Spin Effects in Non-relativistic General Relativity Beyond the Leading Order

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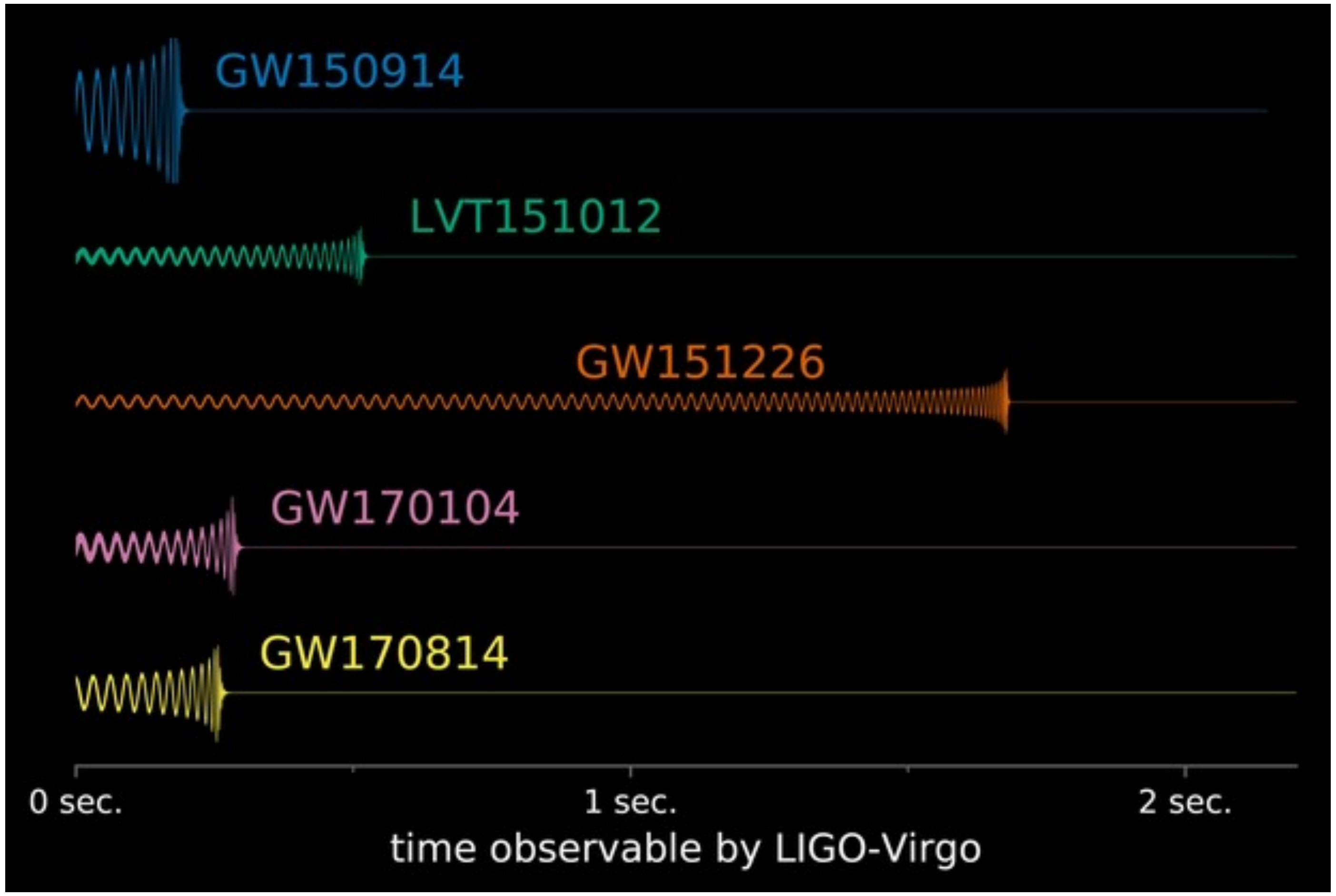
Outline of my talk:

- Motivation
- Overview of the EFT (non-relativistic GR) formalism
- Incorporating spin degrees of freedom
- Computing Observables
- Concluding remarks

