Integration-By-Parts-like identities in Schwinger-Feynman-Lee-Pomeransky parametrization

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May 4, 2020

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Multi-Loop calculations and Reductions

- Multi-loop, higher order calculations are demanded in high energy phenomenology; the number of Feynman integrals combinatorially explodes when we aim at higher order corrections for scattering amplitude calculations.
- Among such a large number of Feynman integrals, not all integrals are linearly independent, i.e., there exists a set of non-trivial linear relations.
- A standard way of generating linear relations among families of Feynman integrals relies upon Integration-By-Parts identities (Chetyrkin et al. 1981) over, for example, momentum space.
- In this talk, we will present another approach, namely a construction of IBP-like identities over Schwinger-Feynman parameters (Chen 2020; Sameshima 2019).

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Feynman Integrals

Feynman integrals are given as *d-dimensional momentum integrals*, which are direct translation of Feynman diagrams. Let us fix a generic diagram, and consider the associated Feynman integral:

$$J = \int \frac{d^d l_1}{(2\pi)^d} \cdots \int \frac{d^d l_L}{(2\pi)^d} \frac{Num(l)}{D_1 \cdots D_K} \,,$$

where the numerator is a polynomial of loop momenta and kinematic invariants.

We consider the following scalar integrals, in stead, adding auxiliary denominators:

$$J(\vec{\nu}) := \left(\prod_{j=1}^{L} \int \frac{d^d l_j}{(2\pi)^d} \right) \prod_{a=1}^{N} \frac{1}{D_a^{\nu_a}} \,.$$

Schwinger-Feynman Parametrizations

Using Schwinger trick and an identity $1 = \int_0^\infty d\rho \delta \left(\rho - \sum_{j=1}^N x_j \right)$, we obtain the following parametrizations (for simple derivations, e.g., see (Bitoun et al. 2019)).

- Schwinger parametrization:

$$I(\vec{\nu}) = \frac{1}{(4\pi)^{\frac{dL}{2}}} \left(\prod_{a=1}^N \int_{x_a=0}^\infty \frac{dx_a x_a^{\nu_a - 1}}{\Gamma(\nu_a)} \right) \frac{\exp\left(-\frac{\mathcal{F}}{\mathcal{U}}\right)}{\mathcal{U}^{\frac{d}{2}}}$$

- Feynman parametrization:

$$I(\vec{\nu}) = \frac{1}{(4\pi)^{\frac{dL}{2}}} \Gamma(\omega) \left(\prod_{a=1}^N \int_{x_a=0}^1 \frac{dx_a x_a^{\nu_a - 1}}{\Gamma(\nu_a)} \right) \frac{\delta\left(1 - \sum_{j=1}^N x_j\right)}{\mathcal{F}^\omega \mathcal{U}^{\frac{d}{2} - \omega}}$$

- Lee-Pomeransky paramerization:

$$I(\vec{\nu}) = \frac{1}{(4\pi)^{\frac{dL}{2}}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{a=1}^N \int_{x_a=0}^\infty \frac{dx_a x_a^{\nu_a - 1}}{\Gamma(\nu_a)} \right) \mathcal{G}^{-\frac{d}{2}}$$

where $\omega = \sum_{a=1}^N \nu_a - \frac{dL}{2}$, $\mathcal{U}, \mathcal{F}$ and $\mathcal{G} := \mathcal{U} + \mathcal{F}$ are graph polynomials.

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Differentiate then Integrate

Integration-By-Parts identities (Chetyrkin et al. 1981) are the outcomes of the translational invariance of Feynman integrals in momentum space:

$$\int d^d l_1 \cdots d^d l_L \frac{\partial}{\partial l_i} \cdot \left[q_k \prod_{a=1}^{N} \frac{1}{D_a^{\nu_a}} \right] = \text{surface term} = 0\,.$$

The surface term corresponding to $\frac{\partial}{\partial l_i} \cdot q_k$ is zero. Applying partial derivative, we can build a set of linear relations among Feynman integrals.

