Daniel Vagie

University of Oklahoma

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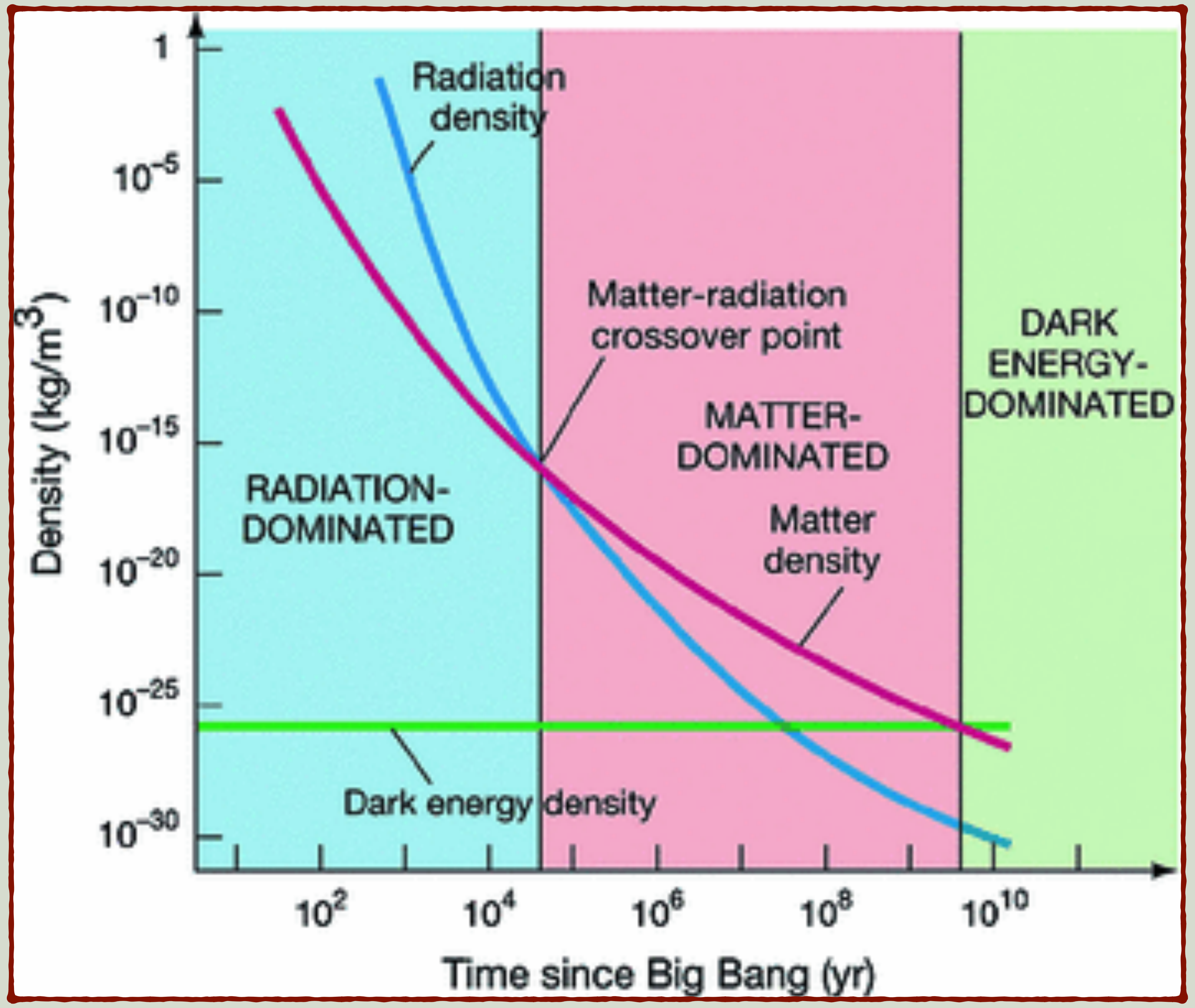
Based on work by
Huaike Guo, Elizabeth Loggia, Graham White, Kuver Sinha, Daniel Vagie

Phase Transitions as a Witness of an Early Matter Dominated Era

Introduction

- Standard view of cosmology suggests RDE -> MDE -> Today
- Theoretical motivation for a modified expansion history
- Gravitational waves produced during PT
- Typically assumed PT happened during RDE with equations in Minkowski spacetime

De Angelis A., Pimenta M. (2018)



- Sound Shell model (Hindmarsh 2019) provides the best model for the acoustic GWs
- Equations in an expanding universe can be rescaled to have Minkowski form
- Suppression in spectrum observed when using Sound Shell model for short phase transitions

Gravitational Waves in Expanding Universe

- FLRW metric: $ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij}(\mathbf{x}))d\mathbf{x}^2$
- Conformal time: $dt = a\, d\eta$
- GW sourced by T.T. part of perturbed energy momentum tensor
- Einstein equation describes time evolution of each Fourier component of GWs
- Solve by method of Green's function

$$h''_q + 2\frac{a'}{a}h'_q + q^2 h_q = 16\pi G a^2 \pi_q^T$$

$\propto v^2$

$$\mathcal{P}_{GW} = \frac{d\Omega_{GW}}{d\ln k} = \frac{1}{24\pi^2 H^2} k^3 P_{\dot{h}}\left(t, k\right)$$

$\langle \dot{h}_{ij}(t, \mathbf{q}) \dot{h}_{ij}(t, \mathbf{q}) \rangle$

Spectral Density of $\dot{h}$

$$\langle \dot{h}_{ij}(t,\mathbf{q})\dot{h}_{ij}(t,\mathbf{k})\rangle = (2\pi)^{-3}\delta^3(\mathbf{k}+\mathbf{q})P_{\dot{h}}$$

$$h_{ij}(t,\mathbf{q}) = 16\pi G \int_{\tilde{\eta}_0}^{\tilde{\eta}} d\tilde{\eta}' G(\tilde{\eta},\tilde{\eta}') \frac{a^2(\eta')\pi_{ij}^T(\eta',q)}{q^2}$$

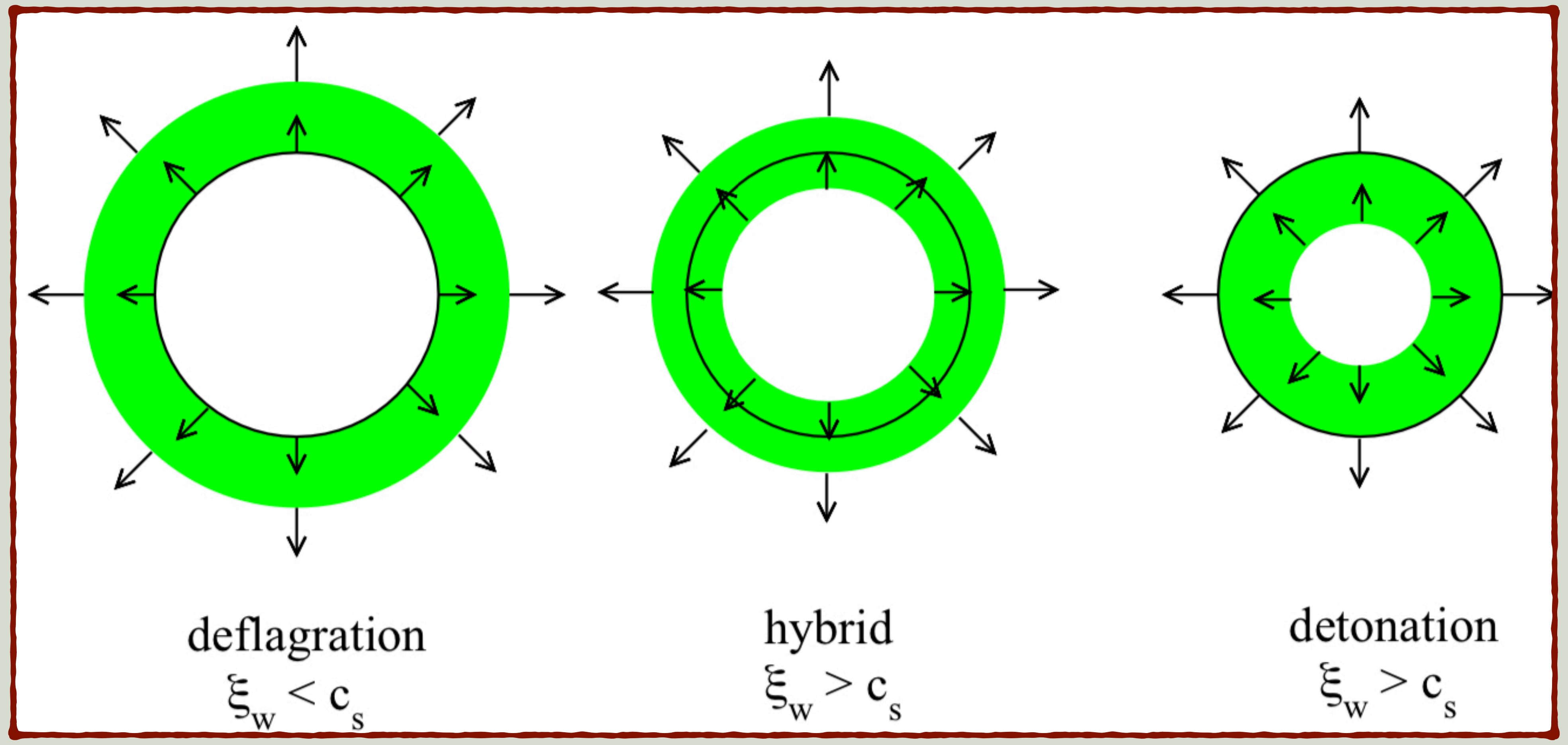
- Average over the random processes generating the GWs
- GW power spectrum depends on 2 point correlator of the T.T. energy momentum tensor
- Model correlator with Sound Shell model
- Ignore highly oscillatory terms $(\tilde{\eta}_1 + \tilde{\eta}_2)$
- $\tilde{\eta} = q\eta$

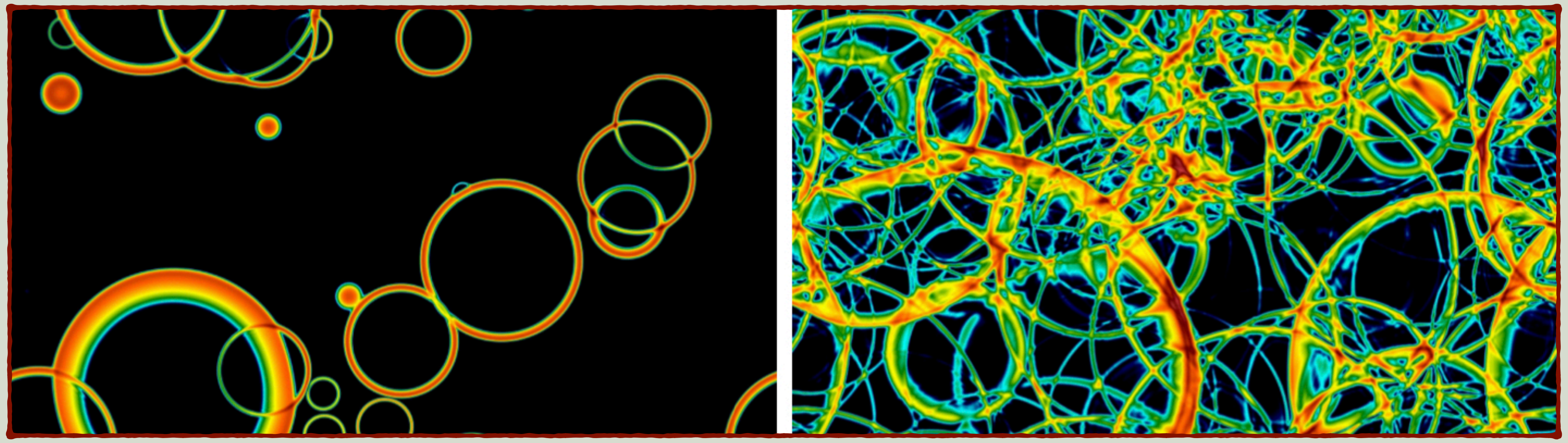
$$\frac{\partial G(\tilde{\eta},\tilde{\eta}_1)}{\partial \tilde{\eta}}\frac{\partial G(\tilde{\eta},\tilde{\eta}_2)}{\partial \tilde{\eta}} = \frac{\tilde{\eta}_1\tilde{\eta}_2}{2} \times \begin{cases} \tilde{\eta}^{-2}\left(1+\tilde{\eta}^{-2}\right)\cos(\tilde{\eta}_1-\tilde{\eta}_2) \\ \tilde{\eta}^{-4}\left(1+3\tilde{\eta}^{-2}+9\tilde{\eta}^{-4}\right)\left((\tilde{\eta}_1-\tilde{\eta}_2)\sin(\tilde{\eta}_1-\tilde{\eta}_2)+(1+\tilde{\eta}_1\tilde{\eta}_2\cos(\tilde{\eta}_1-\tilde{\eta}_2)\right) \end{cases}$$

