Leading Fermionic Three-Loop Electroweak Corrections to EWPOs

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OUTLINE

0. Why precision calculation ?

1. Input Parameters of EWPOs

2. Why leading fermionic 3-loop ?

3. Renormalization

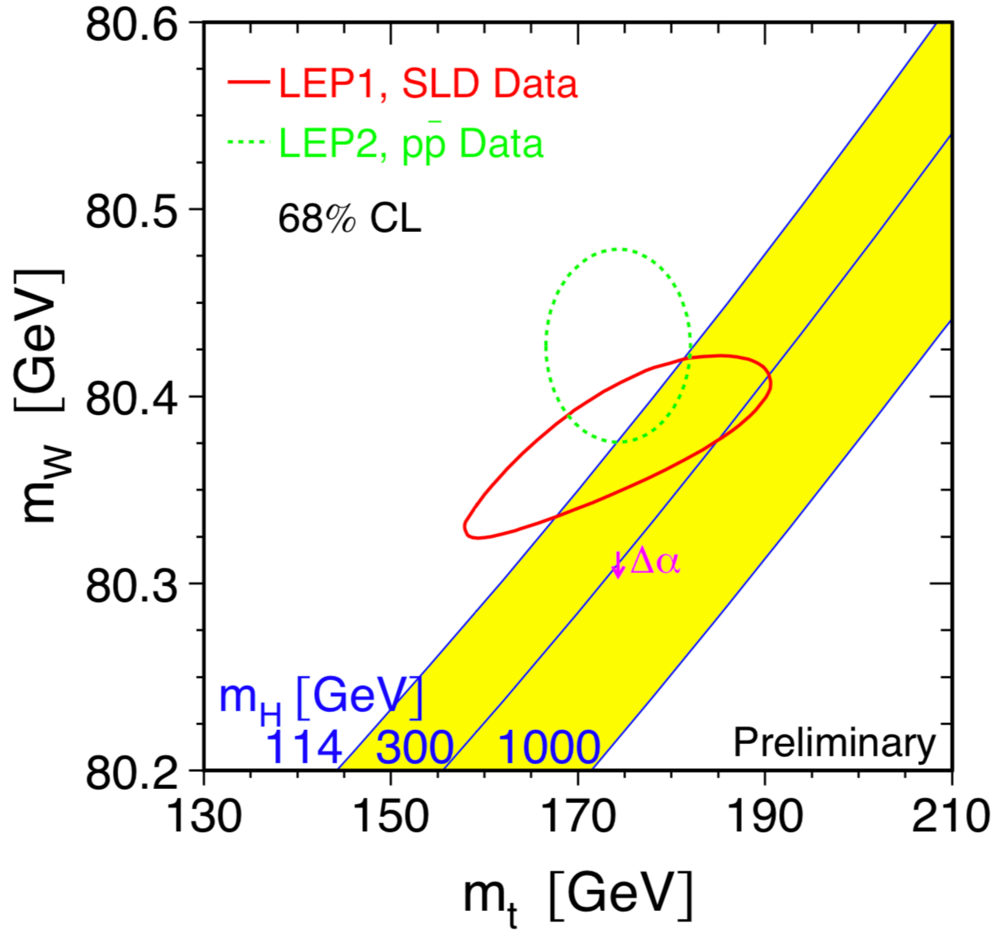
4. Technical Aspects

5. Numerical Results

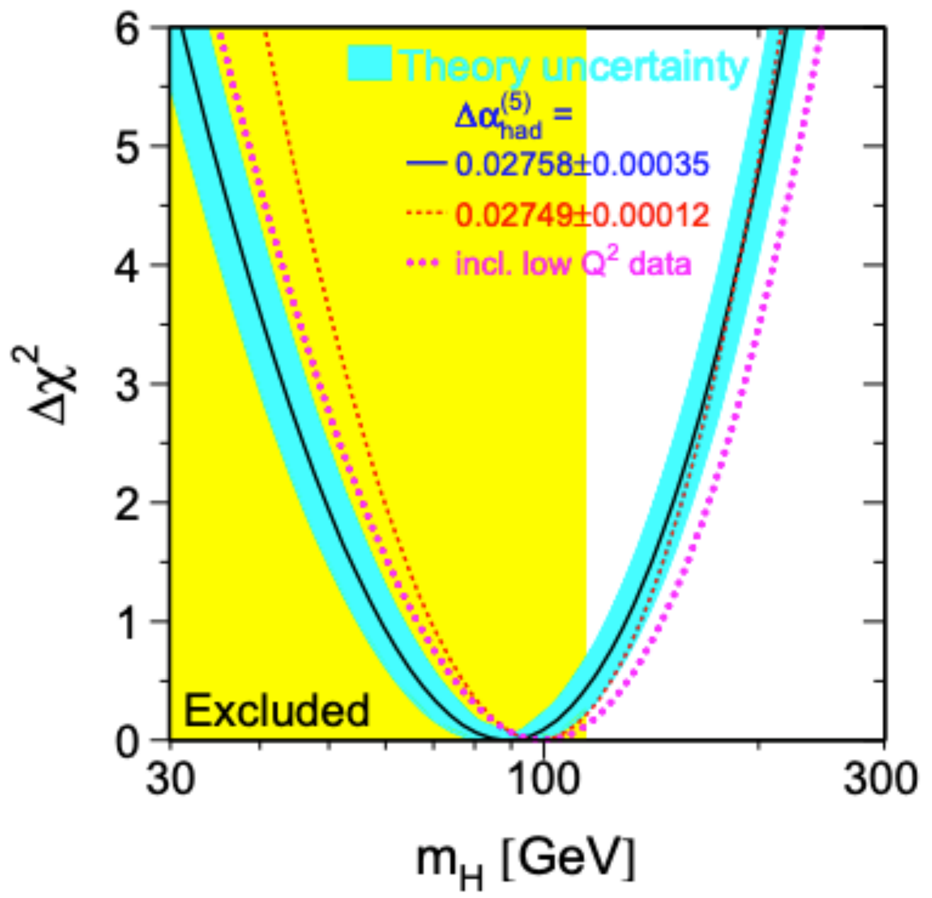
Precision Calculation in SM

1. Why it is important?

Top Mass Indirectly Constrained



Higgs Mass Indirectly Constrained



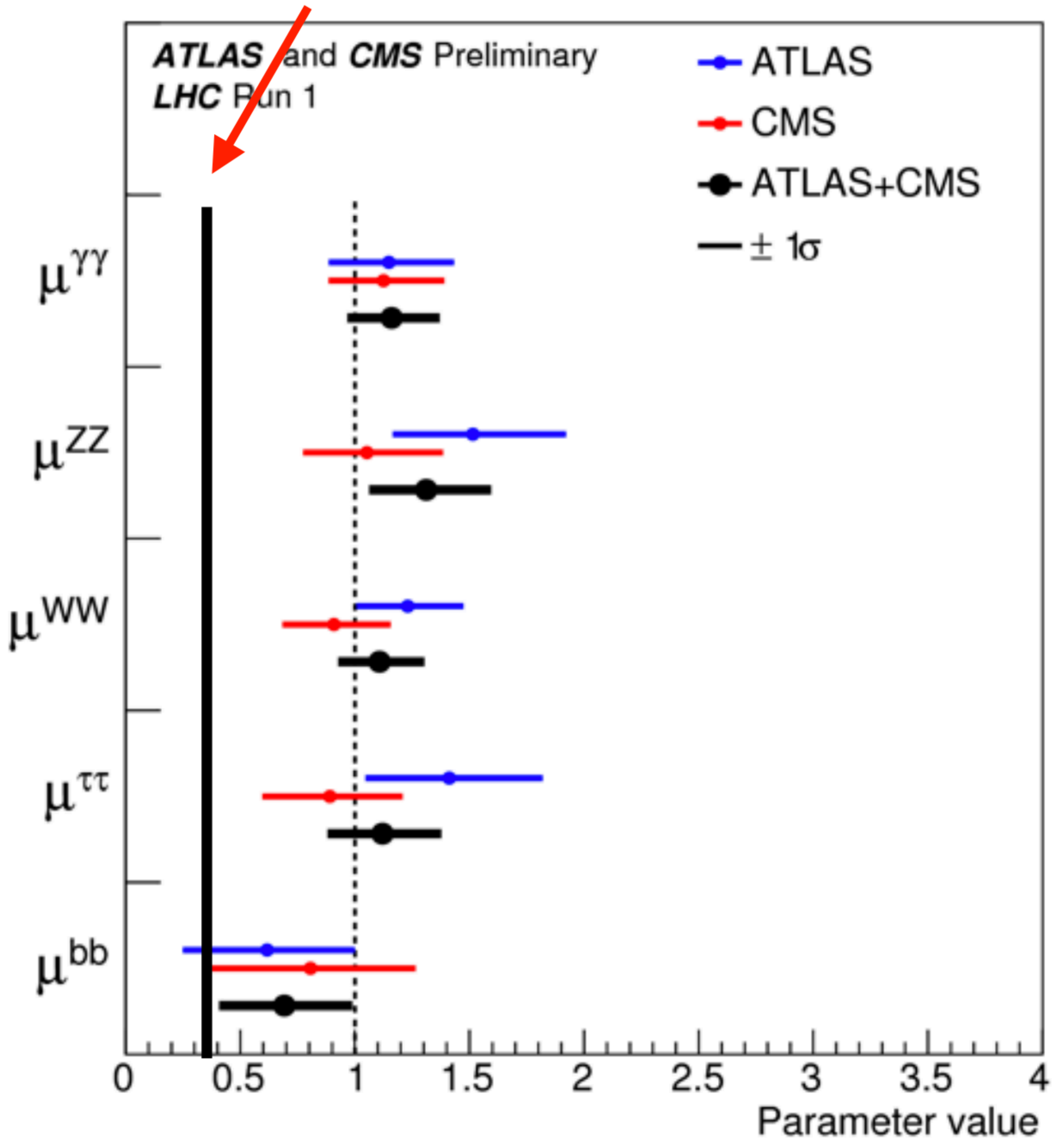
LEPWWG'05

145 GeV < mt < 185 GeV, by 1994!

- Constrainted by quantum correction.
- In agreement with measurements in later experiments.