We follow the same spirit; for a fixed graph and the associated graph polynomial $\mathcal{G}$, we consider:

$$\left(\prod_a \int_{x_a=0}^{\infty} dx_a \right) \partial_i \left[f \mathcal{G}^{-\frac{d}{2}+1} \right] = \text{surface term}\,,$$

where f is a monomial of parameters $[x_1, \cdots, x_N]$. In SF parametrizations, the surface term is in general non-zero (Roman N. Lee et al. 2013). Some early discussions on surface terms are already appeared in literature, e.g., in (Bern et al. 1994).

Non-zero Surface Terms

Let us show an example of non-zero surface term: $f = 1, \partial_1 = \frac{\partial}{\partial x_1}$ case.

Using the fundamental theorem of calculus, we obtain:

$$\int_{x_1=0}^{\infty} dx_1 \partial_1 \mathcal{G}^{-\frac{d}{2}+1} = \mathcal{G}^{-\frac{d}{2}+1}\Big|_{x_1=\infty} - \mathcal{G}^{-\frac{d}{2}+1}\Big|_{x_1=0} = -\,\mathcal{G}^{-\frac{d}{2}+1}\Big|_{x_1=0} \,,$$

where we have used the fact that $\mathcal{G}^{-\frac{d}{2}+1}\Big|_{x_1=\infty} = 0$ in the dimension regularization scheme. Hence, we have the following non-zero surface term:

$$(-) \int dx_2 \cdots \int dx_N \quad \underbrace{\mathcal{G}\big|_{x_1=0}}_{\text{Polynomials of } x_2 \cdots x_N} \quad \mathcal{G}^{-\frac{d}{2}}\Big|_{x_1=0} \,.$$

We interpret it as a linear combination of integrals of D_1-reduced diagrams.

Differential Forms

As a natural generalization of the above formalism, consider a generic $(N-1)$-form

$$f := \sum_i \varphi_i dx_1 \wedge \cdots \widehat{\wedge dx_i} \wedge \cdots \wedge dx_N,$$

where $\varphi_i, 1 \leq i \leq N$ is an arbitrary 0-form. Then we obtain

$$\int_\Gamma d\left(f\mathcal{G}^{-\frac{d}{2}}\right) = \int_\Gamma \sum_i \frac{\partial\left(\mathcal{G}^{-\frac{d}{2}}\varphi_i\right)}{\partial x_i}(-)^{i-1} dx_1 \wedge \cdots \wedge dx_N$$
$$= \int_\Gamma \left(\sum_i (-)^{i-1} \frac{\partial \varphi_i}{\partial x_i}\mathcal{G}^{-\frac{d}{2}} + \left(-\frac{d}{2}\right)\sum_i (-)^{i-1}\varphi_i \frac{\partial \mathcal{G}}{\partial x_i}\mathcal{G}^{-\frac{d}{2}-1}\right)$$
$$\times\, dx_1 \wedge \cdots \wedge dx_N$$

If we apply Stokes' theorem, we will obtain IBP-like linear equations with dimension shift.

Dimension shift and Syzygies

We can remove such dimension shift by adding an additional constrain: a sygyzy. Let $(\phi_0, \phi_1, \cdots, \phi_N)$ be a set of polynomial that satisfies:

$$\phi_0 \mathcal{G} + \sum_i \phi_i \partial_i \mathcal{G} = 0.$$

Constructive algorithms and implementations are known (Cox et al. 2007), hence we obtain the following equation with no dimension shift:

$$\int_\Gamma d\left(f\mathcal{G}^{-\frac{d}{2}}\right) dx_1 \cdots dx_N = \int_\Gamma \left[\sum_{i=1}^N \left(\frac{\partial \phi_i}{\partial x_i}\right) + \frac{d}{2}\phi_0\right] \mathcal{G}^{-\frac{d}{2}} dx_1 \cdots dx_N$$

When we combine the left hand side with Stokes' theorem (within the dimensional regularization scheme), we have the following IBP-like equations in SFLP:

$$(-)\sum_{i=1}^N \int_{\partial_i \Gamma} \left[\phi_i \mathcal{G}^{-\frac{d}{2}}\right]_{x_i=0} d^{N-1}x_i = \int_\Gamma \left[\frac{d}{2}\phi_0 + \sum_{i=1}^N \left(\frac{\partial \phi_i}{\partial x_i}\right)\right] \mathcal{G}^{-\frac{d}{2}} dx_1 \cdots dx_N$$