Acoustic Gravitational Waves

- Important source of GWs
- Colliding sound shells
- Compression waves surrounding the expanding bubbles of the stable phase propagate long after the phase transition
- Computable from relativistic hydrodynamics
- Detectable at LISA

arXiv:1004.4187





arXiv:1705.01783

Sound Shell Model

- Dominate source of shear stress is the local velocity field from the sound waves in plasma
- Fluid velocity field is the linear superposition of single-bubble contributions
- E.O.M of fluid, and hence sound waves of plasma, same as Minkowski spacetime in expanding universe
- Interpret velocity w.r.t. to conformal time
- GWs produced from the propagation of the sound shells
- Computed from the convolution of power spectrum sourced by the velocity field

Velocity Spectral Density

- Velocity field, after most bubbles collide, is obtained by adding all the individual bubble contributions
- Velocity profile becomes initial condition for freely propagating sound waves
- Total number of bubbles nucleated within a Hubble volume with co-moving size V_c is N_b
- Velocity field follows a Gaussian distribution to a good approximation

$$v^i_{\mathbf{q}} = \sum_{n=1}^{N_b} v^{i(n)}_{\mathbf{q}}$$

- Randomness removed by doing an ensemble average

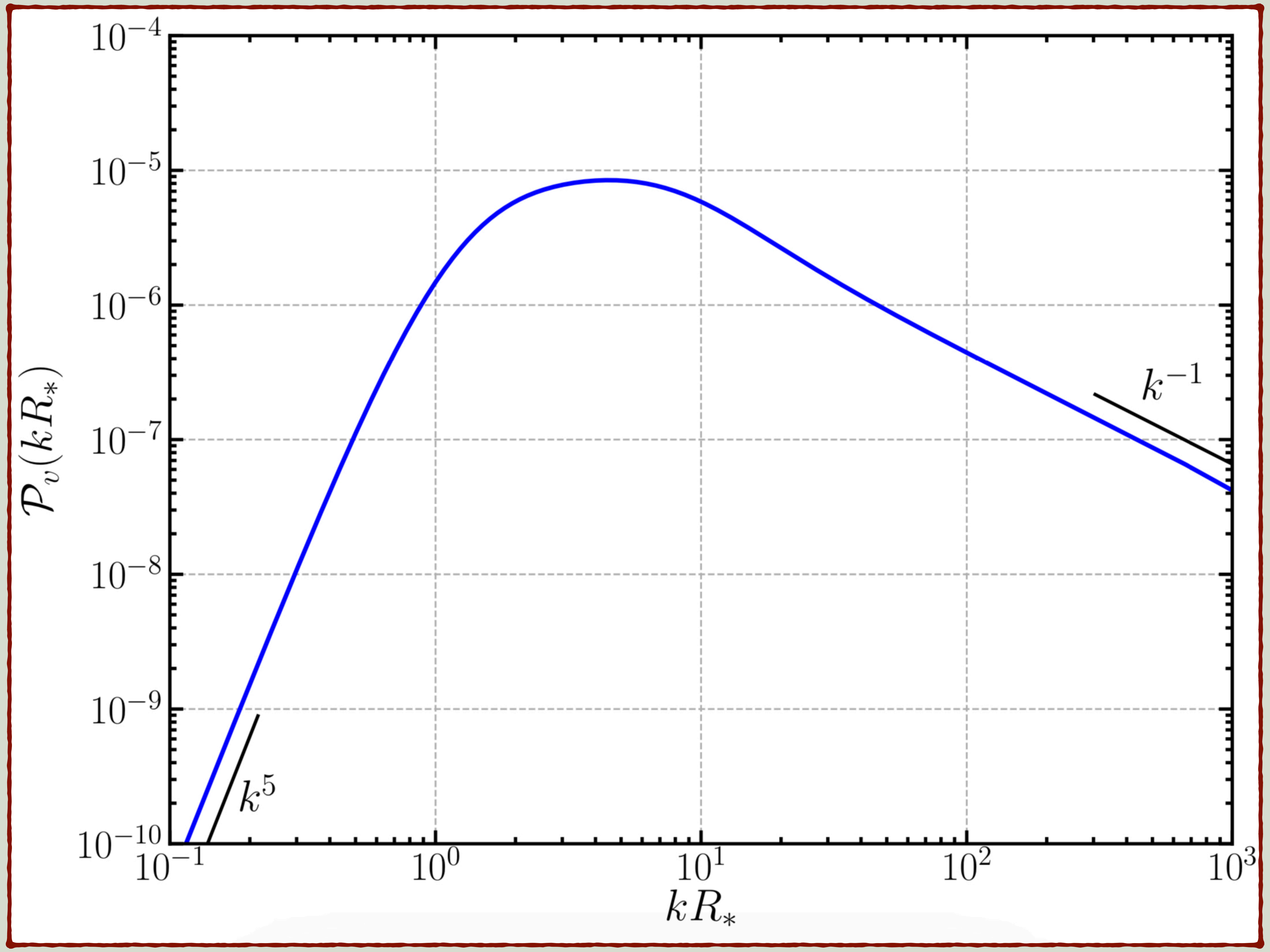
$$\langle v^i_{\mathbf{q}} v^{j*}_{\mathbf{q}} \rangle = \hat{q}^i \hat{q}^j (2\pi)^3 \delta^3(\mathbf{q}_1 - \mathbf{q}_2) \underbrace{\frac{1}{R^3_{*c} \beta^6_c} \int d\tilde{T} \tilde{T}^6 \nu(\tilde{T}) \, |A(\frac{q\tilde{T}}{\beta_c})|^2}_{\equiv P_v(q)},$$

Dimensionless Velocity Power Spectrum

- Causality arguments require $\mathscr{P}_{GW}$ to go as k^9 and k^3 for low/high frequencies
- $\mathscr{P}_v$ should go as k^5 and k^{-1}
- Contains information on the shape of fluid shells, fluid shell thickness, wall speed, and peak amplitude

$$\mathscr{P}_v = 2\frac{(qR_*)^3}{2\pi^2 R_*^3}P_v(q)$$

$$\beta_c = (8\pi)^{1/3}\frac{v_w}{R_{*c}}$$



$\alpha_n = 0.0046, v_w = 0.92, a = 1$

Shear Stress UETC in Expanding Universe

- T.T. component of energy momentum tensor is the source of the GWs from the metric perturbations
- Fluid variables can be rescaled
- Needed to compute $\langle \dot{h}_{ij}(t,\mathbf{q})\dot{h}_{ij}(t,\mathbf{k})\rangle$ to get $P_{\dot{h}}$ and ultimately $\mathscr{P}_{GW}$
- Directly calculated from the velocity power spectrum in the Sound Shell model

$$\langle \pi_{ij}^{T}(\eta_1,\mathbf{k})\pi_{ij}^{T}(\eta_2,\mathbf{q})\rangle = \frac{a_*^8}{a^4(\eta_1)a^4(\eta_2)}\left[\left(\bar{\tilde{\epsilon}}+\bar{\tilde{p}}\right)U_f^2\right]^2 L_f^3\tilde{\Pi}\left(kL_f, k\eta_1, k\eta_2\right)$$

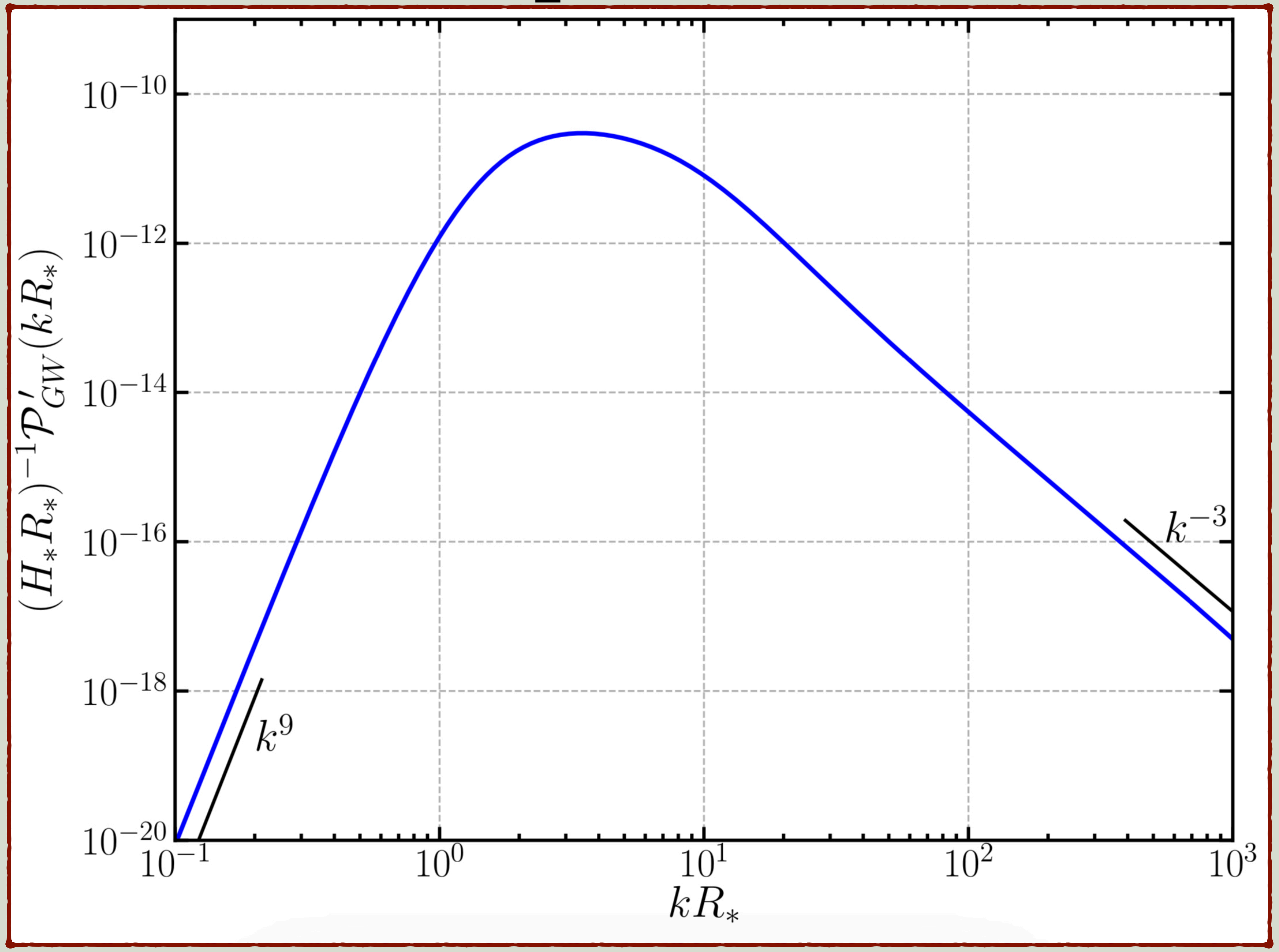
$$\Pi^2\left(k,\eta_1,\eta_2\right) = 4\left(\bar{\epsilon}+\bar{p}\right)^2\int\frac{d^3q}{(2\pi)^3}\frac{q^2}{\tilde{q}^2}(1-\mu^2)^2P_v(q)P_v(\tilde{q})\cos(\omega\eta_-)\cos(\tilde{\omega}\eta_-)$$

Dimensionless GW Power Spectrum

- Assume that the autocorrelation time of the fluid fluctuation is small compared to Hubble time: $x = 1/2(\tilde{\eta}_1 + \tilde{\eta}_2)$ and $z = \tilde{\eta}_1 - \tilde{\eta}_2$, $x \gg z$
- η_* = conformal time at phase transition
- $\mathscr{P}_{GW}$ should go as k^9 and k^{-3} and redshifted by $1/a(\eta)^4$