Compact Object Binaries and LIGO/Virgo/LISA

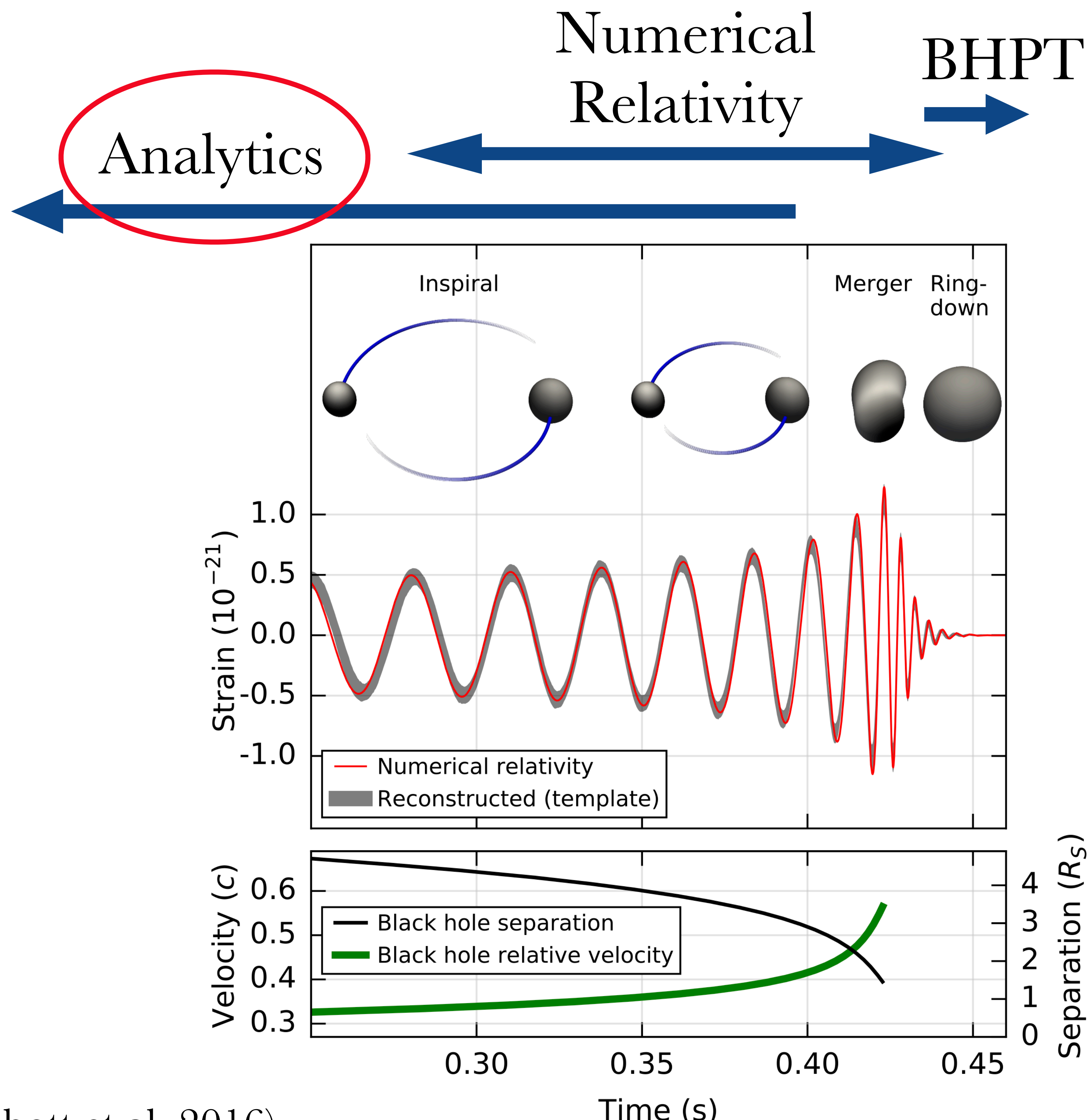
- First GW detection from black hole merger - LIGO 2015
- Dozens of detections since
- Why?
    - Strong field G.R.
    - Structure of Neutron Stars
    - Multi-messenger physics
    - Populations in stellar graveyard (spins, masses, etc)
    - Others?



LIGO/University of Oregon/Ben Farr

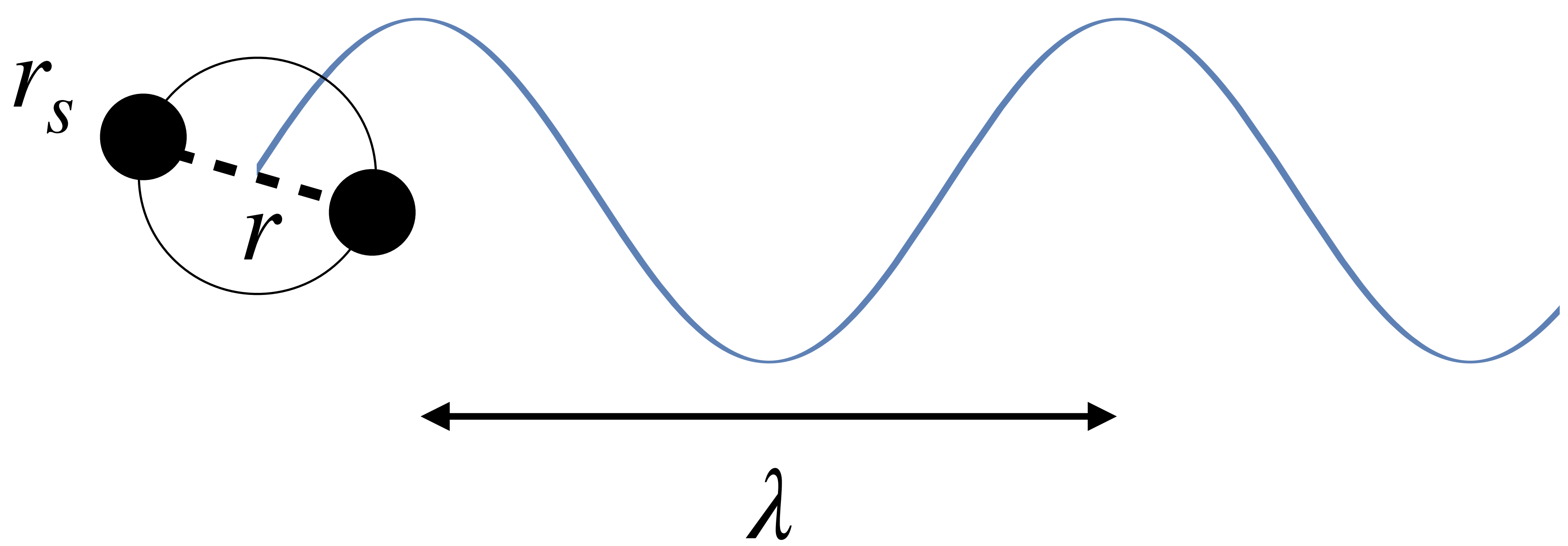
Big Picture: Producing precise templates for GW detectors