Great Success of Precision Physics from History!

without higher-order corrections



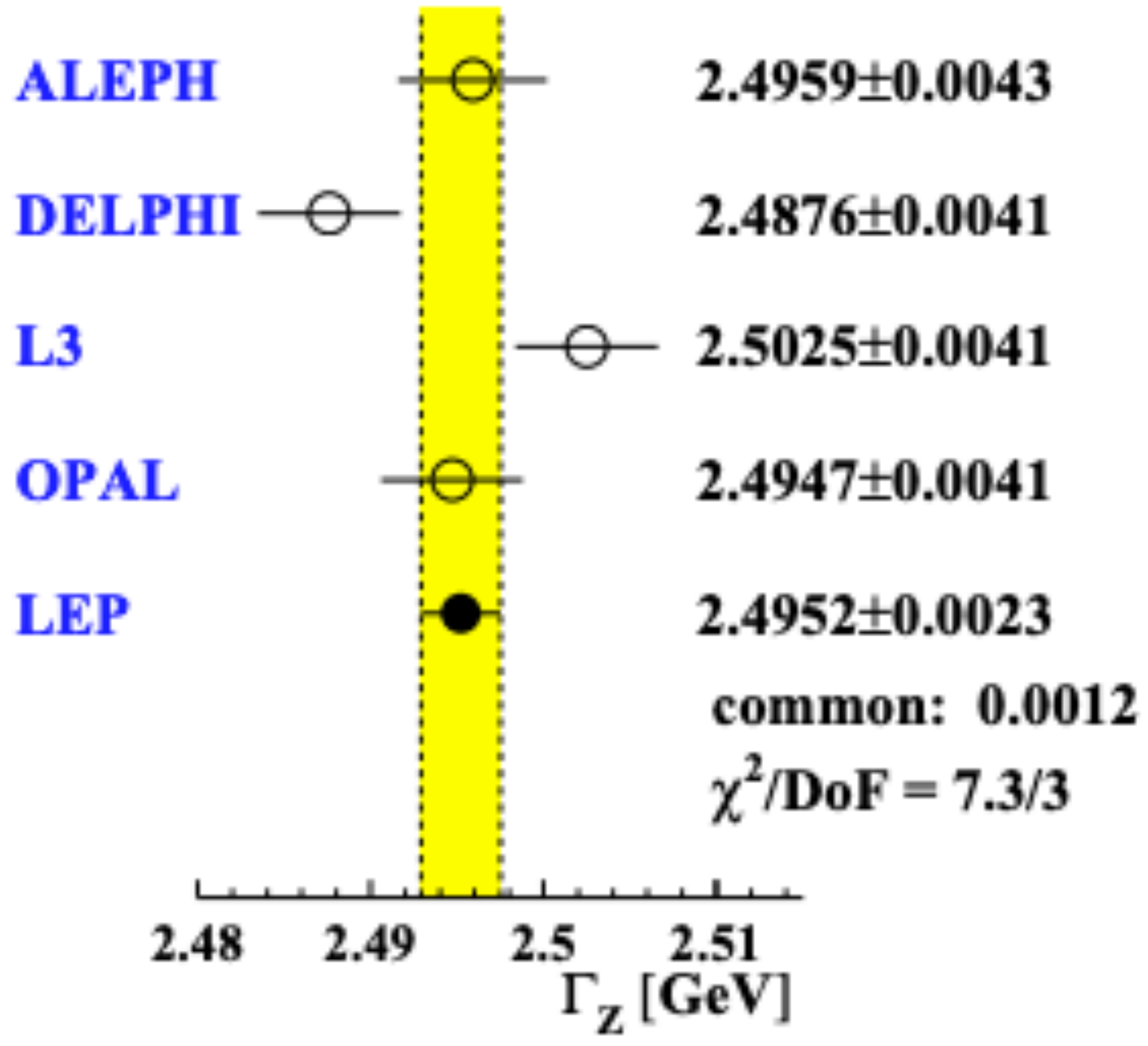
Some Important EWPOs' Introduction

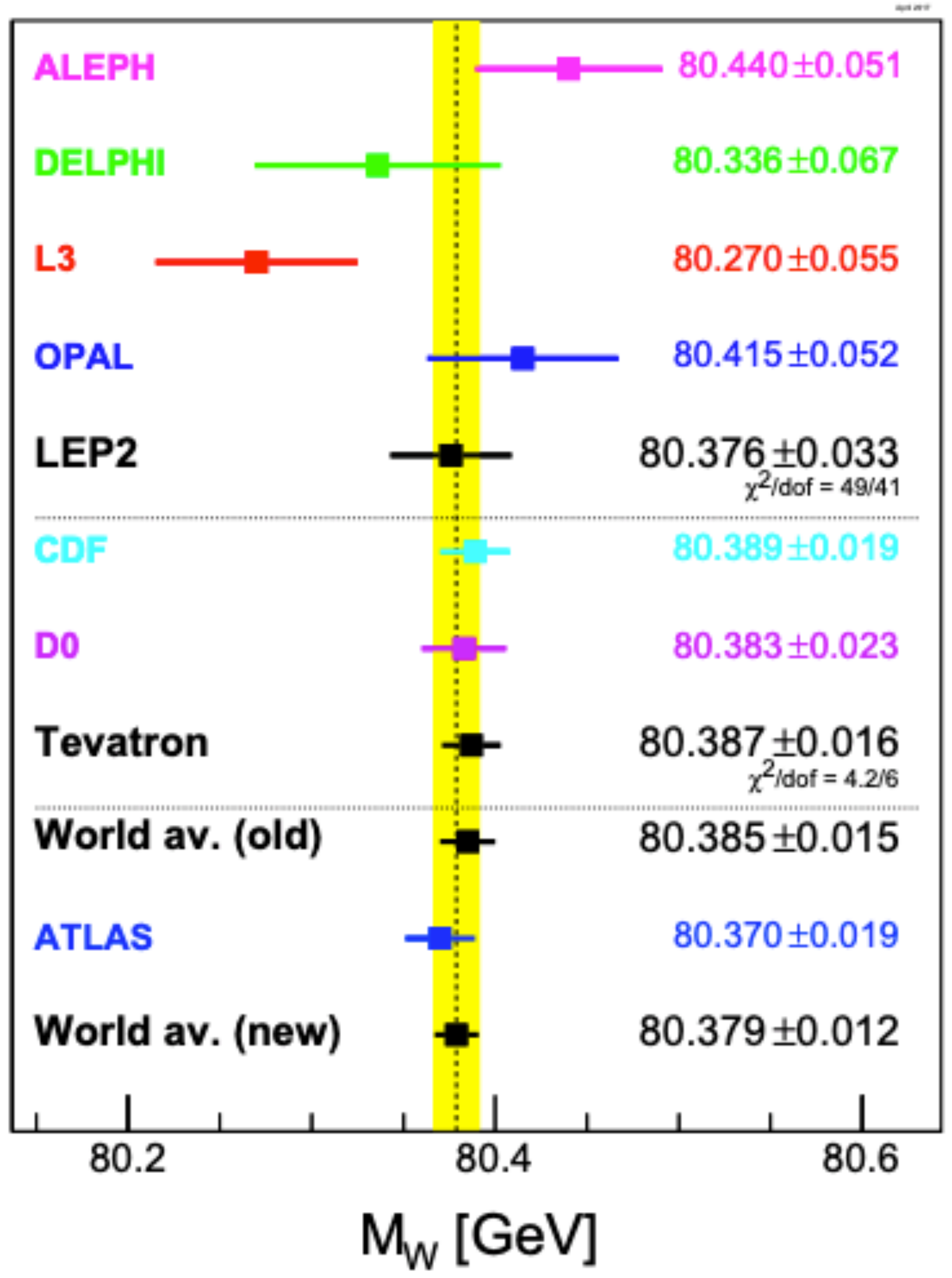
M_W **Can be extracted from muon decay (will discuss in detail later)**

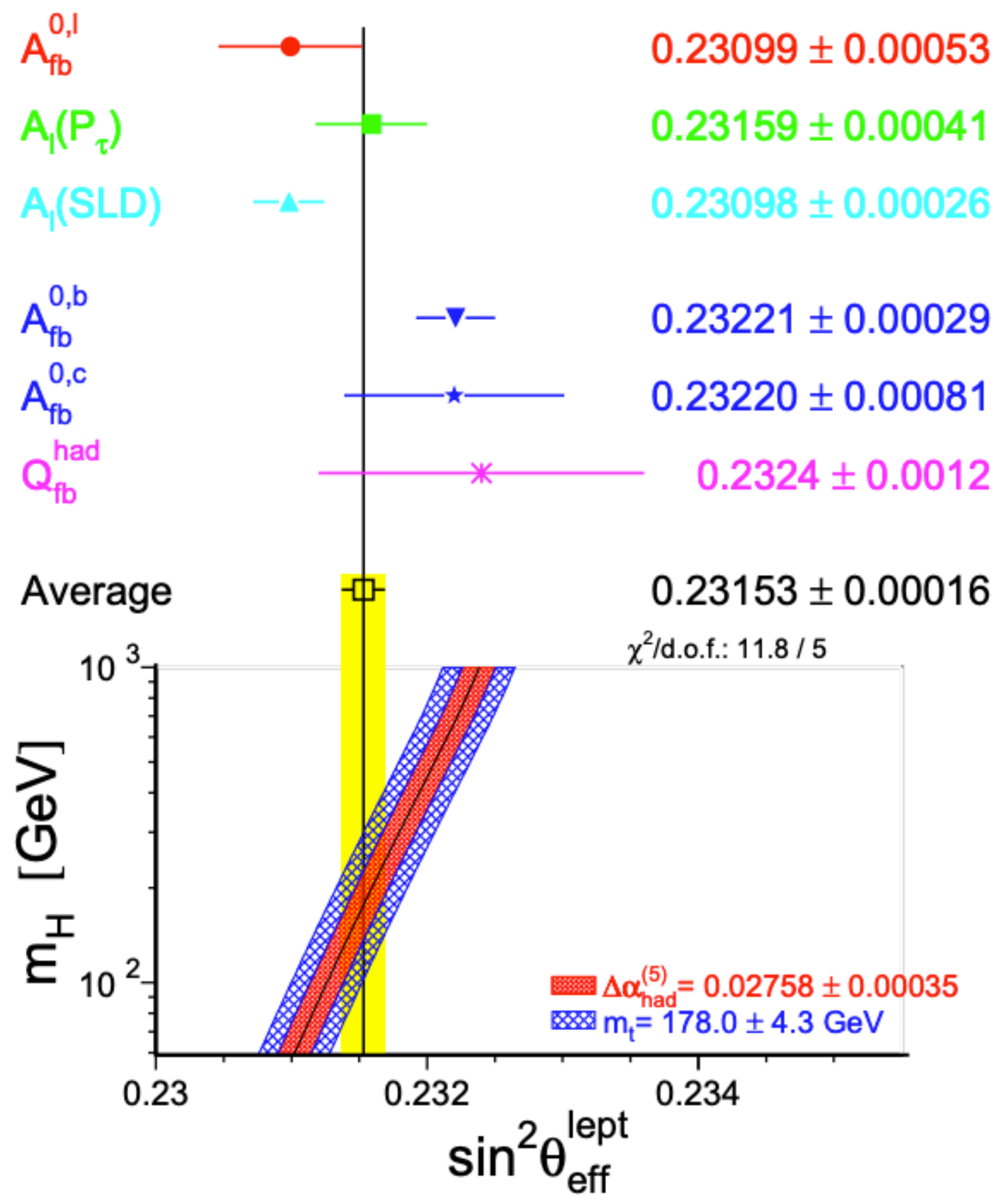
Γ_Z **Can be derived from requiring the Z-pole located at** $M_Z^2 - iM_Z\Gamma_Z$ **,with the help of optical theorem**