Syzygies as Generators

If $(\phi_0, \phi_1, \cdots, \phi_N)$ satisfies the sygyzy condition: $\phi_0 \mathcal{G} + \sum_i \phi_i \partial_i \mathcal{G} = 0$, clearly, for any polynomial e in $[x_1, \cdots, x_N]$ (e.g., $e := x_j$), we obtain another:

$$(e\phi_0, e\phi_1, \cdots, e\phi_N)$$

I.e., we can generate another equation by multiplying polynomials to a given $(\phi_0, \phi_1, \cdots, \phi_N)$:

$$(-)\sum_{i=1}^{N} \int_{\partial_i \Gamma} \left[e\phi_i \mathcal{G}^{-\frac{d}{2}} \right]_{x_i=0} d^{N-1}x_i = \int_{\Gamma} \left[\frac{d}{2} e\phi_0 + \sum_{i=1}^{N} \left(\frac{\partial e\phi_i}{\partial x_i} \right) \right] \mathcal{G}^{-\frac{d}{2}} dx_1 \cdots dx_N$$

Looping over such a polynomial e in the parameter space, we can construct a set of non-trivial equations.

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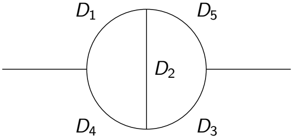
Kite diagram

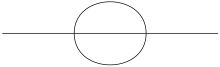
We will show a full reduction of the following Kite diagram:

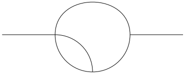


The corresponding Lee-Pomeransky polynomial $\mathcal{G}$ is:

$$p^2 x_3 x_4 x_5 + p^2 x_2 x_4 x_5 + p^2 x_1 x_4 x_5 + x_4 x_5 + p^2 x_2 x_3 x_5 + p^2 x_1 x_3 x_5 + x_2 x_5 \\ + x_1 x_5 + p^2 x_1 x_3 x_4 + x_3 x_4 + p^2 x_1 x_2 x_4 + x_2 x_4 + p^2 x_1 x_2 x_3 + x_2 x_3 + x_1 x_3 + x_1 x_2$$

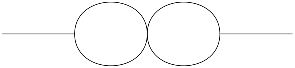
One can obtain, running an IBP reduction automation (e.g., Kira (Maierhöfer et al. 2018)), two Master Integrals:

 $\sim I(1,1,1,0,0)$

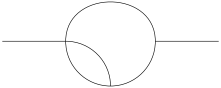
 $\sim I(1,0,1,1,1)$

Subdiagrams

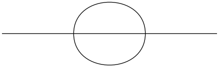
If we contract the central edge D_2 of the kite diagram, we obtain the following glasses diagram:



If we are to contract another edge, we will have a massless tadpole, a scale-less diagram. Such a scale-less diagram is zero. Similarly, we obtain four types of non-zero subdiagrams (moon diagrams):



and a non-zero sunset diagram:



Including the sunset diagram, these are the only surviving (sub-)diagrams.

“Pinching” of edge contraction

This edge contraction, or simply pinching, of a graph-theoretical procedure has appeared in the Surface Term. From the kite to the glasses:

$$\frac{1}{D_1 D_2 D_3 D_4 D_5} \to \frac{1}{D_1 D_3 D_4 D_5}$$

we eliminate the central edge D_2 and obtain a subdiagram.
When we set $\nu_2 = 0$ within LP parametrization, we will immediately face troubles:

$$\int_{\mathbb{R}_>} \frac{dx_1 x_1^{-1}}{\Gamma(0)} \left(\prod_{a \neq 2} \int_{\mathbb{R}_>} dx_a \right) \mathcal{G}^{-\frac{d}{2}} .$$