- Solve Einstein's field equations: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$
- Challenges
  - Difficult to find exact solutions
  - Numerical methods costly
  - High precision waveforms needed for parameter extraction
  - Large template bank needed for detection



(B. P. Abbott et al. 2016)

Strategy: Take advantage of separation of scales!



- Post-Newtonian Approximation beyond Newtonian: $\mathcal{O}(v^n) = \left(\frac{n}{2}\right)$ PN
- By Virial theorem: $\dfrac{G_N m}{r} = v^2 \quad \rightarrow \quad \dfrac{r_s}{r} \sim v^2, \ \dfrac{r}{\lambda} \sim v$

$r_s \equiv$ Size of Compact Object

$r \equiv$ Orbital Radius

$\lambda \equiv$ Gravitational Wave Wavelength

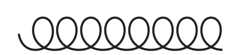
- Start with full GR coupled to point particle:

$$S = S_{\mathrm{matter}}(\dot{x}_i, g, S) + S_{\mathrm{Einstein-Hilbert}}(g) + S_{\mathrm{gauge\ fix}}$$

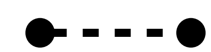
- Expand metric:

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{h^{\mu\nu}}{M_{\mathrm{Pl}}} = \eta^{\mu\nu} + \bar{h}^{\mu\nu} + H^{\mu\nu}$$

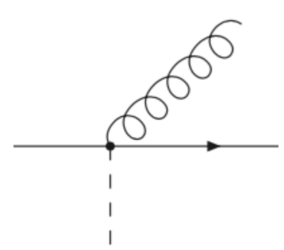
Radiation Modes: $\bar{h}^{\mu\nu}$



Potential Modes: $H^{\mu\nu}$



- Compute Feynman rules:



$$\sim m_i \bar{h} H$$

- Integrate out potential modes:

$$\exp\left[iS_{NRGR}\left[x_a, \bar{h}\right]\right] = \int \mathscr{D}H_{\mu\nu} \exp\left[iS\left(\bar{h} + H, x_a\right)\right]$$

(Goldberger and Rothstein 2005)

Non-relativistic General Relativity - Including Spin DOF

- Goal: find action that reproduces Mathisson-Papapetrou-Dixon Eqn.:

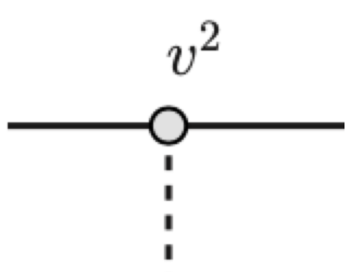
$$\frac{Dp^\mu}{D\tau} = -\frac{1}{2}R^\mu_{\ \nu\alpha\beta}u^\nu S^{\alpha\beta}, \quad \frac{DS^{\mu\nu}}{D\tau} = p^\mu u^\nu - u^\mu p^\nu$$

- (Porto 2006) Use Routhian mechanics + tetrads: $\eta_{IJ} = e^\mu_I e^\nu_J g_{\mu\nu}$

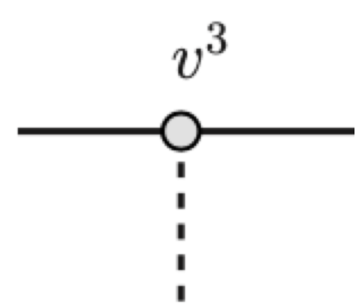
$$\mathcal{R} = -p^\mu u_\mu - \frac{1}{2}\omega_\mu^{ab} S_{ab} u^\mu \qquad (\omega_\mu^{ab} \equiv e^b_\nu \nabla_\mu e^{a\nu})$$

- Expand Routhian using: $e^a_\mu = \delta^a_\mu + \frac{1}{2}\delta^a_\nu (h^\nu{}_\mu - \frac{1}{4}h^\nu{}_\alpha h^\alpha{}_\mu) + \cdots$

- Couplings:



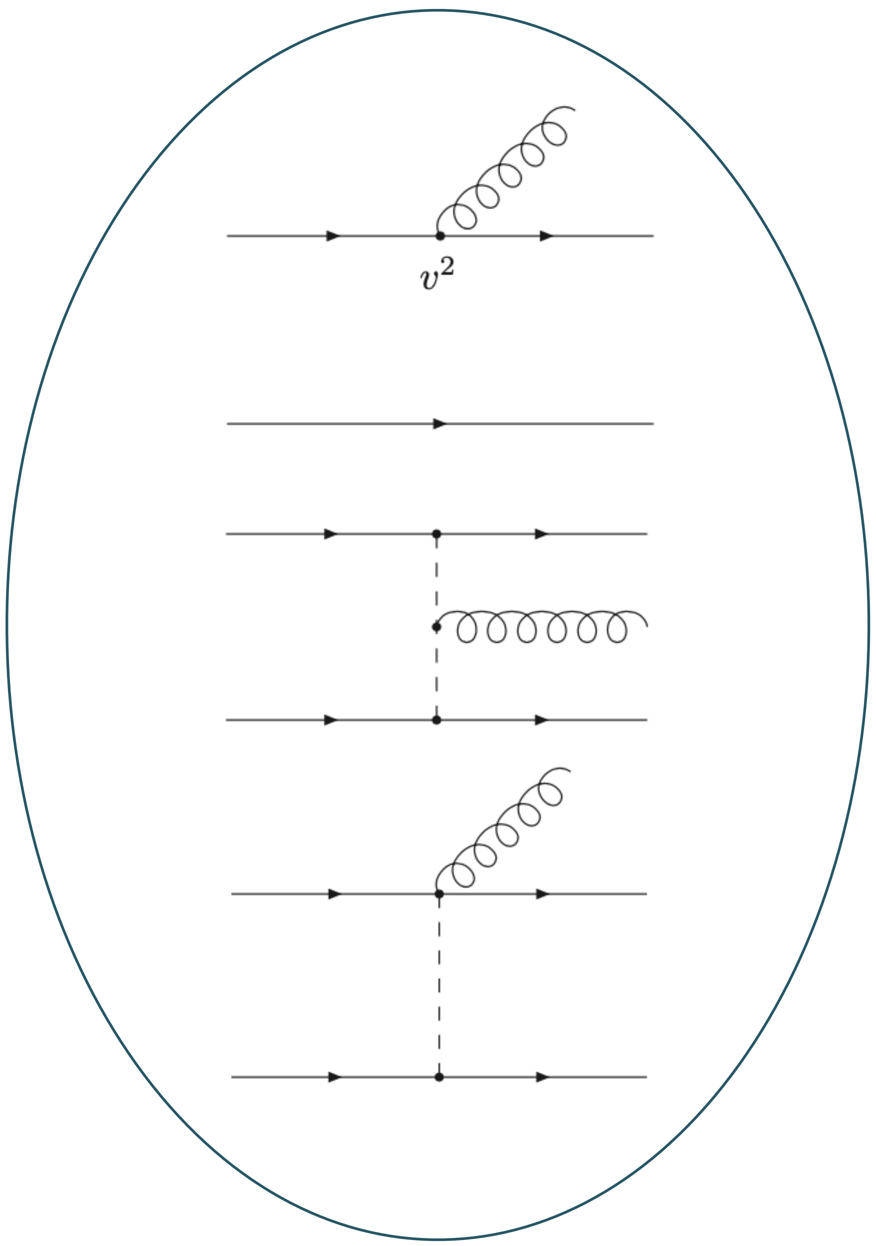
$\sim H_{i0,k}S^{ik}$



$\sim H_{ij,k}S^{ik}v^j + H_{00,k}S^{0k}$

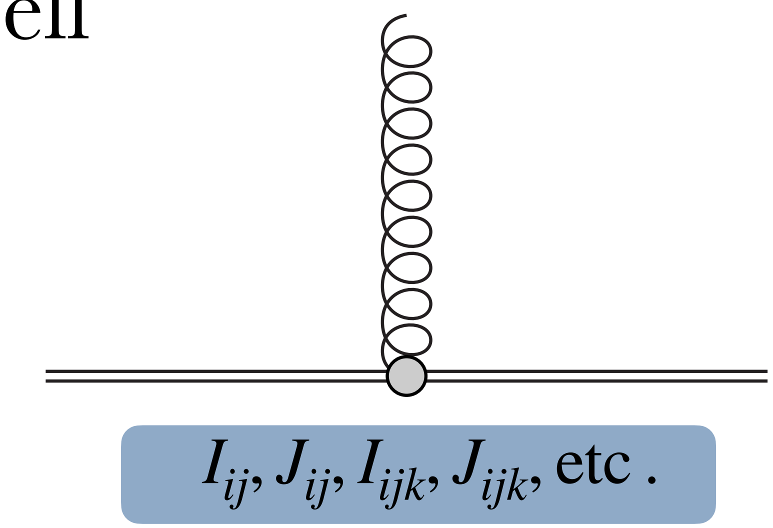
- Power Counting: $S \sim m v_{\text{rot}} r_s < m r_s \sim L v$