$\sin^2\theta^l_{eff}$ **Derived from the asymmetries, which are defined based on** $e^+e^- \to \bar{f}f$ **at Z resonance peak.**

LEPEWWG'06







Some Important EWPOs' Current Status

M_W

- mixed QCD/EW 2-loop corrections √. Djouai, Verzegnassi'87;Djouadi'88; Kniehl,Kühn, Stuar'99;Kniehl,Sirlin'93;Djouadi,Gambino'94
- complete EW 2-loop corrections √. Freitas, Hollik, Walter, Weiglein'00;Awramik,Czakon '02;Onishchenko,Vertin '02
- improvements by 3-loop and 4-loop $\Delta\rho$ √. Avdeev et al.'94; Chetyrkin, Kühn, Steinhauser '95; v.d.Bij et al. '05; Schröder, Steinhauser '06; Faisst et al. '03; Boughezal, Tausk, v.d.Bij '05

$$\Longrightarrow \quad \Delta M_W \sim 4 \text{ MeV}$$

Γ_Z

- complete EW 1-loop and fermionic 2-loop √. Freitas'13'14
- mixed QCD/EW 2-loop corrections √. Djouai, Verzegnassi'87;Halzen Kniehl'91; Djouadi,Gambino'94; Chetyrkin, Kühn'96; Fleischer et al. '92
- leading 3- and 4-loop QCD/Yukawa $O(\alpha_t\alpha_s^2), O(\alpha_t^2\alpha_s), O(\alpha_t^3), O(\alpha_s^3)$ √. v.d.Bij et al. '01;Faisst et al. '03, Schröder et al. '05; Consoli,Hollik, Jegerlehner '89

$$\Longrightarrow \quad \Delta\Gamma_Z \sim 0.5 \text{ MeV}$$

$\sin^2\theta^l_{eff}$

- mixed QCD/EW 2-loop and 3-loop $\Delta\rho$ Corrections as for M_W
- EW complete 2-loop corrections √. Awramik, Czakon, Freitas, Weiglein '04; Hollik,Meier, Uccirati'05;Awramik,Czakon, Freitas '06

$$\Longrightarrow \quad \sin^2\theta^l_{eff} \sim 4.5 \times 10^{-5}$$

2. Motivation

Test of Electroweak Precision Observables Freitas'16

	Current Exp	Current Theory	Main source	CEPC Exp	FCC-ee Exp	ILC Exp
M_W [MeV]	15	4	α^3, $\alpha^2\alpha_s$	1	1	$2.5 - 5$
Γ_Z[MeV]	2.3	0.5	α_{bos}^2, α^3, $\alpha^2\alpha_s$, $\alpha\alpha_s^2$	0.5	0.1	~ 1
$\sin^2\theta_{eff}^l$	1.6×10^{-4}	4.5×10^{-5}	α^3, $\alpha^2\alpha_s$	2.3×10^{-5}	0.6×10^{-5}	1×10^{-5}

- Accuracy of the fit is adequate for the level of precision foreseen for CEPC and FCC-ee.
- The calculation of the next perturbative order for the EWPOs will be necessary!

Improvement has been made

- **EW 2-loop bosonic corrections √.** **Dubovyk, Freitas, Gluza,Riemann, Usovitsch. '18**
- **Leading Order fermonic 3-loop corrections √.** **Chen, Freitas. '20** **This talk!**

Input Parameter in EWPOs

Fundamental parameters from original $SU(2)_L \times U(1)_Y$ $\{g1, g2, \lambda_H, \mu^2, y_f\}$

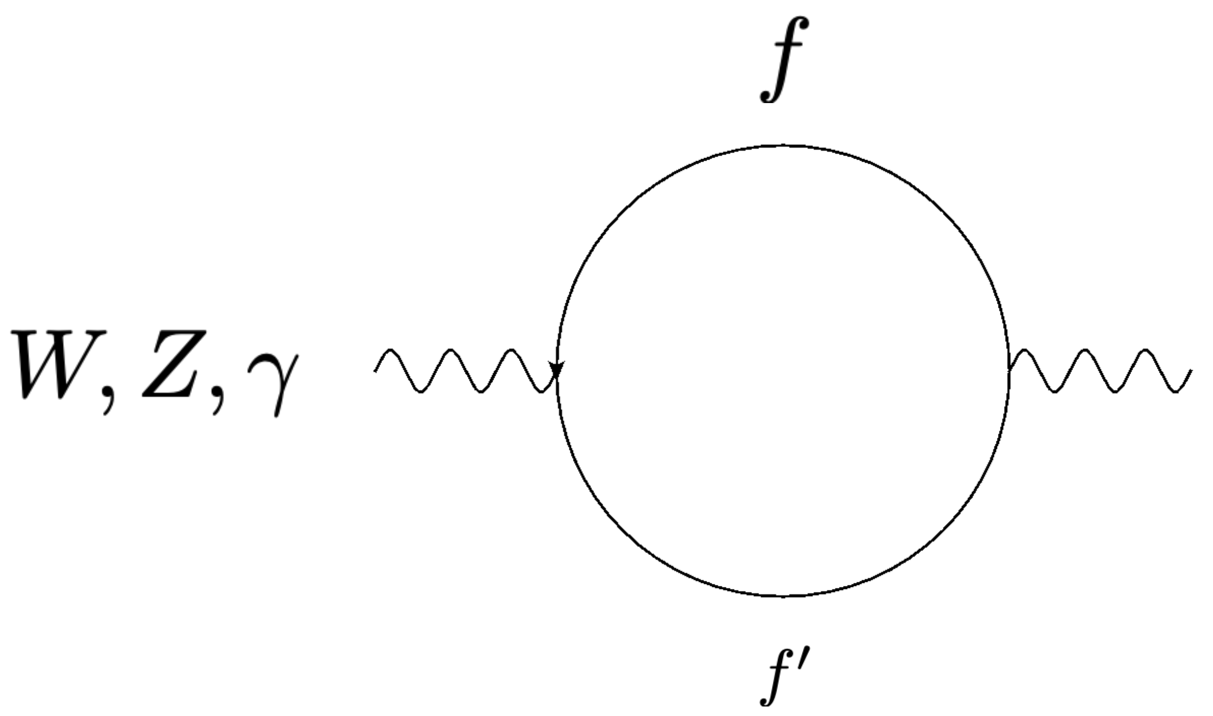
One can reparametrize these parameters into a set of new parameters that has been determined from experiments.
The choice is not unique, different set of IP could be adopted for different purpose.

IP we use $\{\overline{M_W}, \overline{M_Z}, \Delta\alpha, \alpha, G_\mu, m_f\}$ **Where** $\overline{M} = \frac{M}{\sqrt{1+\Gamma^2/M^2}}, \overline{\Gamma} = \frac{\Gamma}{\sqrt{1+\Gamma^2/M^2}}$

and $\Delta\alpha = \Pi_{lf}^{\gamma\gamma}(M_Z^2) - \Pi_{lf}^{\gamma\gamma}(0)$

stems from light-fermion loop contributions in the photon vacuum polarization which contains no-perturbative effects for small momentum.

Why leading-fermionic?



$$W, Z, \gamma \sim \begin{cases} \text{enhanced by power of } m_t & \text{for} f \text{or} f' = t \\ \text{enhanced by large number of flavor numbers} & \text{for f and f' are massless} \end{cases}$$

Numerically enhanced by the power of top mass and the large number of flavors and colors!

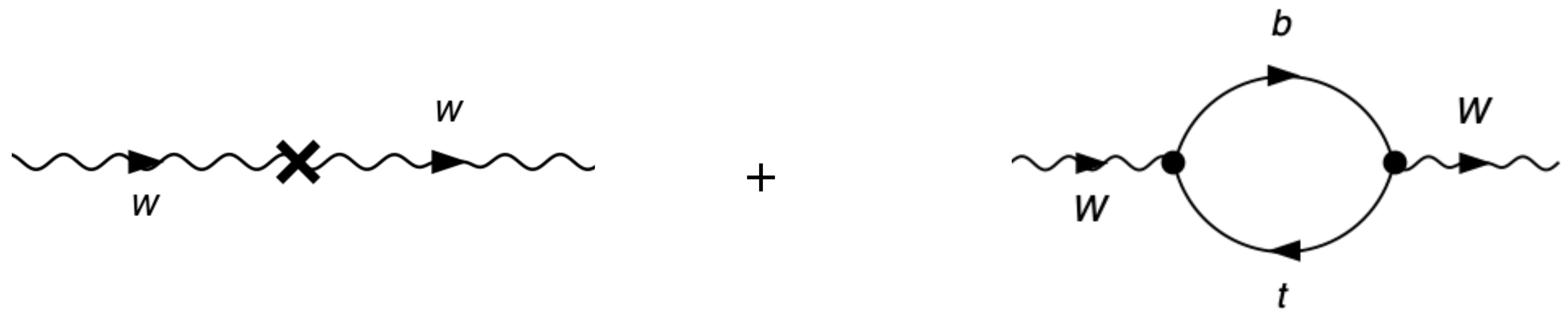
it's also a relatively simple set of diagrams at 3-loop order!