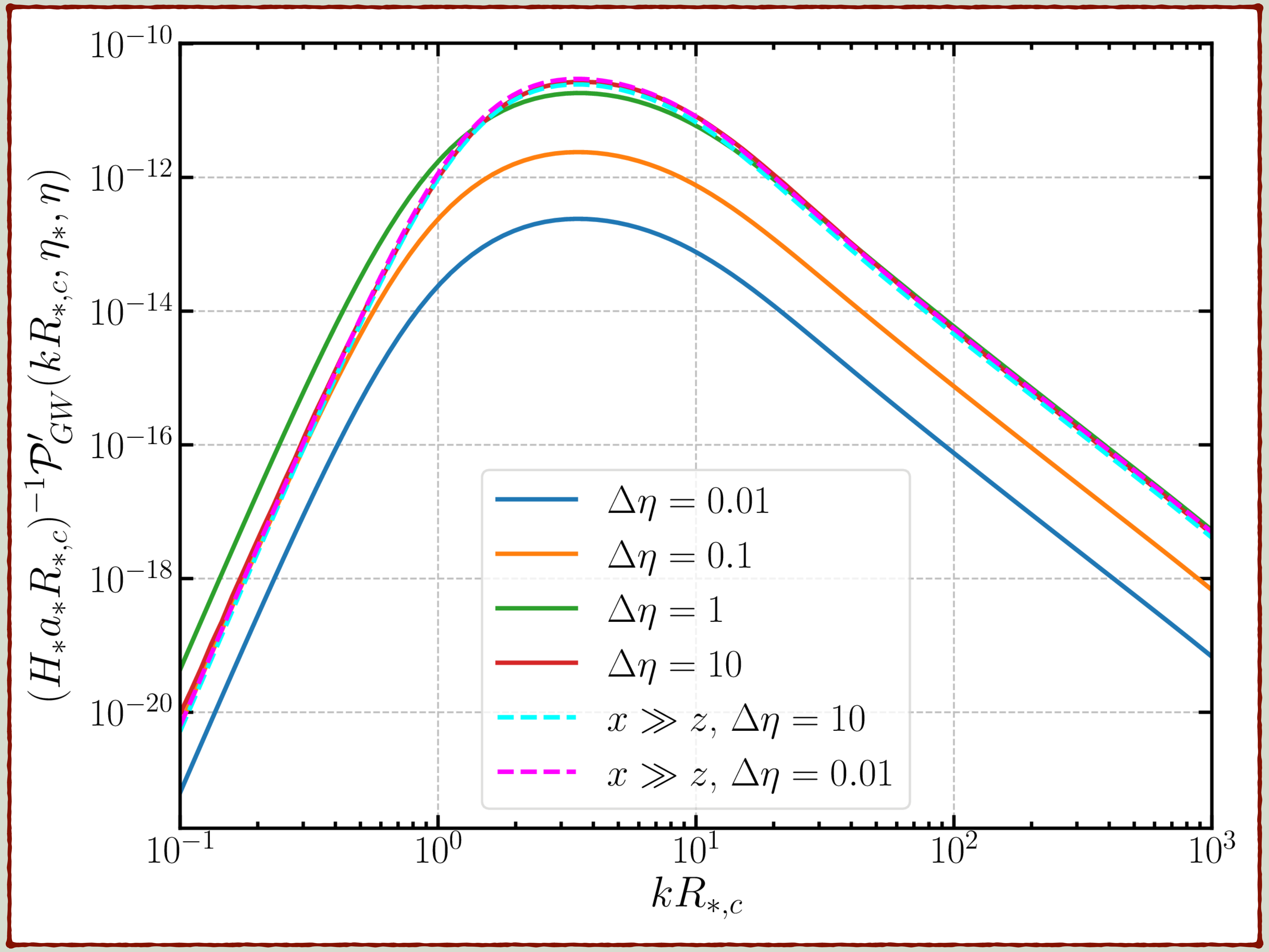
$$\mathscr{P}_{GW} = 3(1+\bar{\omega})^2 \bar{U}_f^2 \left(\frac{a_*^4}{a^4} \frac{H_{*R}^2}{H^2} \right) \left(H_{*R} a_* \eta_*\right) \left(H_{*R} a_* L_f\right) \times$$

$$\frac{\left(kL_f\right)^3}{2\pi^2} \begin{Bmatrix} \tilde{\eta}_*(\tilde{\eta}_*^{-1} + \tilde{\eta}^{-1})(1+\tilde{\eta}^{-2}) \\ 1/3\, \tilde{\eta}_*^3(\tilde{\eta}_*^{-3} + \tilde{\eta}^{-3})(1 + 3\tilde{\eta}^{-2} + 9\tilde{\eta}^{-4}) \end{Bmatrix} \frac{1}{kL_f} \int dz \frac{\cos(z)}{2} \tilde{\Pi}(kL_f)$$



$\alpha_n = 0.0046,\ v_w = 0.92,\ a = 1$

RDE Detonation

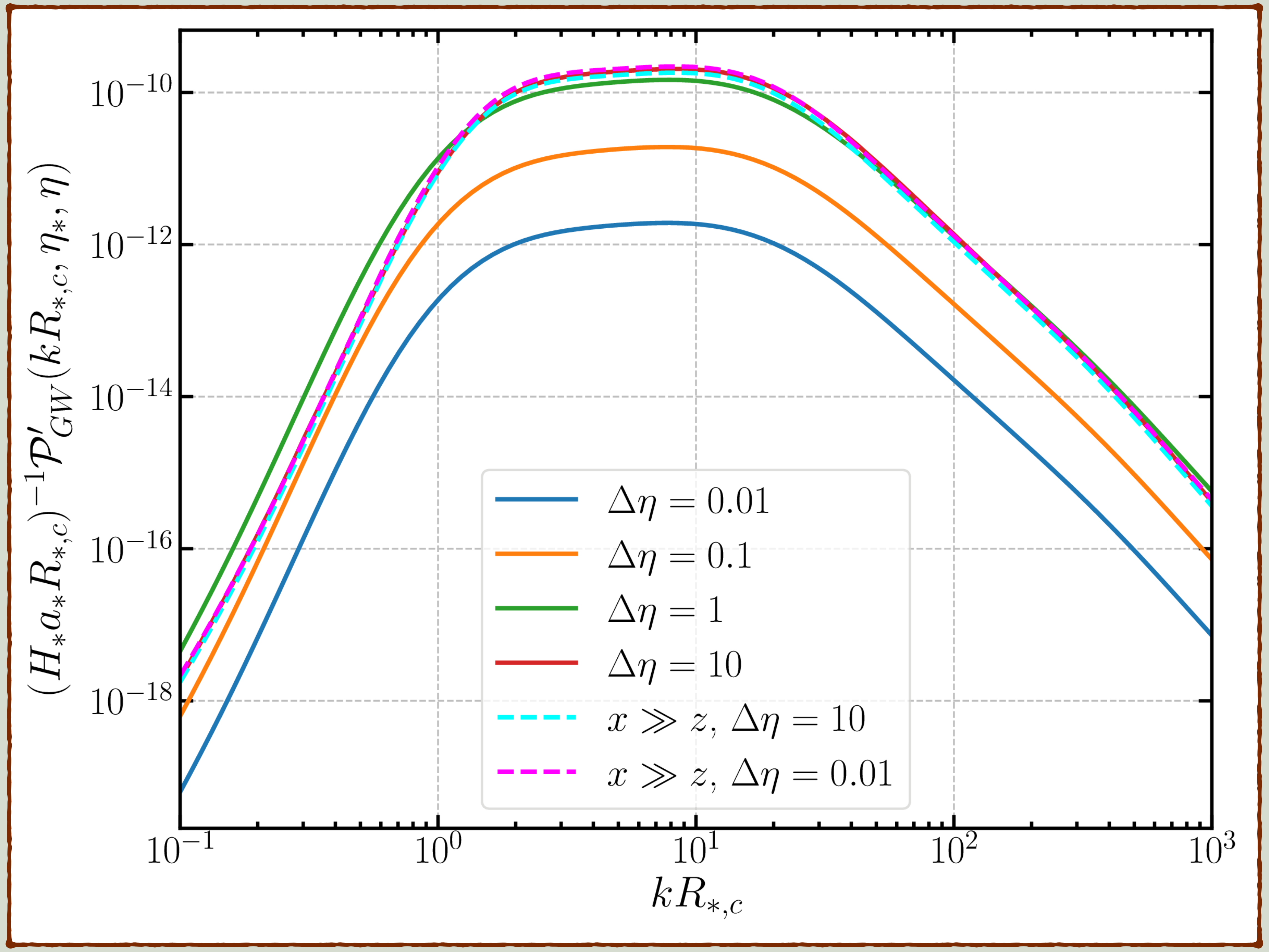


$$\alpha_n = 0.0046,\ v_w = 0.92,\ \Delta\eta = \tilde{\eta} - \tilde{\eta}_*$$

$$\mathcal{P}_{GW} \propto (1 + \tilde{\eta}^{-2}) \int_{\tilde{\eta}_*}^{\tilde{\eta}} dx \int_{z_-}^{z_+} dz \frac{1}{2} \times \frac{\eta_*^2}{\left(x^2 - z^2/4\right)} \cos(z)\tilde{\Pi}(k, q, \tilde{q}, z)$$

- Approximations agree with full calculation for $\eta \gg \eta_*$
- Suppression in spectrum when $\eta_* \sim \eta$
- Slight enhancement for $\eta - \eta_* = 1$ in low frequency regime
- Not yet calculated spectrum observed today $\left(\frac{a_*^4}{a^4}\frac{H_{*,R}^2}{H^2}\right)$

RDE Deflagration

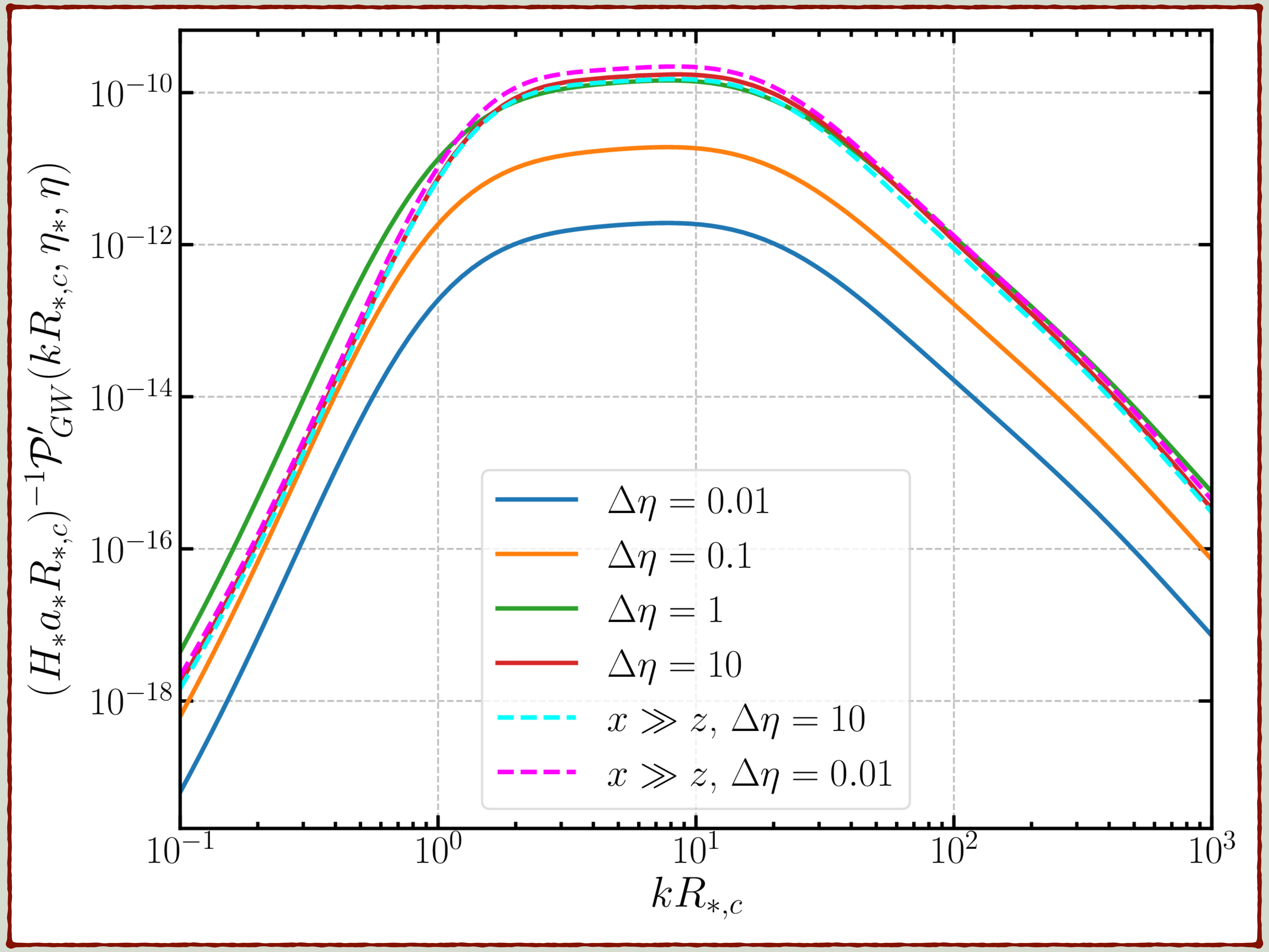


$$\alpha_n = 0.0046,\ v_w = 0.5,\ \Delta\eta = \tilde{\eta} - \tilde{\eta}_*$$

$$\mathcal{P}_{GW} \propto (1 + \tilde{\eta}^{-2}) \int_{\tilde{\eta}_*}^{\tilde{\eta}} dx \int_{z_-}^{z_+} dz \frac{1}{2} \times \frac{\eta_*^2}{\left(x^2 - z^2/4\right)} \cos(z)\tilde{\Pi}(k, q, \tilde{q}, z)$$

