number density which free-streams at $k$. 

![Fraction of DM number density which free-streams at $k$.](image)
The Hot-Fraction Function

• The slope of the transfer function at a given value of $k$ seems to correlate with the total number density of particles which can free-stream at that value of $k$ – particles with momenta $p > k_{\text{FSH}}^{-1}(k)$.

• Motivated by these empirical findings, let us define the “hot-fraction function” $F(k)$ as follows:

$$F(k) = \frac{\int_{-\infty}^{\log k} \tilde{g}(k) d \log k'}{\int_{-\infty}^{\infty} \tilde{g}(k) d \log k'}$$

Fraction of DM number density which free-streams at $k$
The Hot-Fraction Function

- The slope of the transfer function at a given value of \( k \) seems to correlate with the total **number density of particles which can free-stream** at that value of \( k \) – particles with momenta \( p > k_{\text{FSH}}^{-1}(k) \).

- Motivated by these empirical findings, let us define the **“hot-fraction function”** \( F(k) \) as follows:

\[
F(k) = \frac{\int_{-\infty}^{\log k} \tilde{g}(k') d\log k'}{\int_{-\infty}^{\infty} \tilde{g}(k') d\log k'}
\]

where \( \tilde{g}(k) \) is the fraction of DM number density which free-streams at \( k \).
The Hot-Fraction Function

- The slope of the transfer function at a given value of $k$ seems to correlate with the total number density of particles which can free-stream at that value of $k$ – particles with momenta $p > k_{\text{FSH}}^{-1}(k)$.

- Motivated by these empirical findings, let us define the “hot-fraction function” $F(k)$ as follows:

$$F(k) = \frac{\int_{-\infty}^{\log k} \tilde{g}(k) d \log k'}{\int_{-\infty}^{\infty} \tilde{g}(k) d \log k'}$$

$F(k)$ is the fraction of DM number density which free-streams at $k$. The diagram illustrates the behavior of $\tilde{g}(k)$ and $T^2(k)$, with $T^2(k)$ representing the square of the transfer function, and $\tilde{g}(k)$ being the density of particles which can free-stream at $k$. The diagram shows how this density varies with $k$, highlighting the regions where $\tilde{g}(k)$ is significant in comparison to $T^2(k)$. The notation $k_{\text{FSH}}^{-1}(k)$ indicates the inverse of the free-streaming scale, where particles with momenta greater than this scale can free-stream.
The Hot-Fraction Function

- The slope of the transfer function at a given value of $k$ seems to correlate with the total number density of particles which can free-stream at that value of $k$ – particles with momenta $p > k_{FSH}^{-1}(k)$.

- Motivated by these empirical findings, let us define the “hot-fraction function” $F(k)$ as follows:

$$F(k) = \frac{\int_{-\infty}^{\log k} \tilde{g}(k) d \log k'}{\int_{-\infty}^{\infty} \tilde{g}(k) d \log k'}$$

$$\sim k_{FSH}^{-1}$$

The Hot-Fraction Function

$T^2(k) = \frac{\tilde{g}(k)}{N}$

“hot” relative to $k$
A Reconstruction Conjecture

• Our conjecture, then, is that there exists some invertible functional relationship between \( F(k) \) and \( T^2(k) \).

• Empirically, from numerical investigations (using CLASS) of the relationship between these two quantities, we find that

\[
\left| \frac{d \log T^2(k)}{d \log k} \right| \approx F^2(k) + \frac{3}{2} F(k)
\]

• Taking the derivative of both sides, we obtain an approximate analytic expression for reconstructing \( \tilde{g}(k) \) from \( T^2(k) \).

\[
\frac{\tilde{g}(k)}{\mathcal{N}} \approx \frac{1}{2} \left( \frac{9}{16} + \left| \frac{\log T^2(k)}{\log k} \right| \right)^{-1/2} \left| \frac{d^2 \log T^2(k)}{(d \log k)^2} \right|
\]
How Well Does This Work in Practice?

• In order to determine how accurately our reconstruction procedure can reproduce the DM velocity distribution from the transfer function, we need examine how well it works in practice.

• Indeed, we find that our procedure is capable of reproducing the broad-brush features of the DM velocity distribution quite robustly in the context of a concrete example model.*

The Upshot:

Our reconstruction procedure provides a novel and effective method for extracting detailed information about the DM velocity distribution from the matter power spectrum.

Blue: actual $\tilde{g}(k)$ distribution for the model in question
Red: reconstructed $\tilde{g}(k)$ distribution using our procedure

* See talk by Fei Huang immediately following this talk for details.
Summary

- Non-trivial dynamics in the early universe can lead to a complicated – and even multi-modal – DM velocity distribution, which in turn affects the shape of the matter power spectrum.

- In an effort to work backwards, we have studied the relationship between $P(k)$ and the DM velocity distribution and found that the slope of the transfer function at a given $k$ is related to the fraction of the DM number density which can free-stream on the scale $k$.

- Motivated by these results, we have formulated a conjecture for reconstructing the dark-matter velocity distribution from the shape of the matter power spectrum.

- We have shown that this reconstruction conjecture can reliably capture the salient features of the velocity distribution.

- This reconstruction procedure provides a way of probing the internal dynamics within the dark sector which produced the DM through purely gravitational means.