However, the “D_2 absence” can be achieved by setting the corresponding parameter x_2 to be zero:

$$\mathcal{G} \mapsto \left. \mathcal{G} \right|_{x_2=0} .$$

Cauchy’s integral formula gives us the same prescription:

$$\oint_{\circlearrowleft} \frac{dx_2}{2\pi i x_2} \mathcal{G} = \left. \mathcal{G} \right|_{x_2=0} .$$

Tools

We use Singular (Decker et al. 2018) to construct the syzygies

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sygyzies of kite:
[0,0,(p^4)*x3*x5+(-p^4)*x4*x5+(-2*p^2)*x1+(p^2)*x3+(p^2)*x5,(p^4)*x3*x5+(p^4)*x4*x5+(-p^2)*x3+(p^2)*x5-2,(-p^4)*x3*
x5+(-p^4)*x4*x5+(-p^2)*x3+(-2*p^2)*x4+(-p^2)*x5,(2*p^2)*x1+(2*p^2)*x5+2],
[0,0,(2*p^4)*x1*x5+(2*p^4)*x2*x5+(p^4)*x4*x5+(2*p^2)*x1+(p^2)*x2,(-p^4)*x4*x5+(-p^2)*x2+2,(p^4)*x4*x5+(p^2)*x2+(2*p
^2)*x4,(-2*p^4)*x1*x5+(-2*p^4)*x5^2+(-2*p^2)*x1+(-4*p^2)*x5-2],
[0,0,(p^2)*x2*x3+(2*p^2)*x1*x4+(p^2)*x2*x4+(p^2)*x3*x4+(-p^2)*x3*x5+(p^2)*x4*x5+2*x1+x2-x3+2*x4-x5,(p^2)*x2*x3+(p^2
)*x2*x4+(p^2)*x3*x4+(-p^2)*x3*x5+(-p^2)*x4*x5+x2-x3-x5,(-p^2)*x2*x3+(-p^2)*x2*x4+(-p^2)*x3*x4+(p^2)*x3*x5+(p^2)*x4*
x5-x2+x3+x5,(-2*p^2)*x1*x4+(-2*p^2)*x4*x5-2*x1-2*x4],
[0,0,(2*p^2)*x1*x3+(p^2)*x2*x3+(p^2)*x2*x4+(p^2)*x3*x4+(p^2)*x3*x5+(-p^2)*x4*x5+x2+x3+x5,(-p^2)*x2*x3+(-p^2)*x2*x4+
(-p^2)*x3*x4+(p^2)*x3*x5+(p^2)*x4*x5-x2+x3+x5,(p^2)*x2*x3+(p^2)*x2*x4+(p^2)*x3*x4+(-p^2)*x3*x5+(-p^2)*x4*x5+x2-x3-x
5,(-2*p^2)*x1*x3+(-2*p^2)*x3*x5-2*x3-2*x5],
[0,0,(2*p^2)*x1^2+(2*p^2)*x1*x2+(p^2)*x1*x4+x1+x2+x4,(-p^2)*x1*x4+x1+x2-x4,(p^2)*x1*x4-x1-x2-x4,(-2*p^2)*x1^2+(-2*p
^2)*x1*x5-2*x1],
[0,1,(p^2)*x3+(-p^2)*x4,(p^2)*x3+(p^2)*x4+1,(-p^2)*x3+(-p^2)*x4-1,-1],
[0,(p^2)*x1+(p^2)*x5+1,(p^2)*x1+(-p^2)*x5,1,-1,(-p^2)*x1+(-p^2)*x5-1],
[2,-x1,-x1-2*x2-x4,x4,-x4,x1]
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Figure: A set of syzygies constructed with Singular.

and a built-in linear solver of Maxima CAS (Maxima 2017) to solve a linear system of equations.

We use "bottom-up" approach; building a system of equations from lower sectors until we obtain a full reduction within each sector (Laporta algorithm (Laporta 2000). E.g., on the sunset sector, all the surface terms fall into massless tadpoles, hence zeros, and we eventually obtain a master integral $I(1,1,1,0,0)$ in the sunset sector.