- Approximations agree with full calculation for $\eta \gg \eta_*$
- Suppression in spectrum when $\eta_* \sim \eta$
- Slight enhancement for $\eta - \eta_* = 1$ in low frequency regime
- Not yet calculated spectrum observed today

MDE Deflagration



$$\alpha_n = 0.0046,\ v_w = 0.5,\ \Delta\eta = \tilde{\eta} - \tilde{\eta}_*$$

$$\mathcal{P}_{GW} \propto \left(1 + 3\tilde{\eta}^{-2} + 9\tilde{\eta}^{-4}\right) \int_{\eta_*}^{\eta} dx \int_{z_-}^{z_+} dz \frac{1}{2} \times \frac{\eta_*^4}{\left(x^2 - z^2/4\right)^3} \left(z \sin(z) + \left(1 + x^2 - z^2/4\right)\cos(z)\right) \tilde{\Pi}(k, q, \tilde{q}, z)$$

- Approximations agree with full calculation for $\eta \gg \eta_*$
- Suppression in spectrum when $\eta_* \sim \eta$
- Slight enhancement for $\eta - \eta_* = 1$ in low frequency regime
- Not yet calculated spectrum observed today

Conclusion

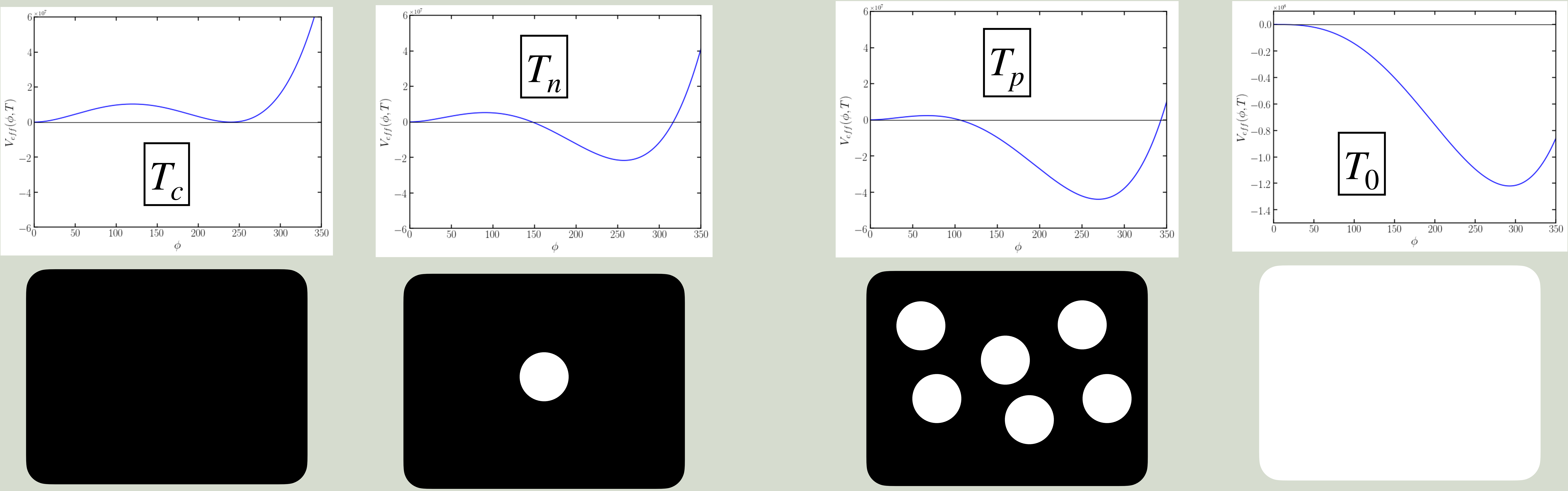
- PT produces GWs through bubble nucleation
- Sound Shell model can be used to calculate the GW spectrum for various cosmological histories
- Suppression for short phase transitions and MDE
- Things to do:
  - Compute the GWs that would be observed today
  - See if there are noticeable deviations for various cosmological histories such as early MDE or Kination

PT can serve as a cosmological witness to non standard cosmological histories which are motived by dark matter and string theory.

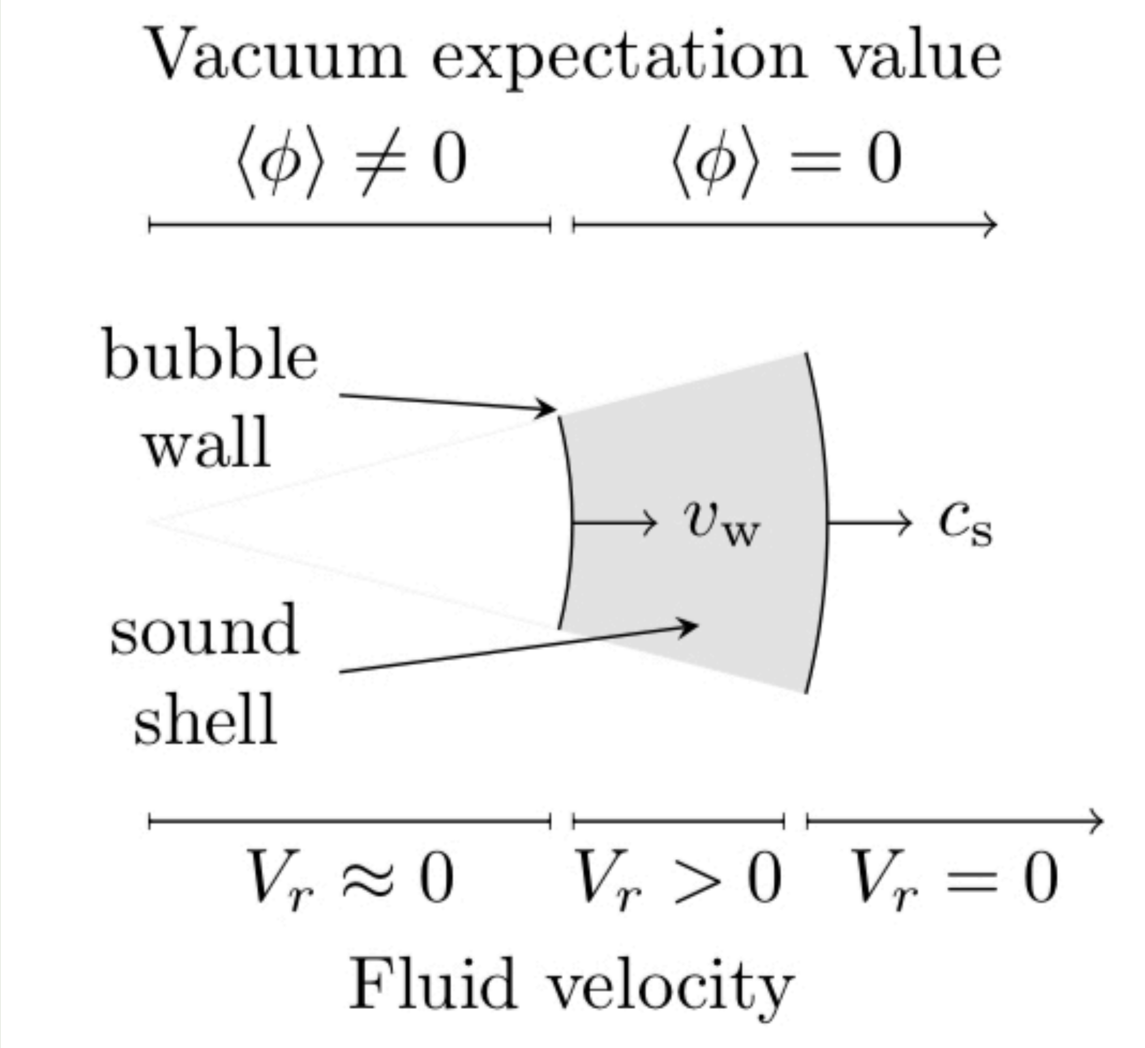
Back Up Slides

Electroweak Phase Transition

- Essential step in EWBG by providing an out of equilibrium environment
- Electroweak symmetry restoration at high T
- First order phase transition proceeds through bubble nucleation
- Dynamics of nucleated bubbles in plasma will generate GWs



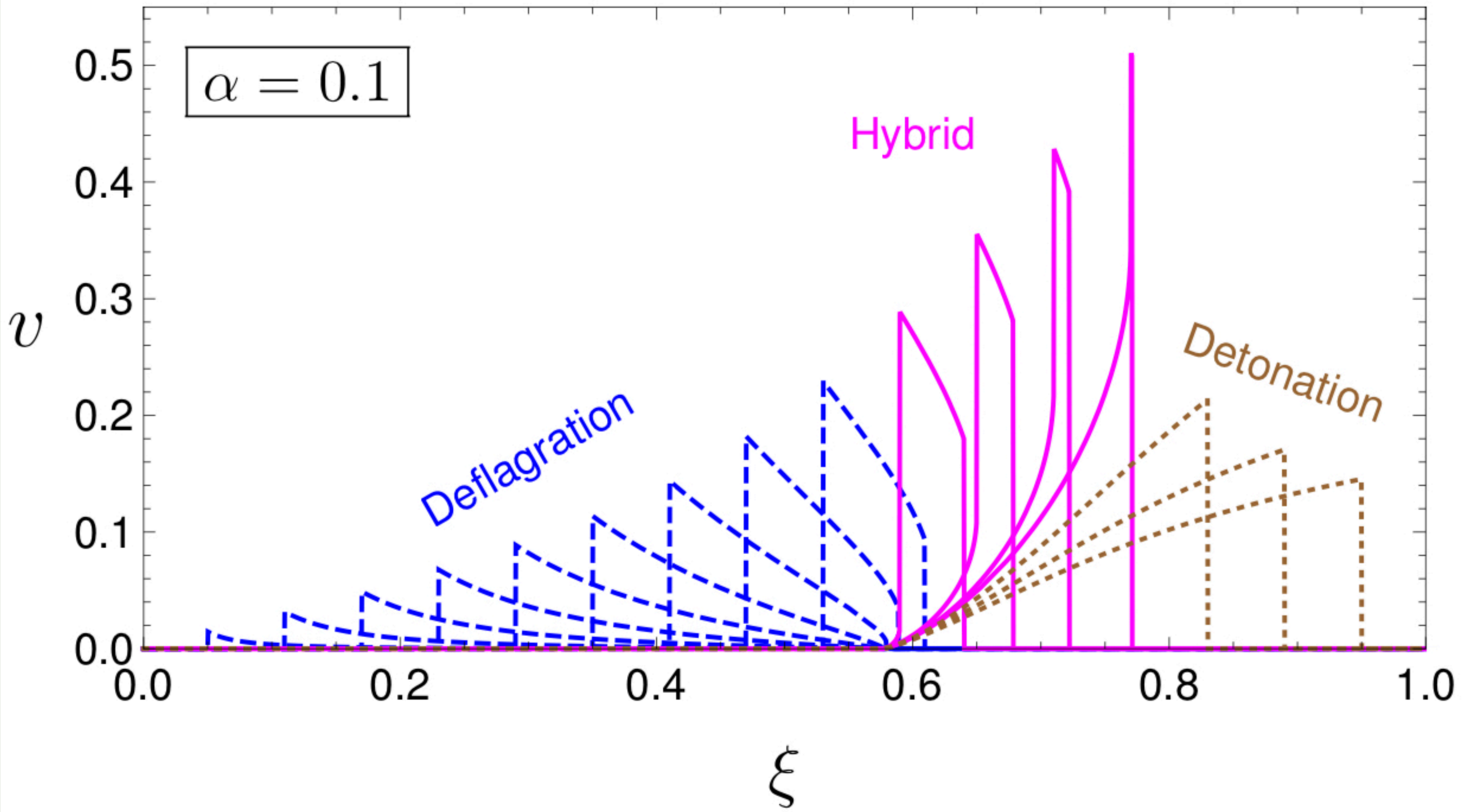
Relativistic Hydrodynamics



- v_w required for EWBG and GW calculations
- Velocity profile computed from boundary conditions at the bubble wall, conserving energy - momentum tensor, and knowledge of the phase transition dynamics (α_n)
- Equations in expanding universe same form as Minkowski space time ($\tilde{\epsilon} = a^4\epsilon$, $\tilde{p} = a^4 p$, and all other quantities in terms of conformal time)