Schematic of EFT



Full theory

Integrate out off-shell potential modes

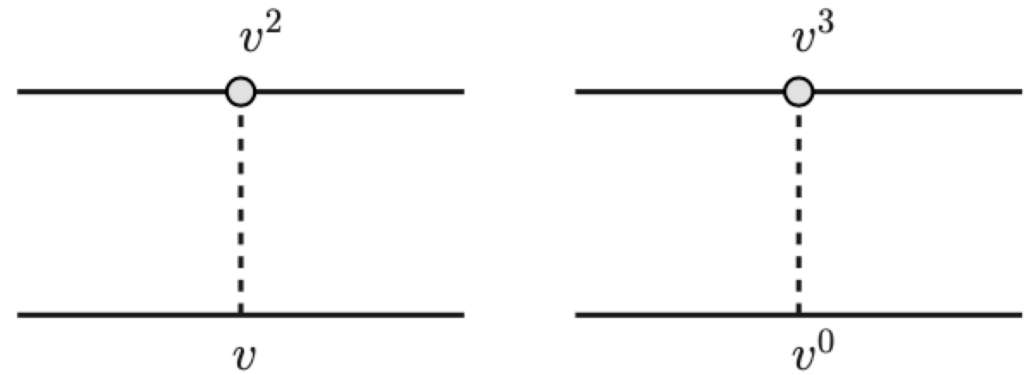


Multipole moments

Long-range theory

1.5PN Spin-Orbit Effects

- 1.5PN diagrams:



- 1.5PN potential from Lagrangian:

$$V^{\rm SO}_{\rm 1.5PN} = \frac{G_N m_2}{r^3} x^j \left(S_1^{j0} + S_1^{jk} \left(v_1^k - 2 v_2^k \right) \right) + 1 \leftrightarrow 2$$

Euler-Lagrange

$$\mathbf{a}^{\rm (cov)}_{\rm SO} = \frac{1}{r^3} \left\{ 6 \hat{\mathbf{n}} \left[(\hat{\mathbf{n}} \times \mathbf{v}) \cdot \left(2\mathbf{S} + \frac{\delta m}{m} \mathbf{\Delta} \right) \right] - \left[\mathbf{v} \times \left(7\mathbf{S} + 3 \frac{\delta m}{m} \mathbf{\Delta} \right) \right] + 3 \dot{r} \left[\hat{\mathbf{n}} \times \left(3\mathbf{S} + \frac{\delta m}{m} \mathbf{\Delta} \right) \right] \right\}$$

- In general: $\mathbf{a} = \mathbf{a}_{\rm Newton} + \mathbf{a}_{\rm 1PN} + \mathbf{a}_{\rm 1.5PN} + \mathbf{a}_{\rm 2PN} + \mathbf{a}_{\rm 2.5PN} + \cdots$

- $S^{\mu\nu}$- 6 DOF, but 3 unphysical
- Spin Supplementary Conditions (SSC)
  - Covariant: $S^{\mu\nu} p_\nu = 0$
  - Newton-Wigner: $S^{\mu 0} - S^{\mu j}\left(\frac{\tilde{p}^j}{\tilde{p}_0 + m}\right) = 0$ ($\tilde{p}^\mu$ in locally flat frame)
- Physically: Coordinate transform + PN shift of C.O.M.
- Equations of motion depend on SSC!
  - $\mathbf{a}_{\mathrm{SO}}^{(\mathrm{cov})} = \frac{1}{r^3}\left\{6\hat{\mathbf{n}}\left[(\hat{\mathbf{n}}\times\mathbf{v})\cdot\left(2\mathbf{S}+\frac{\delta m}{m}\boldsymbol{\Delta}\right)\right] - \left[\mathbf{v}\times\left(7\mathbf{S}+3\frac{\delta m}{m}\boldsymbol{\Delta}\right)\right] + 3\dot{r}\left[\hat{\mathbf{n}}\times\left(3\mathbf{S}+\frac{\delta m}{m}\boldsymbol{\Delta}\right)\right]\right\}$
  - $\mathbf{a}_{\mathrm{SO}}^{(\mathrm{NW})} = \frac{1}{r^3}\left\{\frac{3}{2}\hat{\mathbf{n}}\left[(\hat{\mathbf{n}}\times\mathbf{v})\cdot\left(7\mathbf{S}+3\frac{\delta m}{m}\boldsymbol{\Delta}\right)\right] - \left[\mathbf{v}\times\left(7\mathbf{S}+3\frac{\delta m}{m}\boldsymbol{\Delta}\right)\right] + \frac{3}{2}\dot{r}\left[\hat{\mathbf{n}}\times\left(7\mathbf{S}+3\frac{\delta m}{m}\boldsymbol{\Delta}\right)\right]\right\}$