Renormalization

- **the on-shell (OS) scheme is adopted in this calculation**

In such a scheme, the pole of a massive propagator is defined as $s_0 \equiv \overline{M^2} - i\overline{M\Gamma}$

$\hat{\Sigma}^{WW}_{\mu\nu} \equiv$



$$= -ig_{\mu\nu}(Z_W(k^2 - \overline{M^2_W}) - \delta\overline{M^2_W} Z_W \overline{M^2_W}) + ik_\mu k_\nu Z_W + \Sigma^{WW}_{\mu\nu}(k^2)$$

$$= -i(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})\Sigma^{WW}_T(k^2) - i\frac{k_\mu k_\nu}{k^2}\Sigma^{WW}_L(k^2)$$

The longitudinal part cancels against unphysical amplitude (introduced from Feyman 't Hooft gauge) as a consequence of Slavnov-Taylor identity.

After resummation, propagator looks like

$$D(k^2) = k^2 - s_0 + \hat{\Sigma}_{WW}(k^2)$$

As OS requires $D(s_0) = 0$

$$\Longrightarrow \quad \boxed{\delta M_W^2 = \frac{Re[Z_W(k^2 - \overline{M_W^2} + \hat{\Sigma}_T^{WW}(k^2)_{k^2 \to s_0}]}{Z_W} \quad \Gamma_W = \frac{Im[\hat{\Sigma}_T^{WW}(s_0)_{k^2 \to s_0}]}{Z_W \overline{M_W}}}$$

By recursively using the master equations above, one can derive the mass counter term at arbitrary higher order.

$$\delta\overline{M}^2_{\mathrm{W}(1)} = \mathrm{Re}\,\Sigma_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\,,$$

$$\delta\overline{M}^2_{\mathrm{W}(2)} = \mathrm{Re}\,\Sigma_{\mathrm{W}(2)}(\overline{M}^2_{\mathrm{W}}) + \left[\mathrm{Im}\,\Sigma_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right]\left[\mathrm{Im}\,\Sigma'_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right],$$

$$\begin{aligned}\delta\overline{M}^2_{\mathrm{W}(3)} = {} & \mathrm{Re}\,\Sigma_{\mathrm{W}(3)}(\overline{M}^2_{\mathrm{W}}) + \left[\mathrm{Im}\,\Sigma_{\mathrm{W}(2)}(\overline{M}^2_{\mathrm{W}})\right]\left[\mathrm{Im}\,\Sigma'_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right] \\ & + \left[\mathrm{Im}\,\Sigma_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right]\Big\{\mathrm{Im}\,\Sigma'_{\mathrm{W}(2)}(\overline{M}^2_{\mathrm{W}}) - \left[\mathrm{Im}\,\Sigma'_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right]\left[\mathrm{Re}\,\Sigma'_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right] \\ & \qquad - \tfrac{1}{2}\left[\mathrm{Im}\,\Sigma_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right]\left[\mathrm{Re}\,\Sigma''_{\mathrm{W}(1)}(\overline{M}^2_{\mathrm{W}})\right]\Big\}\,.\end{aligned}$$

Field counter term for the gauge boson drops out since unstable particles can only appear as internal particles in a physical process!

$$\begin{aligned}\delta\overline{M}^2_{\mathrm{Z}(3)} = {} & \mathrm{Re}\,\Sigma_{\mathrm{ZZ}(3)}(\overline{M}^2_{\mathrm{Z}}) + \left[\mathrm{Im}\,\Sigma_{\mathrm{ZZ}(2)}(\overline{M}^2_{\mathrm{Z}})\right]\left[\mathrm{Im}\,\Sigma'_{\mathrm{ZZ}(1)}(\overline{M}^2_{\mathrm{Z}})\right] \\ & + \left[\mathrm{Im}\,\Sigma_{\mathrm{ZZ}(1)}(\overline{M}^2_{\mathrm{Z}})\right]\Big\{\mathrm{Im}\,\Sigma'_{\mathrm{ZZ}(2)}(\overline{M}^2_{\mathrm{Z}}) - \left[\mathrm{Im}\,\Sigma'_{\mathrm{ZZ}(1)}(\overline{M}^2_{\mathrm{Z}})\right]\left[\mathrm{Re}\,\Sigma'_{\mathrm{ZZ}(1)}(\overline{M}^2_{\mathrm{Z}})\right] \\ & \qquad - \tfrac{1}{2}\left[\mathrm{Im}\,\Sigma_{\mathrm{ZZ}(1)}(\overline{M}^2_{\mathrm{Z}})\right]\left[\mathrm{Re}\,\Sigma''_{\mathrm{ZZ}(1)}(\overline{M}^2_{\mathrm{Z}})\right] \\ & \qquad - \frac{\mathrm{Im}\,\Sigma_{\gamma\mathrm{Z}(1)}(\overline{M}^2_{\mathrm{Z}})}{\overline{M}^2_{\mathrm{Z}}}\left[2\,\mathrm{Re}\,\Sigma'_{\gamma\mathrm{Z}(1)}(\overline{M}^2_{\mathrm{Z}}) + \delta Z^{\gamma\mathrm{Z}}_{(1)} + \delta Z^{\mathrm{Z}\gamma}_{(1)}\right]\Big\} \\ & + \frac{\mathrm{Im}\,\Sigma_{\gamma\mathrm{Z}(1)}(\overline{M}^2_{\mathrm{Z}})}{\overline{M}^2_{\mathrm{Z}}}\Big\{2\,\mathrm{Im}\,\Sigma_{\gamma\mathrm{Z}(2)}(\overline{M}^2_{\mathrm{Z}}) - \frac{\mathrm{Im}\,\Sigma_{\gamma\mathrm{Z}(1)}(\overline{M}^2_{\mathrm{Z}})}{\overline{M}^2_{\mathrm{Z}}}\left[\mathrm{Im}\,\Sigma_{\gamma\gamma(1)}(\overline{M}^2_{\mathrm{Z}})\right]\Big\} \\ & + \tfrac{1}{2}\overline{M}^2_{\mathrm{Z}}\,\delta Z^{\gamma\mathrm{Z}}_{(1)}\,\delta Z^{\gamma\mathrm{Z}}_{(2)}\,.\end{aligned}$$

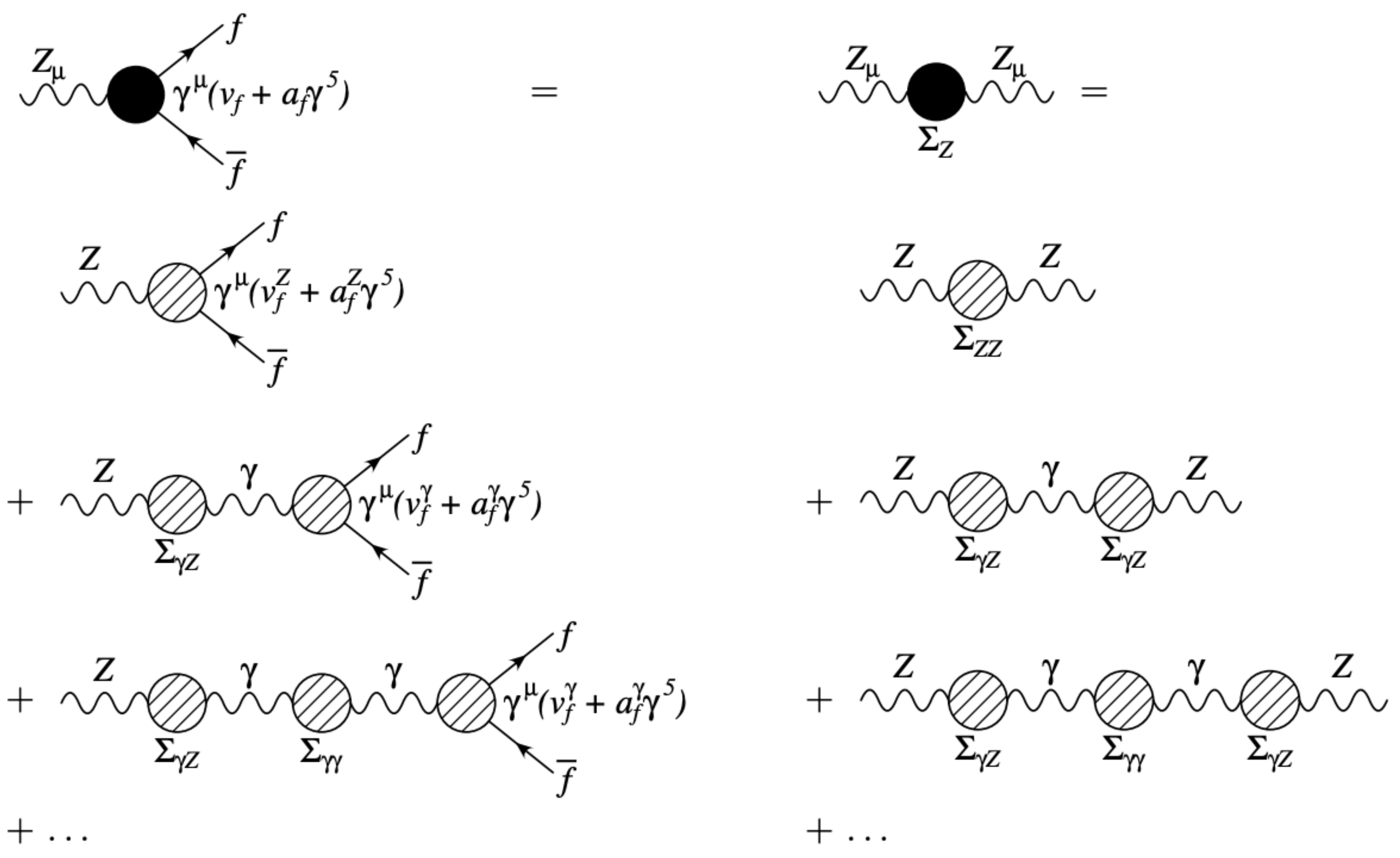
Z-boson mass counter term turns to be more sutble due to the gamma-Z mixing