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial\phi^\mu + (\epsilon + p)U^\mu U^\nu + g^{\mu\nu}p$$

$$2\frac{v}{\xi} = \frac{1 - v\xi}{1 - v^2}\left[\frac{\mu^2}{c_s} - 1\right]\partial_\xi v$$



Gravitational Waves in Expanding Universe

- FLRW metric: $ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij}(\mathbf{x}))d\mathbf{x}^2$
- Conformal time: $dt = a\, d\eta$
- Fourier space: $h_{ij}(t, \mathbf{x}) = \int d^3q e^{i\mathbf{q}\cdot\mathbf{x}} h_{ij}(t, \mathbf{q})$
- GW sourced by T.T. Part of perturbed energy momentum tensor: $\delta T_{ij} = a^2 \pi_{ij}^T + \ldots$
- Einstein equation describes time evolution of each Fourier component of GWs:

$$h_q'' + 2\frac{a'}{a}h_q' + q^2 h_q = 16\pi G a^2 \pi_q^T$$

Power Spectrum

- GW energy density: $\rho_{GW}(t) = \frac{1}{32\pi G}\left\langle \dot{h}_{ij}(t,\mathbf{x})\dot{h}_{ij}(t,\mathbf{x})\right\rangle$
- Power spectrum: $\left\langle \dot{h}_{ij}(t,\mathbf{q})\dot{h}_{ij}(t,\mathbf{k})\right\rangle = (2\pi)^{-3}\,\delta^3\left(\mathbf{k}+\mathbf{q}\right)P_{\dot{h}}\left(k,t\right)$
- Dimensionless energy density fraction: $\Omega_{GW} = \frac{\rho_{GW}}{\rho_c}$

$$\mathscr{P}_{GW} = \frac{d\Omega_{GW}}{d\ln k} = \frac{1}{24\pi^2 H^2}k^3 P_{\dot{h}}\left(t,k\right)$$

- $\mathscr{P}_{GW} \sim \frac{1}{a^4}$ for deep in the horizon
- Use fluid-scalar system to build model for $P_{\dot{h}}$ - Sound Shell Model

Fluid and Scalar Field System

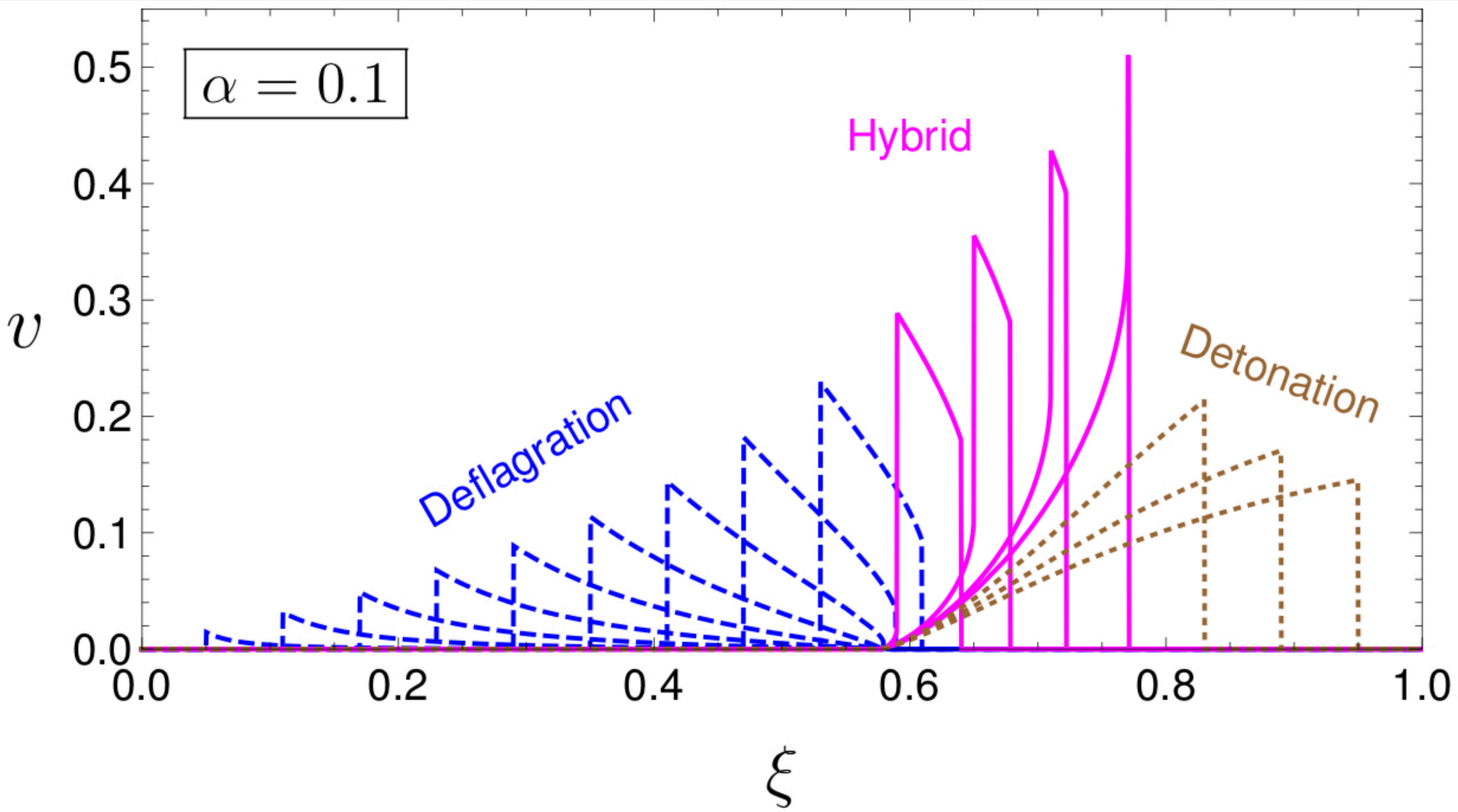
- Long lasting sound waves produced during phase transition dominate source of GWs
- Numerical simulations based on fluid-scalar system in Minkowski Space
- Connect equations to FLRW metric
- $T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial \phi^\mu + (\epsilon + p) U^\mu U^\nu + g^{\mu\nu} p$
- Scalar/Vector components of the conservation of energy momentum equation in a scalar universe can be rescaled to match Minkowski form ($\tilde{\epsilon} = a^4 \epsilon, \tilde{p} = a^4 p$, and all other quantities in terms of conformal time)
- Can neglect scalar field and analytically determine the velocity profiles of the plasma

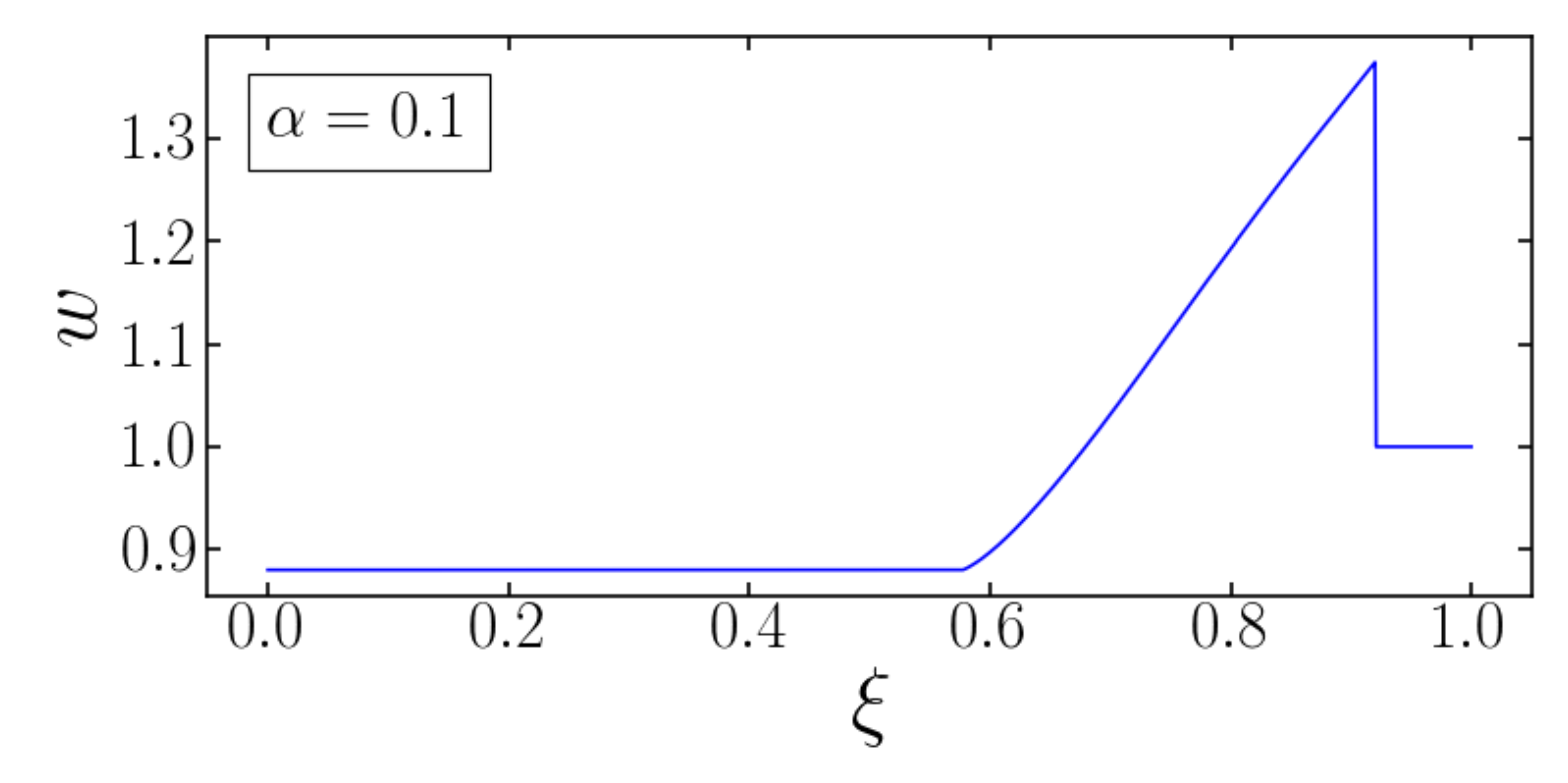
Fluid System

$$\left(a^4 S^i\right)' + \nabla \cdot \left(a^4 S^i \mathbf{v}\right) + \partial_i \left(a^4 p\right) = 0$$

$$\left(a^4 \epsilon \gamma\right)' + \left[\gamma' + \nabla \cdot (\gamma \mathbf{v})\right] \left(a^4 p\right) + \nabla \cdot \left(a^4 \epsilon \gamma \mathbf{v}\right) = 0$$

$$\gamma^2 \left(v' + \frac{1}{2} \hat{\mathbf{v}} \cdot \nabla v^2\right) \left[a^4 \left(\epsilon + p\right)\right] + v \left(a^4 p\right)' + \hat{\mathbf{v}} \cdot \nabla \left(a^4 p\right) = 0$$