NLO Spin-Orbit Potential: (Porto 2010)

$$
\begin{aligned}
V_{\rm so}^{\rm NLO} = \frac{Gm_2}{r^3}\Bigg[\Bigg\{ & S_1^{i0}\left(2\mathbf{v}_2^2 - 2\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{3}{2r^2}\left(\mathbf{v}_2\cdot\mathbf{r}\right)^2 - \frac{1}{2}\mathbf{a}_2\cdot\mathbf{r}\right) + \left(2\mathbf{v}_1\cdot\mathbf{v}_2 + \frac{3\left(\mathbf{v}_2\cdot\mathbf{r}\right)^2}{r^2} - 2\mathbf{v}_2^2 + \mathbf{a}_2\cdot\mathbf{r}\right)S_1^{ij}\mathbf{v}_2^j \\
& -\left(\frac{3}{2r^2}\left(\mathbf{v}_2\cdot\mathbf{r}\right)^2 + \frac{1}{2}\mathbf{a}_2\cdot\mathbf{r}\right)S_1^{ij}\mathbf{v}_1^j + 2S_1^{ij}\mathbf{a}_2^j\mathbf{v}_2\cdot\mathbf{r} + r^2 S_1^{ij}\mathbf{a}_2^j\Bigg\}\mathbf{r}^i \\
& + S_1^{i0}\left(\left(\mathbf{v}_1 - \mathbf{v}_2\right)^i\mathbf{v}_2\cdot\mathbf{r} - \frac{3}{2}\mathbf{a}_2^i r^2\right) + S_1^{ij}\mathbf{v}_2^i\mathbf{v}_1^j\mathbf{v}_2\cdot\mathbf{r} - r^2 S_1^{ij}\mathbf{a}_2^j\mathbf{v}_2^i - \frac{1}{2}r^2 S_1^{ij}\mathbf{a}_2^j\mathbf{v}_1^i\Bigg] \\
& + \frac{G^2 m_2}{r^4}\mathbf{r}^i\left[-\left(m_1 + 2m_2\right)S_1^{i0} + \left(m_1 - \frac{m_2}{2}\right)S_1^{ij}\mathbf{v}_1^j + \frac{5m_2}{2}S_1^{ij}\mathbf{v}_2^j\right]
\end{aligned}
$$

Impose SSC

$\mathbf{a}_{\rm reduced}$

CM corrections

Frame corrections

→ $\mathbf{a}_{\rm 2.5PN\ SO}$

- Energy flux:

$$\dot{E} = \sum_{\ell=2}^{\infty} \frac{G(\ell+1)(\ell+2)}{\ell(\ell-1)\ell!(2\ell+1)!!} \left\langle \left(\frac{d^{\ell+1}}{dt^{\ell+1}} I^L \right)^2 \right\rangle + \sum_{\ell=0}^{\infty} \frac{4G\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!} \left\langle \left(\frac{d^{\ell+1}}{dt^{\ell+1}} J^L \right)^2 \right\rangle$$

- Waveform:

$$h_{ij}^{TT}(x) = -\frac{4G}{|\mathbf{x}|} \Lambda_{ij,k_{\ell-1}k_\ell} \left[\sum_{\ell=2}^{\infty} \frac{1}{\ell!} \partial_t^\ell I^L (t_{\rm ret})\, \mathbf{n}^{L-2} - \sum_{\ell=2}^{\infty} \frac{2\ell}{(\ell+1)!} \epsilon^{ab(k_\ell} \partial_t^\ell J^{k_{\ell-1})bL-2} (t_{\rm ret})\, \mathbf{n}^{aL-2} \right]$$

- Compute Energies from potentials:

$$E_{\rm N} = \mu \left(\frac{1}{2} v^2 - \frac{m}{r} \right)$$

$$E_{\rm 1.5PN\ SO} = \frac{1}{r^3} \mathbf{L}_{\rm N} \cdot \left(\mathbf{S} + \frac{\delta m}{m} \boldsymbol{\Delta} \right)$$

- Orbital Frequency (circular orbits):

$$r\omega^2 = \langle \hat{\mathbf{n}} \cdot \mathbf{a} \rangle$$

(Kidder 1995, Ross 2012)

- For quasi-circular orbit:

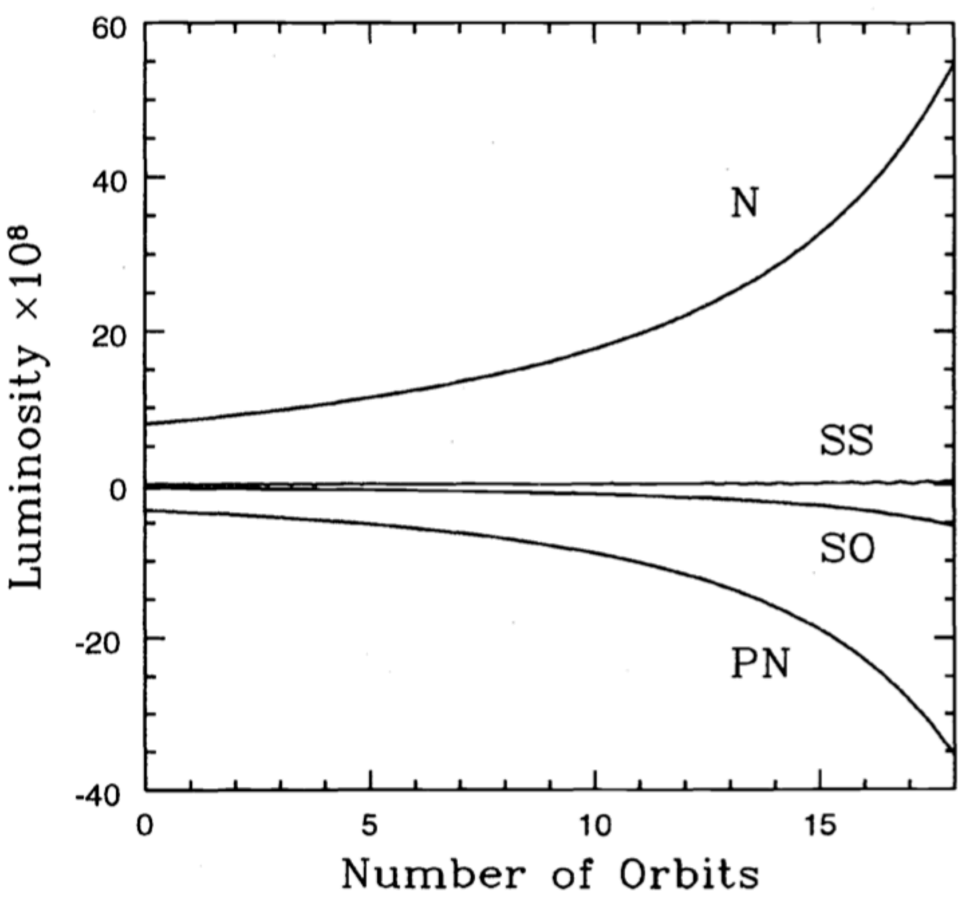
1. $r\omega^2 = \langle \hat{\mathbf{n}} \cdot \mathbf{a} \rangle$ 2. $\dot{r} = \dfrac{dE/dt}{dE/dr}$ 3. $\dfrac{\dot{\omega}}{\omega} = -\dfrac{2}{3}\dfrac{\dot{r}}{r}$

Accumulated orbital phase observed by LIGO:

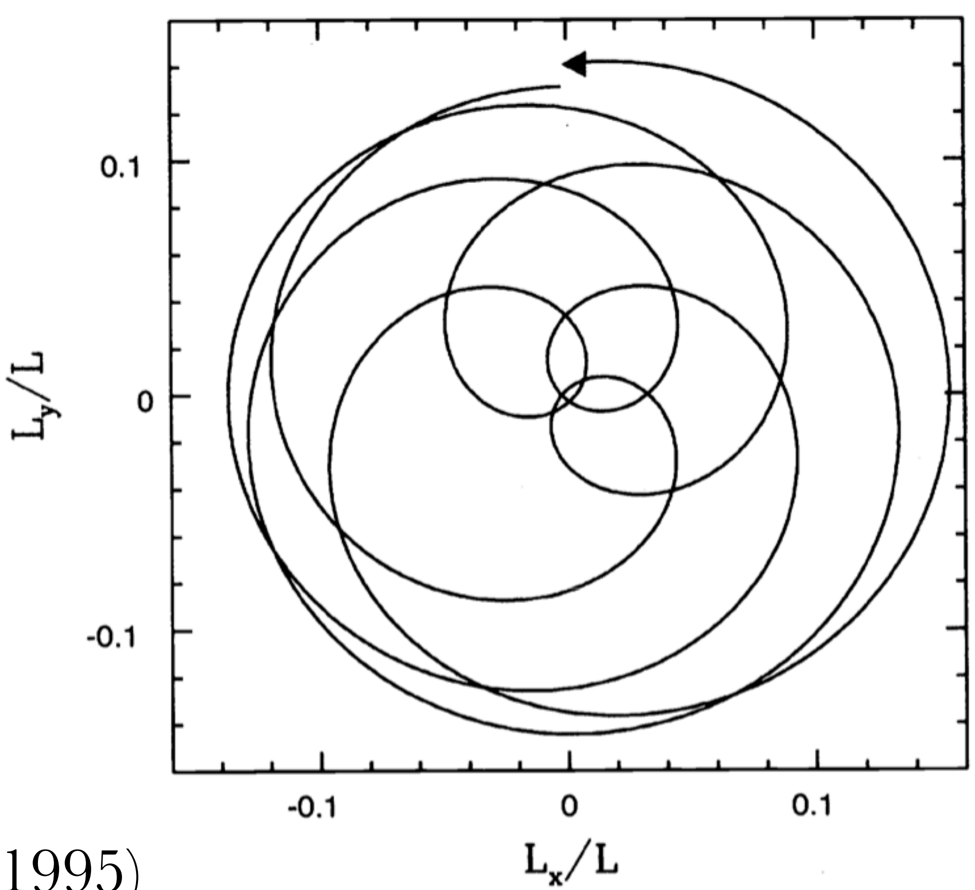
$$\Psi = \int \omega dt = \int_{\omega_i}^{\omega_f} \frac{\omega}{\dot{\omega}} d\omega$$

Need spin accelerations!