Charge Renormalization: **Slavnov-Taylor Identity**

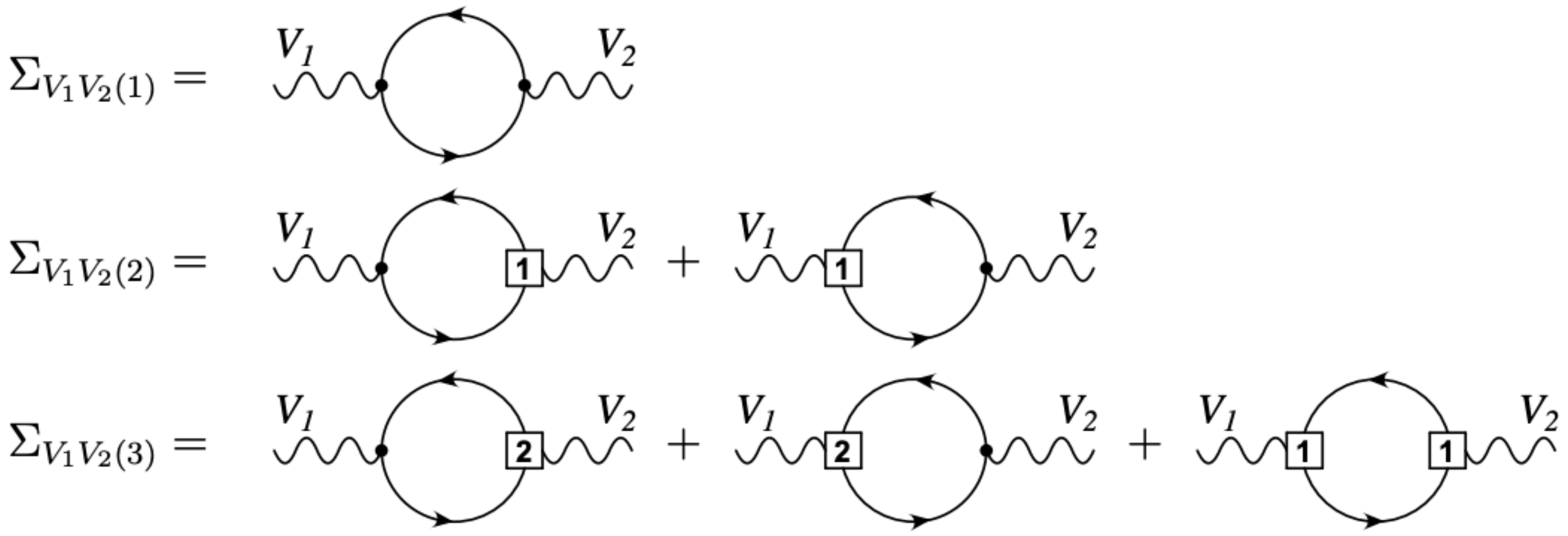
$$Z_e = \left(\sqrt{Z_{\gamma\gamma}} + \frac{s_W}{c_W}\sqrt{Z_{Z\gamma}}\right)^{-1}$$

Weak-mixing angle Renormalization:

$$s_{\mathrm{W}} + \delta s_{\mathrm{W}} = \sqrt{1 - \frac{\overline{M}_{\mathrm{W}}^2 + \delta\overline{M}_{\mathrm{W}}^2}{\overline{M}_{\mathrm{Z}}^2 + \delta\overline{M}_{\mathrm{Z}}^2}}$$



Decomposition of Zff vertex and Z self-energy into 1-PI building blocks

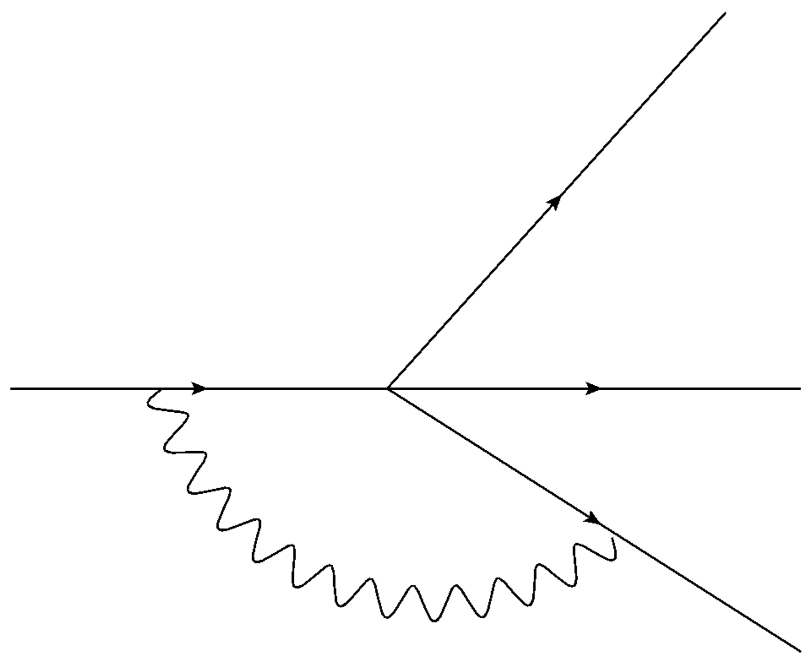


Loop/sub-loop at different order,
boxed number denotes a counter term of loop order n

1. M_W

In 4- Fermi theory

$$\Gamma_{\mu\to\nu_\mu e\nu_e} = \frac{G_\mu^2 m_\mu^2}{192\pi^3} F(\frac{m_e^2}{m_\mu^2})(1+\delta q)$$

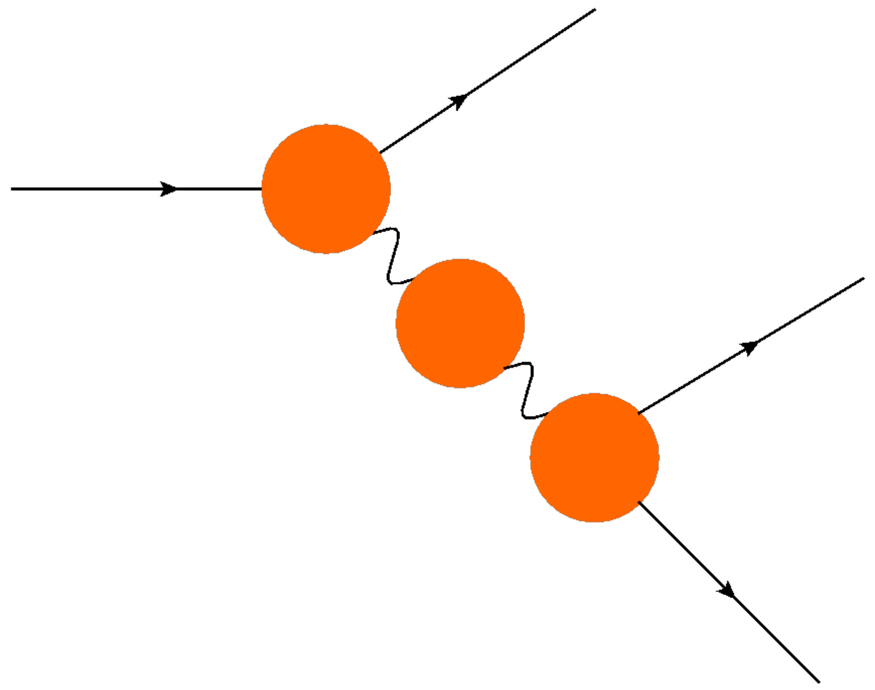


In SM, it involves exchange of W-boson. One can find a relation between 4-Fermi G_μ and SM parameters.

$$G_\mu = \frac{\pi\alpha}{\sqrt{2} s_W^2 \overline{M}_W^2}(1+\Delta r)$$

Δr Features the contributions of radiative corrections

Taking G_μ as am input



$$\overline{M_W^2} = \overline{M_Z^2}(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \frac{\alpha\pi}{\sqrt{2} G_\mu \overline{M_Z^2}}(1+\Delta r)})$$

An implicit relation that needs recursive procedure

2. $\sin^2\theta^f_{eff}$

Unpolarized x-section at lowest order

$$\frac{d\sigma}{d\cos\theta} = \frac{N_C^f G_F^2 M_Z^4}{16\pi}\frac{s}{(s-M_Z^2)^2+s^2\Gamma_Z^2 M_Z^{-2}}\left[(g_v^{e2}+g_a^{e2})(g_v^{f2}+g_a^{f2})(1+\cos^2\theta)+2g_v^e g_a^e g_v^f g_a^f\cos\theta\right]$$

Defining $\sigma_F = \int_0^1 \frac{d\sigma}{d\cos\theta}\,d\cos\theta, \qquad \sigma_B = \int_{-1}^0 \frac{d\sigma}{d\cos\theta}\,d\cos\theta,\quad A^f_{\mathrm{FB}} \equiv \frac{\sigma_F-\sigma_B}{\sigma_F+\sigma_B}$

$$A_f \equiv \frac{2g_v^f g_a^f}{g_a^{f2}+g_v^{f2}} = \frac{1-4|Q_f|\sin^2\theta^f_{\mathrm{eff}}}{1-4|Q_f|\sin^2\theta^f_{\mathrm{eff}}+8|Q_f|^2\sin^4\theta^f_{\mathrm{eff}}}$$

$$A^f_{\mathrm{FB}} = \frac{3}{4}A_e A_f$$

We get $$\sin^2\theta^f_{eff} = \frac{1}{4|Q_f|}\left(1-Re\frac{g_V^f(\overline{M_Z})}{g_A^f(\overline{M_Z})}\right)$$