Dynamics of Phase Transition

- Changes to the dynamics of the Phase Transition in an expanding universe
- Bubble nucleation rate, fraction of false vacuum, unbroken area of the walls at a certain time, bubble final radius, lifetime distributions, bubble number density, and β/H_n
- Changes due to Hubble parameter through scale factor
- Number of nucleated bubbles

$$N_b = \int_{t_1}^{t_2} p(t)g(t_c, t)a^3 V_{\mathrm{cov}}(t)dt$$

Bubble Nucleation Rate

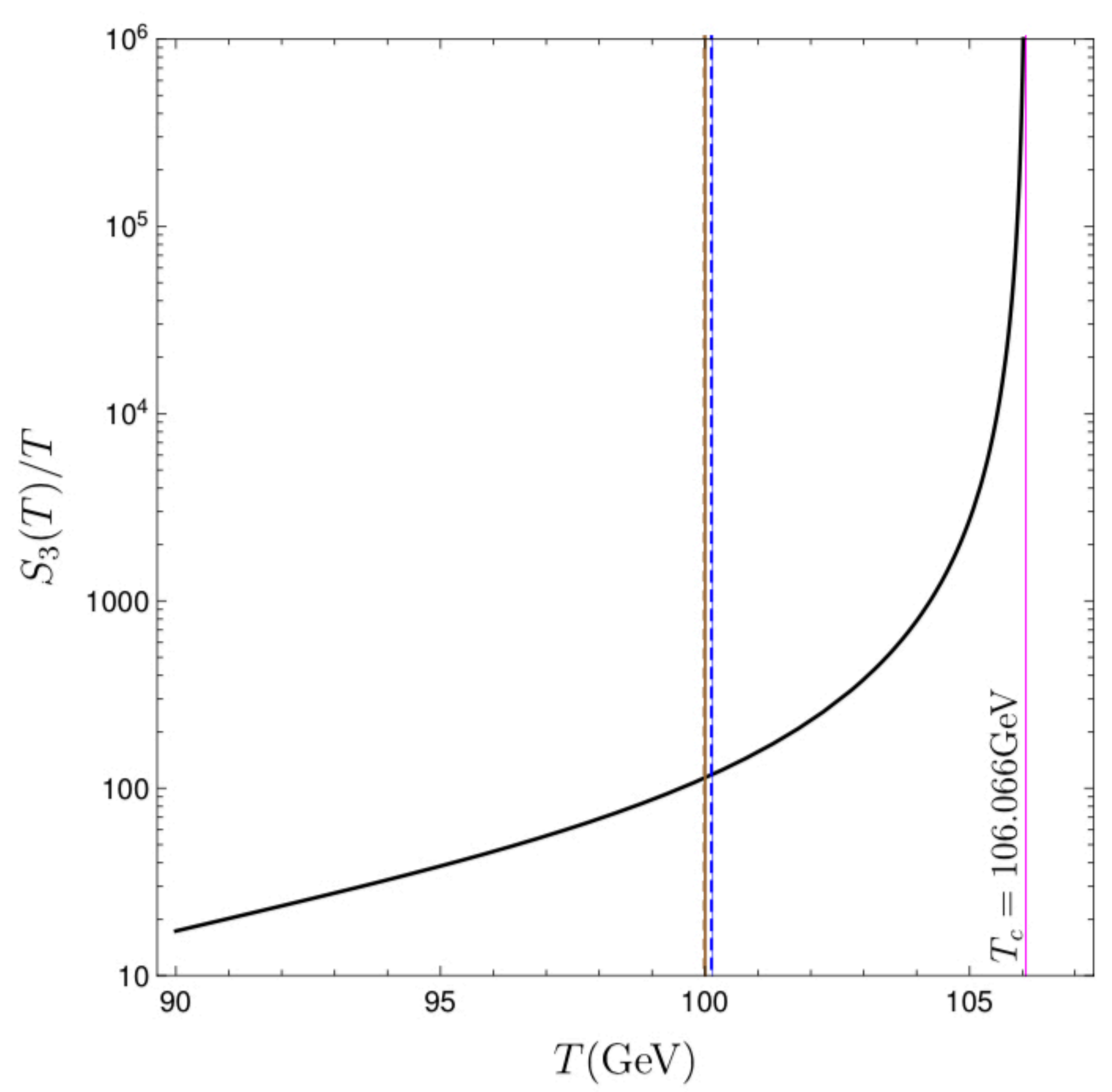
$$p(t) = \begin{cases} p_0 \exp\left[-\frac{S_3(T_0)}{T_0} + \beta(t - t_0)\right] \\ p_0 \exp\left[-\frac{S_3(T_0)}{T_0} - \frac{1}{2}\beta_2^2(t - t_0)^2\right] \end{cases}$$

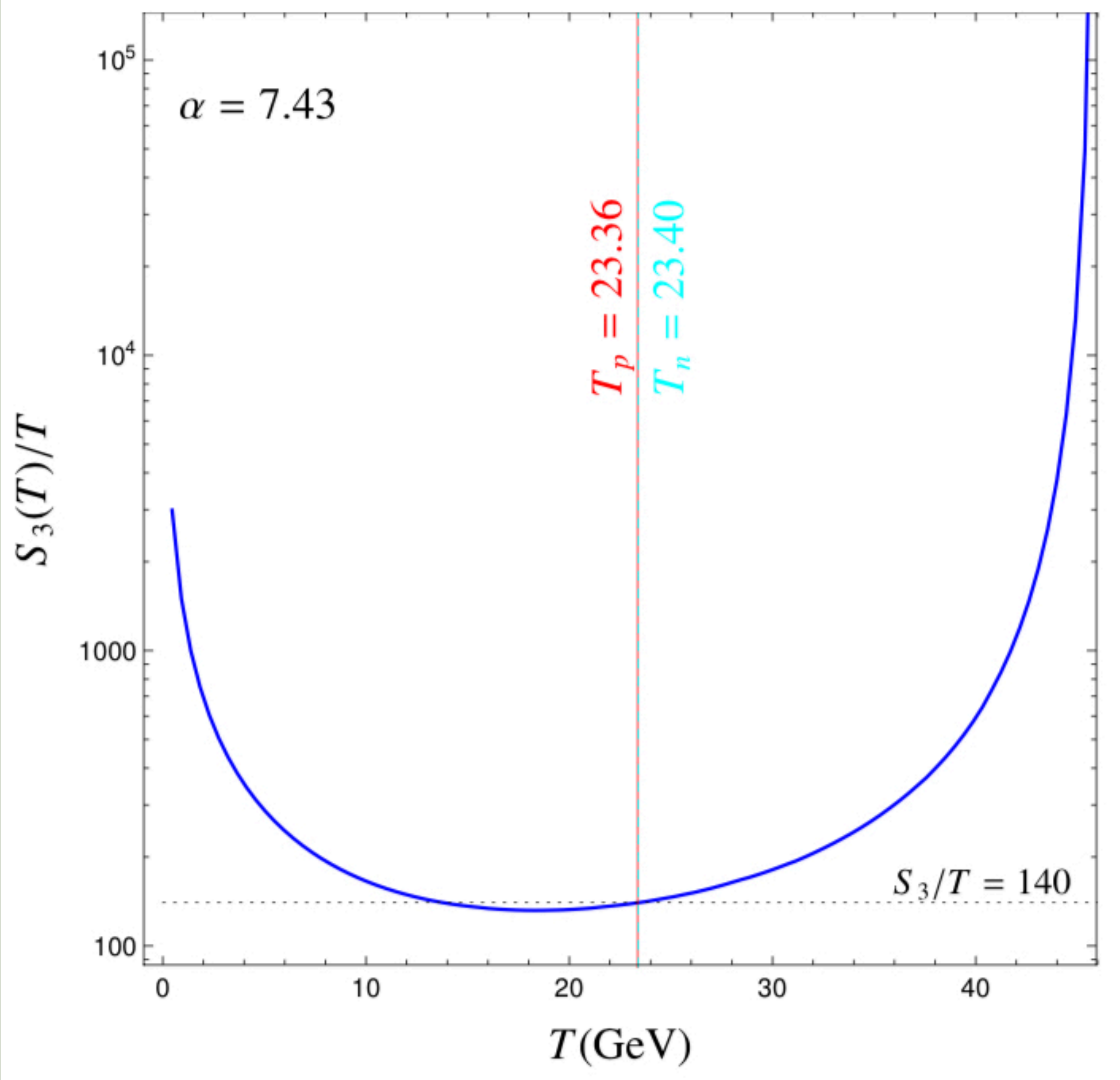
$$S_3\left(\vec{\phi}, T\right) = 4\pi \int dr r^2 \left[\frac{1}{2}\left(\frac{d\vec{\phi}(r)}{dr}\right)^2 + V\left(\vec{\phi}, T\right)\right]$$

$$\left.\frac{d\vec{\phi}(r)}{dr}\right|_{r=0} = 0, \quad \vec{\phi}(r = \infty) = \vec{\phi}_{\text{out}}$$

No T = 0 barrier

Exponential





T = 0 barrier

Instantaneous

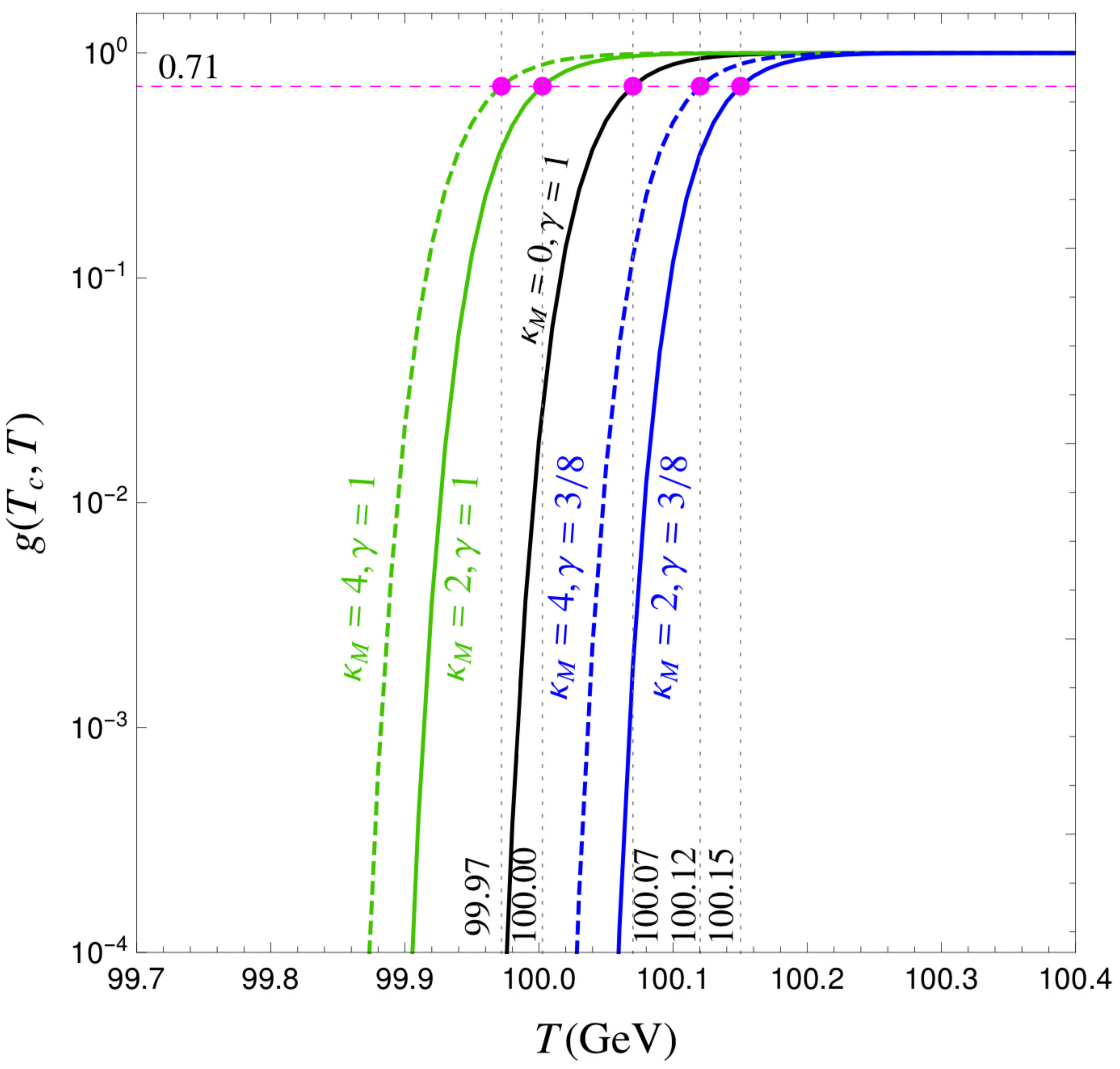
False Vacuum Fraction

$$g(t_c, t) = \exp\left[-\frac{4\pi}{3}\int_{t_c}^{t} dt' p(t') a^3(t') r(t', t)^3\right] \equiv \exp\left[-I(t)\right]$$

T dependence of $g\left(t_c, t\right)$ through scale factor

$$r\left(t', t\right) = \int_{t'}^{t} dt'' \frac{v_w}{a\left(t''\right)} \qquad \frac{a}{a_c} = \left(\frac{T_c}{T}\right)^{-1/\gamma}$$

$$H(T)^2 = \frac{8\pi G}{3}\rho_{R,c}\left(\frac{a_c}{a}\right)^3\left(\kappa_M + \frac{a_c}{a}\right)$$