- We need templates with spin:
  - Corrections to orbital frequency
  - Precession of orbital angular momentum
  - Spin finite size effects enter early!



(Kidder 1995)

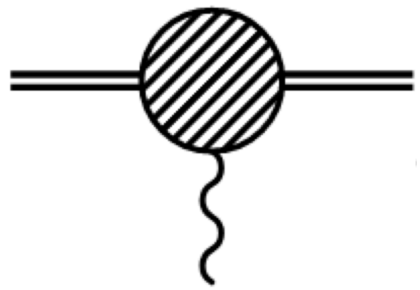


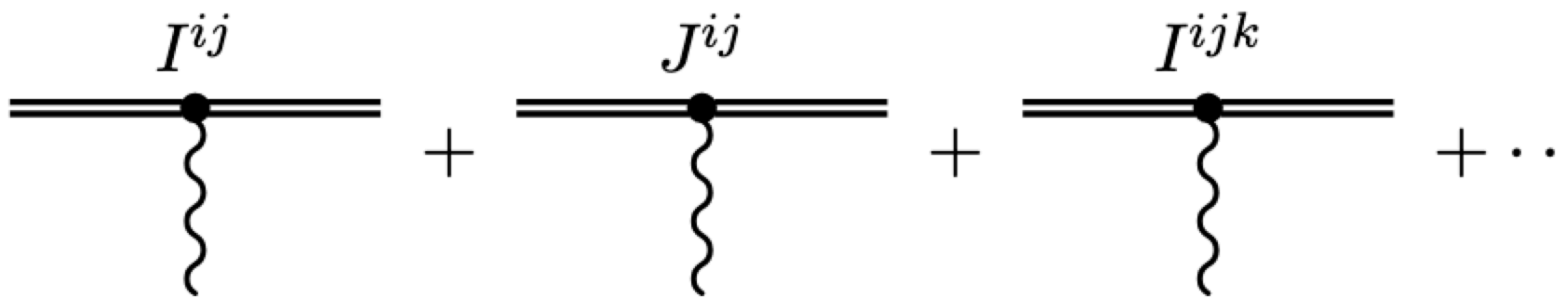
- NLO spin-orbit effects from EFT in the works
  - Templates with spin
  - New results for orbital frequency
- Increased confidence in results with other formalisms
- Other projects:
  - GR from Scattering Amplitudes (including spin!)
  - Finite size effects from EFT

Thank you!

BACK-UP SLIDES

- Calculate Power Loss in EFT

$i\mathcal{A}_h(\mathbf{k}) =$ 

$=$ 

$$P = \underbrace{\frac{G_N}{5}\left\langle\left(\frac{d^3}{dt^3}I^{ij}(t)\right)^2\right\rangle + \frac{16G_N}{45}\left\langle\left(\frac{d^3}{dt^3}J^{ij}(t)\right)^2\right\rangle}_{\text{contribute at 1.5PN}} + \frac{G_N}{189}\left\langle\left(\frac{d^4}{dt^4}I^{ijk}(t)\right)^2\right\rangle + \cdots$$

- Pieces that contribute at 1.5PN:

- $$P_{1.5} \sim \left\{ \left(a^S_{1.5}, Q_0\right), \left(a_0, Q^S_{1.5}\right), \left(a_0, J^S_{0.5}\right), \left(P_0, \delta x^S_{1.5}\right) \right\}$$

  - $Q_0 = (I^{ij}_0, I^{ij}_0)$

  - $Q_{1.5} = (I^{ij}_0, I^{ij}_{S(1.5)})$

  - $J_{0.5} = (J^{ij}_0, J^{ij}_{S(0.5)})$

$$P_{\mathrm{SO}} = -\frac{8}{15}\frac{G_N m^2 \nu}{r^6}\left\{ \mathbf{L}_{\mathrm{N}} \cdot \left[\mathbf{S}\left(78\dot{r}^2 - 80v^2 - 8\frac{G_N m}{r}\right) + \frac{\delta m}{m}\Delta\left(51\dot{r}^2 - 43v^2 + 4\frac{G_N m}{r}\right)\right]\right\}$$

$$S = -\underbrace{\sum_a m_a \int d\tau_a}_{S_{\rm pp}} - \underbrace{2M_{\rm Pl}^2 \int d^4x \sqrt{g}\, R}_{S_{\rm EH}} + \underbrace{M_{\rm Pl}^2 \int d^4x \sqrt{\bar{g}}\, \Gamma_\mu \Gamma^\mu}_{S_{\text{gauge fix}}} + \cdots$$

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\eta(m\omega)^{5/3}\left(1 - \frac{1}{336}(743 + 924\eta)(m\omega)^{2/3} - \frac{1}{12}\sum_{i=1,2}\left[\chi_i\left(\hat{\mathbf{L}}_{\rm N}\cdot\hat{\mathbf{s}}_i\right)\left(113\frac{m_i^2}{m^2} + 75\eta\right)\right](m\omega)\right)$$