Where

$$g_V^f = Z_e(\sqrt{Z_{ZZ}}v_f^Z - Q\sqrt{Z_{\gamma Z}}) - v_f^\gamma\frac{\hat\Sigma_{\gamma Z}}{s+\hat\Sigma_{\gamma\gamma}}$$

$$g_A^f = Z_e(\sqrt{Z_{ZZ}}a_f^Z) - a_f^\gamma\frac{\hat\Sigma_{\gamma Z}}{s+\hat\Sigma_{\gamma\gamma}}$$

3. Partial width $\Gamma[Z \to f\bar{f}]$

Within OS scheme $(s_0 - \overline{M}_Z^2) + \Sigma_T^Z(s_0) = 0$ **where** $s_0 \equiv \overline{M}_Z^2 - i\overline{M}_Z\overline{\Gamma}_Z$ $\quad \Sigma_T^Z(s) = \Sigma_T^{ZZ}(s) - \frac{[\Sigma_T^{\gamma Z}(s)]^2}{s + \Sigma_T^{\gamma\gamma}(s)}$

Requiring s0 to be the pole of Z propagator

$$\overline{\Gamma}_Z = \frac{1}{\overline{M}_Z}\mathrm{Im}\{\Sigma_T^Z(s_0)\}$$
$$= \frac{1}{\overline{M}_Z}\Big[\mathrm{Im}\{\Sigma_T^Z(\overline{M}_Z^2)\} - \overline{M}_Z\overline{\Gamma}_Z\,\mathrm{Re}\{\Sigma_T^{Z\prime}(\overline{M}_Z^2)\} - \tfrac{1}{2}\overline{M}_Z^2\overline{\Gamma}_Z^2\mathrm{Im}\{\Sigma_T^{Z\prime\prime}(\overline{M}_Z^2)\} + \mathcal{O}(\overline{\Gamma}_Z^3)\Big]$$

Optical Theorem is applied here.

$$\mathrm{Im}\,\Sigma_T^Z = \frac{1}{3\overline{M}_Z}\sum_f\sum_{\text{spins}}\int d\Phi\;\left(\mathcal{R}_V^f|v_f|^2 + \mathcal{R}_A^f|a_f|^2\right)$$

$$\overline{\Gamma}_Z = \sum_f \overline{\Gamma}_f\,, \qquad \overline{\Gamma}_f = \frac{N_c^f\overline{M}_Z}{12\pi}\Big[\mathcal{R}_V^f F_V^f + \mathcal{R}_A^f F_A^f\Big]_{s=\overline{M}_Z^2}\,,$$

$$\begin{aligned}
F_V^f = {} & v_{f(0)}^2 + 2\,\mathrm{Re}\,(v_{f(0)}v_{f(1)}) - v_{f(0)}^2\,\mathrm{Re}\,\Sigma'_{Z(1)} \\
& + 2\,\mathrm{Re}\,(v_{f(0)}v_{f(2)}) + |v_{f(1)}|^2 - 2\,\mathrm{Re}\,(v_{f(0)}v_{f(1)})\,\mathrm{Re}\,\Sigma'_{Z(1)} \\
& \quad + v_{f(0)}^2\Big[(\mathrm{Re}\,\Sigma'_{Z(1)})^2 - \mathrm{Re}\,\Sigma'_{Z(2)} - \tfrac{1}{4}(\delta Z_{(1)}^{\gamma Z})^2 - \tfrac{1}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)})\Big] \\
& + 2\,\mathrm{Re}\,(v_{f(0)}v_{f(3)} + v_{f(1)}^*v_{f(2)}) - \Big[2\,\mathrm{Re}\,(v_{f(0)}v_{f(2)}) + |v_{f(1)}|^2\Big]\,\mathrm{Re}\,\Sigma'_{Z(1)} \\
& \quad + 2\,\mathrm{Re}\,(v_{f(0)}v_{f(1)})\,\Big[(\mathrm{Re}\,\Sigma'_{Z(1)})^2 - \mathrm{Re}\,\Sigma'_{Z(2)} - \tfrac{1}{4}(\delta Z_{(1)}^{\gamma Z})^2 - (\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)})\Big] \\
& \quad + v_{f(0)}^2\Big[-(\mathrm{Re}\,\Sigma'_{Z(1)})^3 + 2(\mathrm{Re}\,\Sigma'_{Z(2)})(\mathrm{Re}\,\Sigma'_{Z(1)}) - \mathrm{Re}\,\Sigma'_{Z(3)} \\
& \qquad - \tfrac{1}{2}\delta Z_{(1)}^{\gamma Z}\,\delta Z_{(2)}^{\gamma Z} + \tfrac{1}{2}(\mathrm{Re}\,\Sigma'_{Z(1)})(\delta Z_{(1)}^{\gamma Z})^2 - \tfrac{1}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(2)}) \\
& \qquad + \tfrac{3}{2}(\mathrm{Im}\,\Sigma_{Z(1)})(\mathrm{Re}\,\Sigma'_{Z(1)})(\mathrm{Im}\,\Sigma''_{Z(1)}) + \tfrac{1}{6}(\mathrm{Im}\,\Sigma_{Z(1)})^2(\mathrm{Re}\,\Sigma'''_{Z(1)})\Big],
\end{aligned}$$

$\mathcal{R}_{V,A}$ feature the split-up QED and QCD radiative corrections into final-state.

Technical Aspects

1. Using FeynArts(T. Hahn'01) and FeynCalc V. Shtabovenko, R. Mertig and F. Orellana'16) to carry out diagrams/amplitudes and Dirac Algebra/tensor reduction. **But 3-loop sub diagrams need to be carried out by putting amplitudes in FeynCalc manually.**

Generating diagrams-> convert them to FeynCalc-> Project out Transverse amplitude-> Dirac Algebra/tensor reduction.

Feynman rules at higher order are not implemented in FeynArts yet (only up to 1-loop), might worth an effort to do it.

2. All loop integrals can be written as 1-loop scalar master integrals (Passarino-Veltman) and their derivatives up to second order.

3. All light fermions are treated massless in this calculation, so are the CKM mixing due to the negligible impact.

4. Exact agreement at 2-loop was found between this work and hep-ph/0004091;0202131;0407317;1310.2256
except the second term in the following.

$$\mathrm{Re}\,\Sigma'_{ZZ(2)}(s) - \frac{d}{ds}\left(\frac{[\mathrm{Im}\,\Sigma_{\gamma Z(1)}(s)]^2}{s}\right)$$

The numerical impact of this missing term is shown in next section

$$\frac{289\, D^4\, \text{B0Re}\left(m_Z^2, m_t^2, m_t^2\right)^3 \alpha^3 \sqrt{1-\frac{m_W^2}{m_Z^2}} \sqrt{\frac{m_Z^2-m_W^2}{m_Z^2}}\, m_Z^{20}}{24\,576\, (D-1)^3 \pi^3 m_W^6 \left(m_W^2-m_Z^2\right)^5 \left(m_Z^2-4 m_t^2\right)\left(m_Z^2-m_W^2\right)} - \frac{289\, D^3\, \text{B0Re}\left(m_Z^2, m_t^2, m_t^2\right)^3 \alpha^3 \sqrt{1-\frac{m_W^2}{m_Z^2}} \sqrt{\frac{m_Z^2-m_W^2}{m_Z^2}}\, m_Z^{20}}{3072\, (D-1)^3 \pi^3 m_W^6 \left(m_W^2-m_Z^2\right)^5 \left(m_Z^2-4 m_t^2\right)\left(m_Z^2-m_W^2\right)} + \cdots 237948 \cdots + \frac{512\, A_0\left(m_t^2\right)^2 \text{B0Re}\left(m_Z^2, m_t^2, m_t^2\right) \alpha^3 m_t^2 m_W^8 \sqrt{\frac{m_Z^2-m_W^2}{m_Z^2}}}{27\, (D-1)^3 \pi^3 \left(1-\frac{m_W^2}{m_Z^2}\right)^{3/2} \left(m_Z^2-m_W^2\right)^2 m_Z^{10}}$$

```
dZAff2xuqloopleft =
 1/2 * 4 * 12 * Pi * SMP["alpha_fs"].DiracTrace[ (-DiracGamma[Momentum[q, D], D]).DiracGamma[LorentzIndex[ν, D], D].
    ( dgV2 + dgV1 (1/2 dZZZ1 + dZe1) + gV (1/2 dZZZ2 - 1/8 dZZZ1^2 + 1/2 dZe1 dZZZ1 + dZe2) - Q/2 dZAZ2 - Q/2 dZe1 dZAZ1 - GA5
     (dgA2 + dgA1 (1/2 dZZZ1 + dZe1) + gA (1/2 dZZZ2 - 1/8 dZZZ1^2 + 1/2 dZZZ1 dZe1 + dZe2))) . (DiracGamma[Momentum[k - q, D], D]) .
   DiracGamma[LorentzIndex[μ, D], D].(Q) ] * FeynAmpDenominator[PropagatorDenominator[Momentum[q, D], 0],
  PropagatorDenominator[-Momentum[k, D] + Momentum[q, D], 0]] +
 1/2 * 4 * 6 * Pi * SMP["alpha_fs"].DiracTrace[ (-DiracGamma[Momentum[q, D], D] + SMP["m_t"]).DiracGamma[LorentzIndex[ν, D], D].
    ( dgV2 + dgV1 (1/2 dZZZ1 + dZe1) + gV (1/2 dZZZ2 - 1/8 dZZZ1^2 + 1/2 dZe1 dZZZ1 + dZe2) - Q/2 dZAZ2 - Q/2 dZe1 dZAZ1 - GA5
     (dgA2 + dgA1 (1/2 dZZZ1 + dZe1) + gA (1/2 dZZZ2 - 1/8 dZZZ1^2 + 1/2 dZZZ1 dZe1 + dZe2))) . (DiracGamma[Momentum[k - q, D], D] + SMP["m_t"]) .
   DiracGamma[LorentzIndex[μ, D], D].(Q) ] * FeynAmpDenominator[PropagatorDenominator[Momentum[q, D], SMP["m_t"]],
  PropagatorDenominator[-Momentum[k, D] + Momentum[q, D], SMP["m_t"]]] /. {vf → (Iw - 2 SW^2 Q)/(2 CW SW), af → Iw/(2 CW SW)} /.
 {Iw → 1/2, Q → 2/3, CW → SMP["cos_W"], SW → SMP["sin_W"]} /. {SMP["sin_W"] → Sqrt[1 - SMP["m_W"]^2/SMP["m_Z"]^2], SMP["cos_W"] → SMP["m_W"]/SMP["m_Z"]} // Contract // FCTraceFactor //
 DiracSimplify;
transversedZAff2xuqloopleft =
 1/(D - 1)
  (Pair[LorentzIndex[μ, D], LorentzIndex[ν, D]] - Pair[LorentzIndex[μ, D], Momentum[k, D]]
    Pair[LorentzIndex[ν, D], Momentum[k, D]] / Pair[Momentum[k, D], Momentum[k, D]]) . dZAff2xuqloopleft // Contract // DiracSimplify;
ΣsubdZAff2xuqloopleft = TID[-i/(16 π^4) transversedZAff2xuqloopleft /. DiracTrace → Tr, q, ToPaVe → True] /. Pair[Momentum[k, D], Momentum[k, D]] → k^2;
```