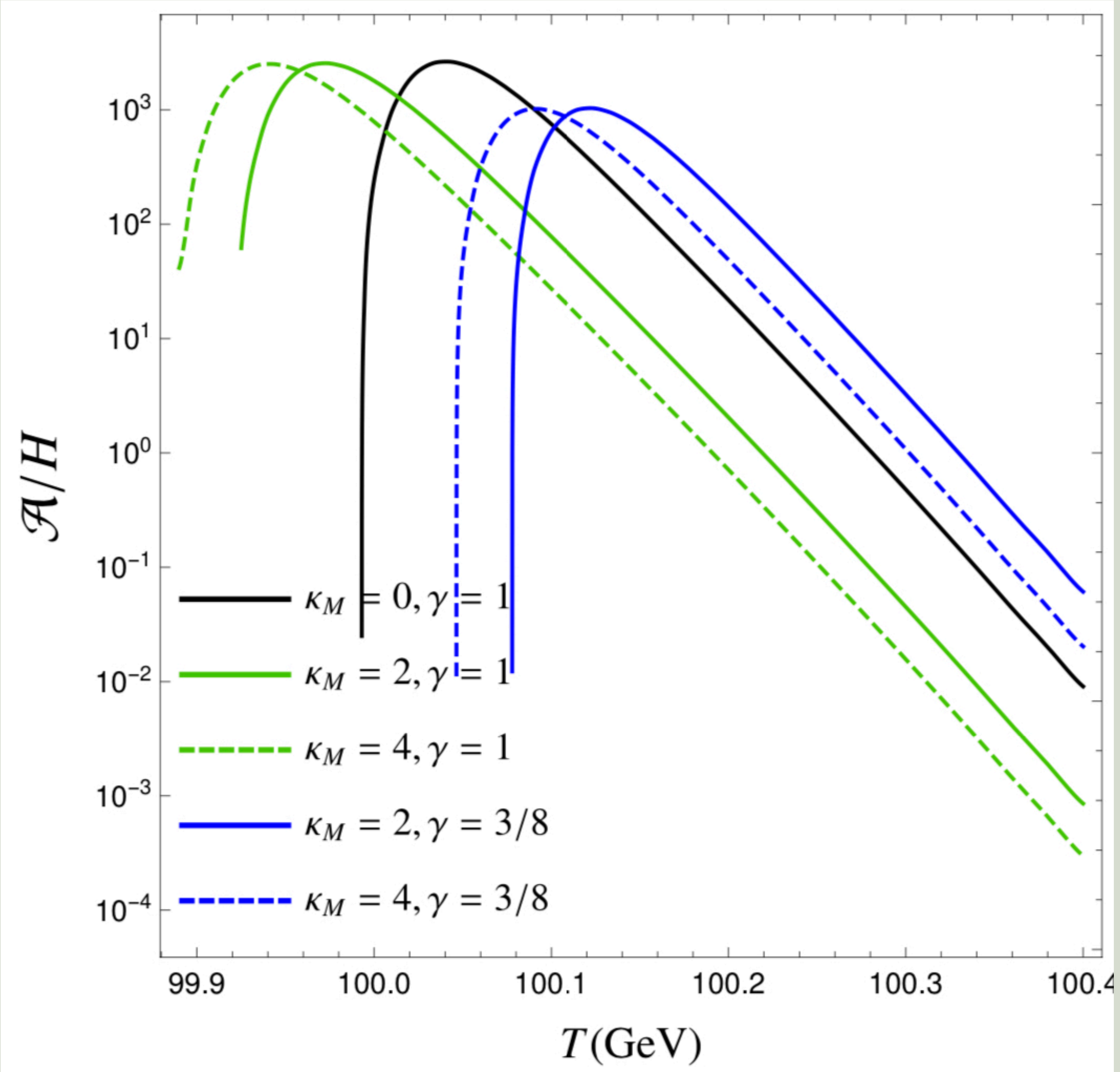
Unbroken Bubble Wall Area

$$dg(t_0, t) = \frac{dV_{\text{False}}}{V_{\text{All}}} = -\mathcal{A}_c(t)v_w\frac{dt}{a}$$

$$\mathcal{A} = \frac{1}{a}\mathcal{A}_c$$

$$\boxed{\mathcal{A} = \frac{\gamma H(T)T}{v_w}\frac{dg\left(T_C, T\right)}{dT}}$$

- Area increases as bubbles form and expand
- Area decreases as bubbles collide and $V_{\text{False}} \to 0$



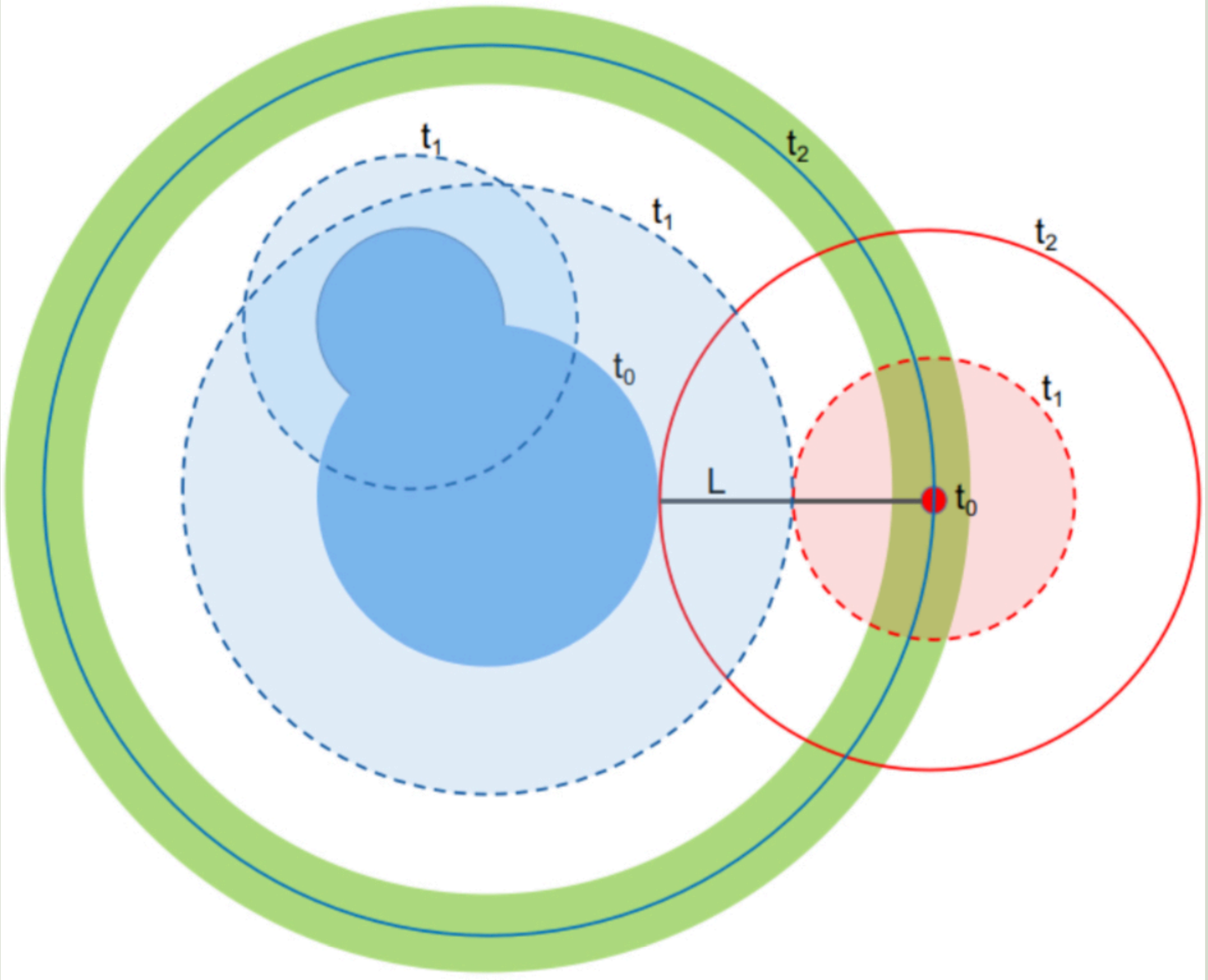
Bubble Lifetime Distribution

$$\boxed{\tilde{n}_{b,c}\left(\eta_{\text{LT}}\right) = v_w \int_{t_c}^{\infty} dt' p(t') a^3(t') \mathscr{A}_c\left(t', v_w \eta_{\text{LT}}\right)}$$

$$r = \int_{t_i}^{t_f} dt \frac{v_w}{a(t)} = v_w \eta_{LT}$$

$$\eta' - \eta_c = \int_{t_c}^{t'} \frac{dt}{a(t)} = \frac{1}{a_c} \int_{T'}^{T_c} \frac{dT''}{T''} \frac{1}{\gamma H\left(T''\right)} \left(\frac{T_c}{T''}\right)^{-1/\gamma} \equiv \Delta_\eta\left(T', T_c\right)$$

$$\eta_{\text{LT}} + \left(\eta' - \eta_0\right) = \Delta_\eta\left(T, T_c\right)$$



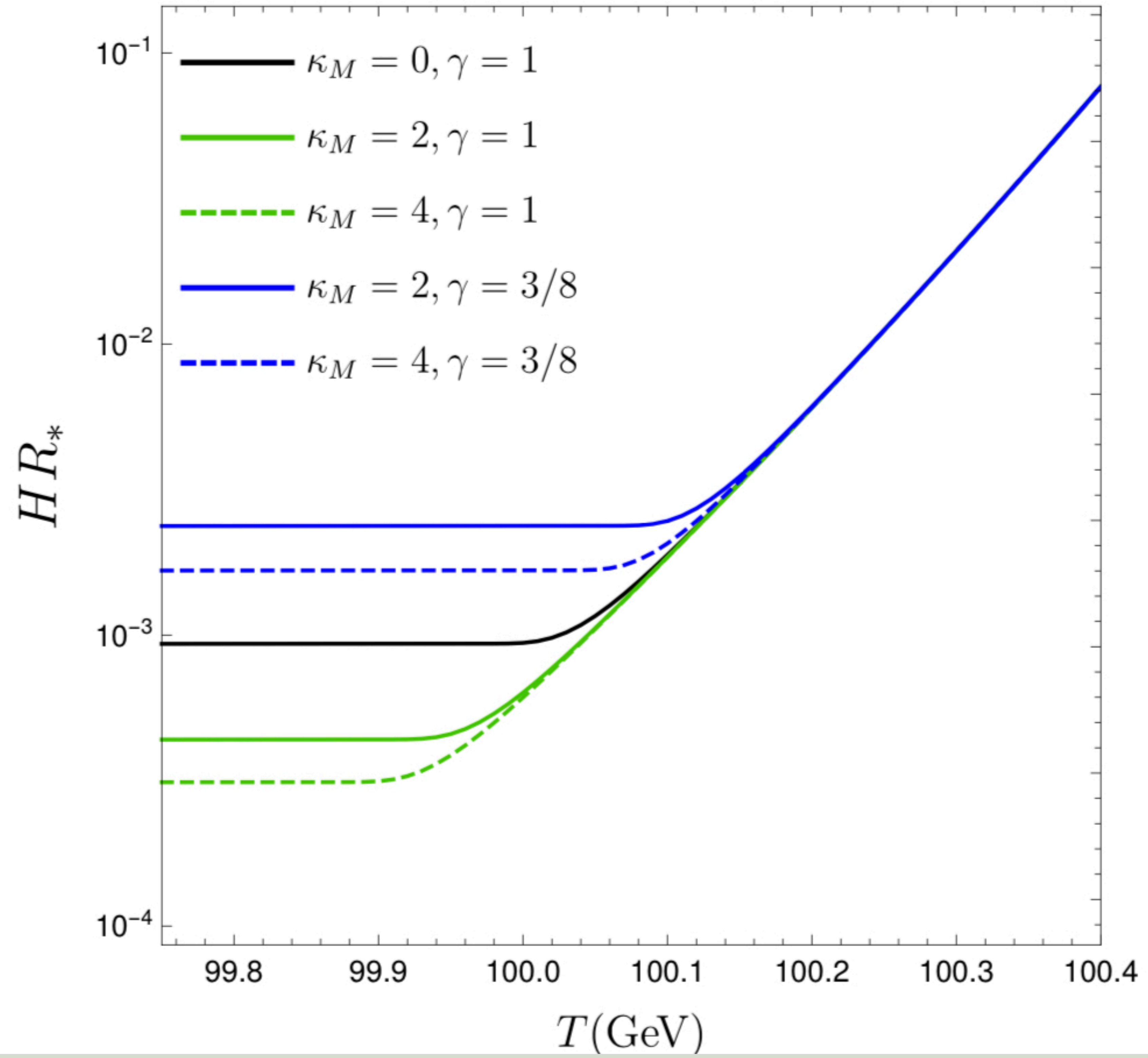
$t \to t_2$, and $t' \to t_0$

Mean Bubble Separation

$$R_*(t) = \left(\frac{V_{\text{Physical}}}{N_b(t)}\right)^{1/3} = \left(\frac{1}{n_b(t)}\right)^{1/3}$$

$$n_b = \frac{N_b}{V_{\text{Physical}}} \qquad \frac{d\left[n_b a^3\right]}{dt} = p(t) g\left(t_c, t\right) a^3(t)$$

$$n_b(T) = \left(\frac{T}{T_c}\right)^{(3/\gamma)} \int_T^{T_c} \frac{dT'}{T} \frac{1}{\gamma H(T')} \bar{p}_0 T'^4 \exp\left[-\frac{S_c(T')}{T'}\right] g\left(T_c, T\right) \left(\frac{T_c}{T'}\right)^{3/\gamma}$$



- R_* increases for delayed false vacuum fraction

Nucleation Temperature

Probability that one bubble nucleates per Hubble volume at T_n

$$\int_{t_c}^{t_n} dt \frac{p(t)}{H(t)^3} = 1$$

$T \propto a^{-\gamma}, a = c_a t^n, H = \dot{a}/a$

$$\int_{T_n}^{T_c} \frac{dT}{T} \frac{p(T)}{\gamma H(T)^4} = 1$$

Radiation dominated universe

$$\int_{T_n}^{T_c} \frac{dT}{T} \left(\frac{90}{8\pi^3 g_*}\right)^2 \left(\frac{m_P}{T}\right)^4 \exp\left[-\frac{S_3(T)}{T}\right] = 1$$

- MDE and $\gamma = 1$: same form as RDE but with different H(T). $H^{\text{MDE}}(T) > H^{\text{RDE}}(T)$. Harder to satisfy criteria and lower T_n
- MDE and $\gamma = 3/8$: criteria easier to satisfy within same Hubble Volume.

Percolation Temperature

Temperature at which the true vacuum is 30 % of the total volume

$$\boxed{p(t_p) = 0.7, \text{ or } I(t_p) \approx 0.34}$$

Strong super-cooling (VDU) $\left.\frac{1}{a^3(t)V_{\text{False}}}\frac{d\left[a^3(t)V_{\text{False}}\right]}{dt}\right|_{t=t_p} < 0$

Inverse Time Duration

$$\frac{S_3}{T} = \frac{S_3}{T}\bigg|_{t_n} + \frac{d\left(S_3/T\right)}{dT}\bigg|_{t_n}(t - t_n) \qquad \frac{\beta}{H_n} = -\frac{1}{H_n}\frac{dT}{dt}\frac{d\left(S_3/T\right)}{dT}\bigg|_{t_n}$$

$$a = c_n t^n$$

$$s_R(T)a^3 = \text{const.}$$

<u>No injection</u>

$$s_R \propto T^3 \rightarrow T \propto 1/a \propto t^{-n}$$

<u>Injection</u>

$$T \propto a^{-3/8}$$

<u>In general</u>

$$T = c_T t^{-n\gamma}$$

$$\boxed{\frac{\beta}{H_n} = \gamma T \frac{d\left(S_3/T\right)}{dT}\bigg|_{t=t_n}}$$

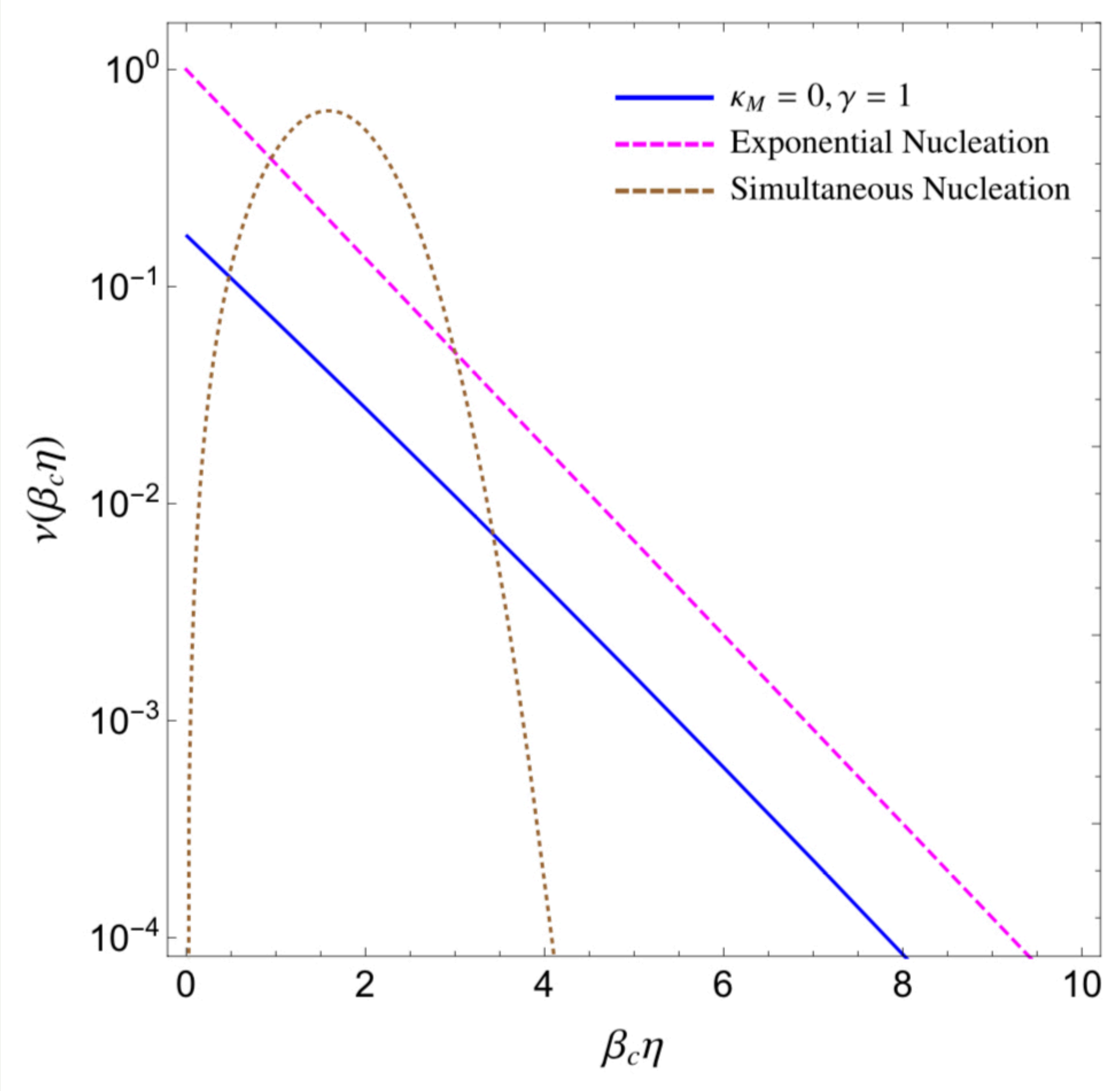
Bubble Lifetime Distribution

$$\nu\left(\tilde{T}\right) = \int_{T_*}^{T_c} \frac{dT'}{T'} \frac{1}{\gamma H(T')} \bar{p}_0 T'^4 \exp\left[-\frac{S_3(T')}{T'}\right] R_*(T')^3 \bar{\mathscr{A}}_c\left(T\left(T', \tilde{T}\right)\right)$$

- Normalized to 1
- Related to the probability distribution of lifetimes

$$n(T_i)dT_i = \frac{\beta}{R_*^3}\nu(\beta T_i)dT_i$$

- Exponential or Simultaneous bubble nucleation



One Bubble

Before collision

$$v^i(\eta < \eta_c, \mathbf{x}) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\left[\tilde{v}^i_{\mathbf{q}}(\eta)e^{i\mathbf{q}\cdot\mathbf{x}} + \tilde{v}^{i*}_{\mathbf{q}}(\eta)e^{-i\mathbf{q}\cdot\mathbf{x}}\right]$$

After collision

$$v^i(\eta > \eta_c, \mathbf{x}) = \int \frac{d^3q}{(2\pi)^3}\left[v^i_{\mathbf{q}}e^{-i\omega\eta+\mathbf{q}\cdot\mathbf{x}} + v^{i*}_{\mathbf{q}}e^{i\omega\eta-i\mathbf{q}\cdot\mathbf{x}}\right]$$

- Equations governing sound waves are 2nd order and require two initial conditions
- Initial conditions for $\tilde{v}^i_{\mathbf{q}}$ and $\tilde{v}^{i'}_{\mathbf{q}}$ at η_{bc}
- Force term in equation governing $\tilde{v}^{i'}_{\mathbf{q}}$ - calculate it from the energy fluctuations

$$\lambda(x) = \frac{e(x) - \bar{e}}{\bar{\omega}}$$

$$\tilde{\lambda}'_{\mathbf{q}} + iq^j\tilde{v}_{\mathbf{q}}{}^j = 0$$

$$\tilde{v}^{j'}_{\mathbf{q}} + ic_s^2 q^j \tilde{\lambda}_{\mathbf{q}} = 0$$

N-th Bubble

- $\tilde{v}^i_{\mathbf{q}}(\eta)$ and $\tilde{v}^{i'}_{\mathbf{q}}(\eta)$ at η_{bc} from self-similar invariant profile of bubble

$$\mathbf{v}^n(\eta, \mathbf{x}) = \hat{\mathbf{R}}(\mathbf{x}) v(\xi) \text{ where } \begin{cases} \mathbf{R} \equiv \mathbf{x} - \mathbf{x}^{(n)} \\ \xi \equiv |\mathbf{R}^{(n)}|/T^{(n)} \\ T^{(n)}(\eta) \equiv \eta - \eta^{(n)} \end{cases}$$

- n-th bubble's contribution to the Fourier coefficient of the sound waves

$$\tilde{v}^{j(n)}_{\mathbf{q}} \eta_{bc} = e^{-i\mathbf{q}\cdot x^{(n)}} \left(T^{(n)}\right)^3 i\hat{z}^j f'(z)\Big|_{\eta=\eta_{bc}}$$

$$\tilde{\lambda}^{(n)}_{\mathbf{q}} = e^{-i\mathbf{q}\cdot \mathbf{x}^{(n)}} \left(T^{(n)}\right)^3 l(z)\Big|_{\eta=\eta_{bc}}$$

$$v^j_{\mathbf{q}} = \frac{1}{2}\left[\tilde{v}^j_{\mathbf{q}}(\eta_{bc} + c_s \tilde{q}^j \tilde{\lambda}(\eta_{bc})\right] e^{i\omega t_c}$$

→

$$\boxed{v^{j(n)}_{\mathbf{q}} = i\hat{z}^j T^{(n)3}_{bc} e^{i\omega\eta_{bc} - i\mathbf{q}\cdot\mathbf{x}^{(n)}_{bc}} A(z_{bc})}$$

$$A(z_{bc}) = \frac{1}{2}\left(f'(z_{bc}) - ic_s l(z_{bc})\right)$$

Functions for Velocity Field

$$f(z) = \frac{4\pi}{z} \int_0^\infty d\xi v(\xi) \sin(\xi z)$$

$$l(z) = \frac{4\pi}{z} \int_0^\infty d\xi \lambda_\xi \sin(\xi z)$$

$v(\xi)$ and $\lambda(\xi)$ invariant profiles from solving fluid equations of motion

Velocity Power Spectrum

- Velocity field, after most bubbles disappear, is obtained by adding all the individual bubble contributions
- Total number of bubbles nucleated within a Hubble volume with comoving size V_c is N_b
- Velocity field follows a Gaussian distribution to a good approximation

$$v^i_{\mathbf{q}} = \sum_{n=1}^{N_b} v^{i(n)}_{\mathbf{q}}$$

- Randomness removed by doing an ensemble average

$$\langle v^i_{\mathbf{q}} v^{j*}_{\mathbf{q}} \rangle = \hat{q}^i \hat{q}^j (2\pi)^3 \delta^3(\mathbf{q}_1 - \mathbf{q}_2) \underbrace{\frac{1}{R^3_{*c}\beta^6_c} \int d\tilde{T}\tilde{T}^6 \nu(\tilde{T}) \, | A(\frac{q\tilde{T}}{\beta_c}) |^2}_{\equiv P_v(q)},$$

Gravitational Spectral Density

$$\langle \dot{h}_{ij}(t,\mathbf{q})\dot{h}_{ij}(t,\mathbf{k})\rangle = (2\pi)^{-3}\delta^3(\mathbf{k}+\mathbf{q})P_{\dot{h}}(k,t)$$

$$P_{\dot{h}} = \frac{a_*^6}{a^4(\eta)}\left[16\pi G(\bar{\tilde{\epsilon}} + \bar{\tilde{p}}U_f^2\right]^2 L_f^3 \left\{\begin{matrix} 1+\tilde{\eta}^{-2} \\ 1+3\tilde{\eta}^{-2}+9\tilde{\eta}^4 \end{matrix}\right\} \int_{\tilde{\eta}_*}^{\tilde{\eta}} dx \int dz$$

$$\tilde{P}_{GW}(kL_f) = \frac{1}{kL_f}\int dz \frac{\cos z}{2}\tilde{\Pi}\left(\tilde{L}_f, z\right) \quad \times \frac{1}{2}\left\{\begin{matrix} \dfrac{\tilde{\eta}_*^2}{x^2 - z^2/4} \\ \dfrac{\tilde{\eta}_*^4}{(x^2 - z^2/4)^3} \end{matrix}\right\} \left\{\begin{matrix} \cos z \\ z\sin z + (1 + x^2 - z^2/4)\cos(z) \end{matrix}\right\} \tilde{\Pi}^2(\tilde{L}_f, \tilde{\eta}_1, \tilde{\eta}_2)$$