Numerical Results

Input Parameters

$$
\begin{aligned}
M_\mathrm{Z} &= 91.1876\ \mathrm{GeV} \\
\Gamma_\mathrm{Z} &= 2.4952\ \mathrm{GeV}
\end{aligned}
\Bigg\} \Rightarrow\ \overline{M}_\mathrm{Z} = 91.1535\ \mathrm{GeV}
$$

$$
\begin{aligned}
M_\mathrm{W} &= 80.358\ \mathrm{GeV} \\
\Gamma_\mathrm{W} &= 2.089\ \mathrm{GeV}
\end{aligned}
\Bigg\} \Rightarrow\ \overline{M}_\mathrm{W} = 80.331\ \mathrm{GeV}
$$

$$
\begin{aligned}
m_\mathrm{t} &= 173.0\ \mathrm{GeV} \\
m_{f\neq\mathrm{t}} &= 0 \\
\alpha &= 1/137.035999084 \\
\Delta\alpha &= 0.05900 \\
G_\mu &= 1.1663787 \times 10^{-5}\ \mathrm{GeV}^{-2}
\end{aligned}
$$

W-Mass

$$\Delta r_{(3)} = 2.50 \times 10^{-5}.$$

$$\Delta\overline{M}_{\mathrm{W}(3)} \approx \frac{\pi\alpha\overline{M}_\mathrm{Z}^2}{2\sqrt{2}G_\mu\overline{M}_\mathrm{W}(\overline{M}_\mathrm{Z}^2 - 2\overline{M}_\mathrm{W}^2)}\Delta r_{(3)} = -0.389\ \mathrm{MeV}.$$

2. $\sin^2 \theta^f_{eff}$

$$\Delta \sin^2 \theta^f_{\mathrm{eff},(3)} = 1.34 \times 10^{-5} \qquad [M_\mathrm{W} \text{ as indep. input}].$$

$$\Delta' \sin^2 \theta^f_{\mathrm{eff},(3)} = \Delta \sin^2 \theta^f_{\mathrm{eff},(3)} - \frac{\Delta \overline{M}^2_{\mathrm{W}(3)}}{\overline{M}^2_\mathrm{Z}} = 2.09 \times 10^{-5} \qquad [M_\mathrm{W} \text{ from } G_\mu].$$

3. Partial width $\Gamma[Z \to f\bar{f}]$

$$\begin{aligned}
\Delta\overline{\Gamma}_{f,(3)} &= N_c^f \left[0.105\,(I_3^f)^2 - 0.105\,I_3^f Q_f + 0.046\,Q_f^2\right] \text{ MeV}, \\
\Delta\overline{\Gamma}_{\ell,(3)} &= 0.019 \text{ MeV}, \\
\Delta\overline{\Gamma}_{\nu,(3)} &= 0.026 \text{ MeV}, \\
\Delta\overline{\Gamma}_{\mathrm{d},(3)} &= 0.041 \text{ MeV}, \\
\Delta\overline{\Gamma}_{\mathrm{u},(3)} &= 0.035 \text{ MeV}, \\
\Delta\overline{\Gamma}_{\mathrm{tot},(3)} &= 0.331 \text{ MeV},
\end{aligned} \qquad [M_\mathrm{W} \text{ as indep. input}]$$

$$\Delta'\overline{\Gamma}_{f,(3)} = \Delta\overline{\Gamma}_{f,(3)} - \frac{\Delta \overline{M}^2_{\mathrm{W}(3)}}{\overline{M}_\mathrm{Z}} \times \frac{\alpha N_c^f}{6 s_\mathrm{W}^4 c_\mathrm{W}^4}\left[(2s_\mathrm{W}^2 - 1)(I_3^f)^2 + 2s_\mathrm{W}^4 Q_f (Q_f - I_3^f)\right]$$

$$\begin{aligned}
\Delta'\overline{\Gamma}_{f,(3)} &= N_c^f \left[0.090\,(I_3^f)^2 - 0.108\,I_3^f Q_f + 0.048\,Q_f^2\right] \text{ MeV}, \\
\Delta'\overline{\Gamma}_{\ell,(3)} &= 0.017 \text{ MeV}, \\
\Delta'\overline{\Gamma}_{\nu,(3)} &= 0.022 \text{ MeV}, \\
\Delta'\overline{\Gamma}_{\mathrm{d},(3)} &= 0.029 \text{ MeV}, \\
\Delta'\overline{\Gamma}_{\mathrm{u},(3)} &= 0.024 \text{ MeV}, \\
\Delta'\overline{\Gamma}_{\mathrm{tot},(3)} &= 0.255 \text{ MeV}.
\end{aligned} \qquad [M_\mathrm{W} \text{ from } G_\mu]$$

4.Missing Terms from previous work

Exact agreement at 2-loop was found between this work and hep-ph/0004091;0202131;0407317;1310.2256

$$\mathrm{Re}\,\Sigma'_{\mathrm{ZZ}(2)}(s) - \frac{d}{ds}\left(\frac{[\mathrm{Im}\,\Sigma_{\gamma\mathrm{Z}(1)}(s)]^2}{s}\right)$$

The second term, which stems from gamma-Z mixing, was missed. We evaluated the numerical impact of this term

$$\Delta\overline{\Gamma}_{f,(2)}\Big|_{\text{this work}} - \Delta\overline{\Gamma}_{f,(2)}\Big|_{\text{Ref. [2,3]}} = -N_c^f\,(v_{f(0)}^2 + a_{f(0)}^2)\,\overline{M}_{\mathrm{Z}}\frac{25\alpha^2(3-8s_{\mathrm{W}}^2)^2}{3888\pi s_{\mathrm{W}}^2 c_{\mathrm{W}}^2}$$

$$= \begin{cases} -0.0028\ \mathrm{MeV} & \text{for } f=\ell, \\ -0.0056\ \mathrm{MeV} & \text{for } f=\nu, \\ -0.0126\ \mathrm{MeV} & \text{for } f=d, \\ -0.0098\ \mathrm{MeV} & \text{for } f=u, \\ -0.0830\ \mathrm{MeV} & \text{for } f=\mathrm{tot}. \end{cases}$$

Summary

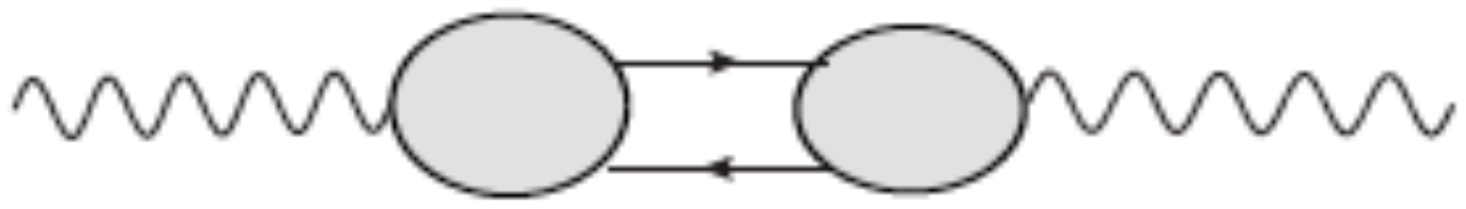
1. Electroweak precision measurements at future lepton colliders require three-loop EW correction.

**2. Some of the most important EWPOs are calculated at leading order fermionic 3-loop.
The results are small as expected but nevertheless important to precision measurements at the next level.**

**3. Missing term were found from previous paper. Although its numerical impact is small,
it is necessary to identify and correct for maintaining the consistency of calculation.
It also shows the importance of having multiple individual calculations in this field.**

Thank you!

- However, the $\Delta\alpha$ contains contributions of quarks with $k^2 < \Lambda_{QCD}$, which considered as the non-perturbative regime.
- One way of dealing with this is by using dispersion relations to extract the hadronic contribution from some other experiment.



$$F(q^2) = F(q_0^2) + \frac{q^2 - q_0^2}{\pi} \int_{M^2}^{\infty} \frac{ds}{s - q_0^2} \frac{ImF(s)}{s - q^2 - i\epsilon}$$

Reflecting to $\Delta\alpha^{(5)}_{hadron}$, one can have

$$Re\frac{d\Sigma^{\gamma\gamma}(s)}{ds} - \frac{d\Sigma^{\gamma\gamma}(0)}{ds} = \frac{s}{\pi} Re \int_{s_0}^{\infty} ds' \frac{Im\Sigma^{\gamma\gamma\prime}(s')}{s'(s' - s - i\epsilon)}$$

Then by using the optical theorem one gets

$$Im\Sigma^{\gamma\gamma\prime}(s) = \frac{s}{e^2}\sigma_{tot}(e^+e^- \to \gamma^* \to hadrons)(s),$$

Error Estimate

Perfector method. J. Erler'05, U. Baur et al. '01

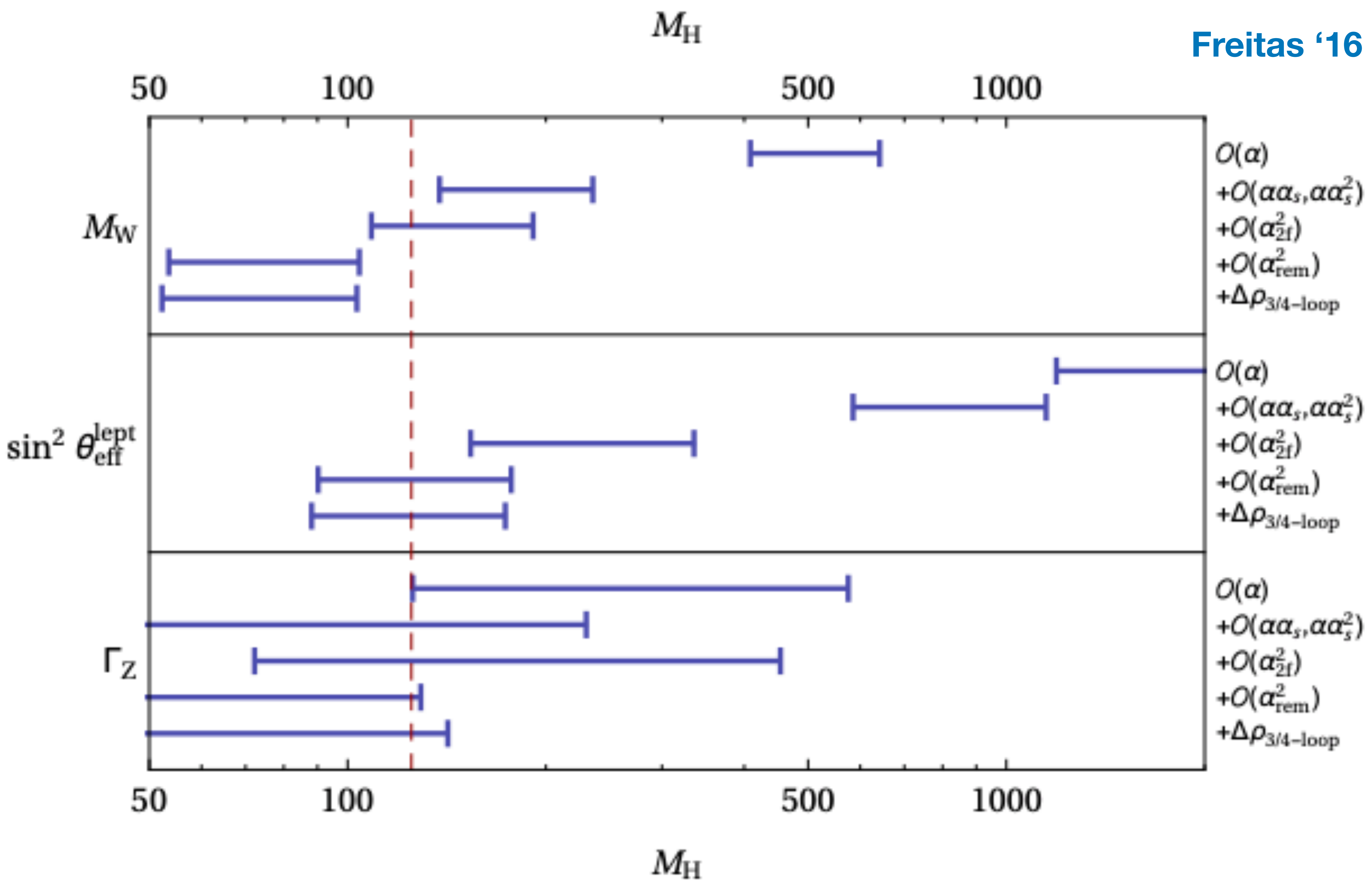
$$\mathcal{O}(\alpha_{\rm bos}) \sim \Gamma_Z \alpha^2 \approx 0.13 \text{ MeV},$$
$$\mathcal{O}(\alpha^3) - \mathcal{O}(\alpha_t^3) \sim \Gamma_Z \alpha \alpha_t^2 \approx 0.12 \text{ MeV},$$
$$\mathcal{O}(\alpha^2\alpha_s) - \mathcal{O}(\alpha_t^2\alpha_s) \sim \Gamma_Z \frac{\alpha\alpha_t n_q}{\pi}\alpha_s(m_t) \approx 0.23 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_s^2) - \mathcal{O}(\alpha_t\alpha_s^2) \sim \Gamma_Z \frac{\alpha n_q}{\pi}\alpha_s^2(m_t) \approx 0.35 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_s^3) - \mathcal{O}(\alpha_t\alpha_s^3) \sim \Gamma_Z \frac{\alpha n_q}{\pi}\alpha_s^3(m_t) \approx 0.04 \text{ MeV}.$$

Geometric series extrapolation method. M. Awramik, M. Czakon and A. Freitas'06, Freitas,Hollik, Walter,Weiglein '02,Freitas '14

$$\mathcal{O}(\alpha_{\rm bos}) \sim [\mathcal{O}(\alpha_{\rm bos})]^2 \approx 0.10 \text{ MeV},$$
$$\mathcal{O}(\alpha^3) - \mathcal{O}(\alpha_t^3) \sim \frac{\mathcal{O}(\alpha^2)}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_t^2)] \approx 0.26 \text{ MeV},$$
$$\mathcal{O}(\alpha^2\alpha_s) - \mathcal{O}(\alpha_t^2\alpha_s) \sim \frac{\mathcal{O}(\alpha\alpha_s)}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha^2) - \mathcal{O}(\alpha_t^2)] \approx 0.30 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_s^2) - \mathcal{O}(\alpha_t\alpha_s^2) \sim \frac{\mathcal{O}(\alpha\alpha_s)}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha\alpha_s) - \mathcal{O}(\alpha_t\alpha_s)] \approx 0.23 \text{ MeV},$$
$$\mathcal{O}(\alpha\alpha_s^3) - \mathcal{O}(\alpha_t\alpha_s^3) \sim \frac{\mathcal{O}(\alpha\alpha_s^2)}{\mathcal{O}(\alpha)}[\mathcal{O}(\alpha\alpha_s) - \mathcal{O}(\alpha_t\alpha_s)] \approx 0.035 \text{ MeV}.$$

Back-ups

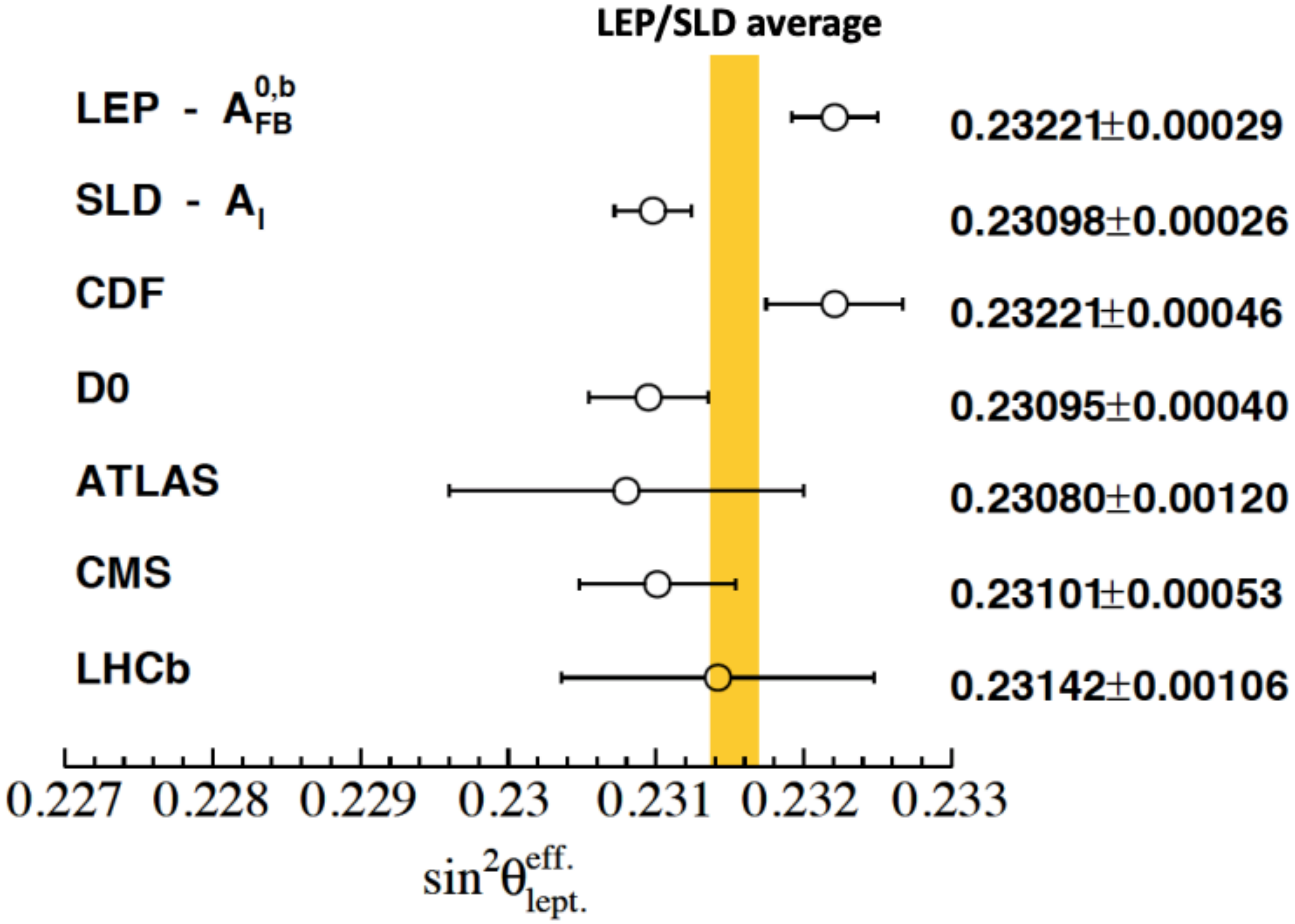
Freitas '16



Back-ups

W. Barter '18

Weak Mixing Angle – current status



Only showing explicitly the most precise measurements at LEP/SLD; overall average from all LEP/SLD measurements shown as gold band.

Tevatron combination reaches uncertainty of 0.00033; compatibility of measurements 2.6%.

Current ATLAS analysis only uses 7 TeV data.

Current CMS analysis uses sophisticated techniques to reduce statistical and PDF uncertainties.

Current LHCb analysis has smaller PDF uncertainties than ATLAS, but a larger stat. unc (lower lumi).