A syzygy on the sunset sector is

$$\left[p^2\,x_3+2, -p^2\,x_1\,x_3-x_3-2\,x_1, p^2\,x_2\,x_3+x_3, -p^2\,x_3^2-x_3\right].$$

When we plug this list into our IBP-like identity formula, we obtain:

$$0=\Gamma\left(\frac{3(d-2))}{2}\right)(d-3)\,I(1,1,1,0,0)+\Gamma\left(\frac{3d-8}{2}\right)\frac{(d-4)\;p^2}{2}I(1,1,2,0,0)$$

It gives a reduction of $I(1,1,2,0,0)$ with respect to $I(1,1,1,0,0)$ of a MI.

In the sunset sector, any integral of of the form $I(\nu_1, \nu_2, \nu_3, 0, 0)$ can be reduced into a linear combination of a MI, $I(1, 1, 1, 0, 0)$.

$$I([1,1,3,0,0]) = \frac{I([1,1,1,0,0])\,(d-3)(3d-10)(3d-8)}{2(d-6)\,p^4},\quad I([1,3,1,0,0]) = \frac{I([1,1,1,0,0])\,(d-3)(3d-10)(3d-8)}{2(d-6)\,p^4},\quad I([3,1,1,0,0]) = \frac{I([1,1,1,0,0])\,(d-3)(3d-10)(3d-8)}{2(d-6)\,p^4},$$

$$I([1,2,2,0,0]) = \frac{I([1,1,1,0,0])\,(d-3)(3d-10)(3d-8)}{(d-4)\,p^4},\quad I([2,1,2,0,0]) = \frac{I([1,1,1,0,0])\,(d-3)(3d-10)(3d-8)}{(d-4)\,p^4},\quad I([2,2,1,0,0]) = \frac{I([1,1,1,0,0])\,(d-3)(3d-10)(3d-8)}{(d-4)\,p^4},$$

$$I([1,1,4,0,0]) = -\frac{I([1,1,1,0,0])\,(d-5)(d-4)(d-3)(3d-10)(3d-8)}{2(d-8)(d-6)\,p^6},\quad I([1,4,1,0,0]) = -\frac{I([1,1,1,0,0])\,(d-5)(d-4)(d-3)(3d-10)(3d-8)}{2(d-8)(d-6)\,p^6},$$

$$I([4,1,1,0,0]) = -\frac{I([1,1,1,0,0])\,(d-5)(d-4)(d-3)(3d-10)(3d-8)}{2(d-8)(d-6)\,p^6},\quad I([1,2,3,0,0]) = -\frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-10)(3d-8)}{2(d-8)(d-6)\,p^6},$$

$$I([1,3,2,0,0]) = -\frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-10)(3d-8)}{2(d-6)\,p^6},\quad I([2,1,3,0,0]) = -\frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-10)(3d-8)}{2(d-6)\,p^6},$$

$$I([2,3,1,0,0]) = -\frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-10)(3d-8)}{2(d-6)\,p^6},\quad I([3,1,2,0,0]) = -\frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-10)(3d-8)}{2(d-6)\,p^6},$$

$$I([3,2,1,0,0]) = -\frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-10)(3d-8)}{2(d-6)\,p^6},\quad I([2,2,2,0,0]) = -\frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-10)(3d-8)}{(d-4)\,p^6},$$

$$I([1,1,5,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-4)(d-3)(3d-14)(3d-10)(3d-8)}{8(d-10)(d-8)\,p^8},\quad I([1,5,1,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-4)(d-3)(3d-14)(3d-10)(3d-8)}{8(d-10)(d-8)\,p^8},$$

$$I([5,1,1,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-4)(d-3)(3d-14)(3d-10)(3d-8)}{8(d-10)(d-8)\,p^8},\quad I([1,2,4,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-8)\,p^8},$$

$$I([1,4,2,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-8)\,p^8},\quad I([2,1,4,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-8)\,p^8},$$

$$I([2,4,1,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-8)\,p^8},\quad I([4,1,2,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-8)\,p^8},$$

$$I([4,2,1,0,0]) = \frac{I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-8)\,p^8},\quad I([1,3,3,0,0]) = \frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{4(d-6)\,p^8},$$

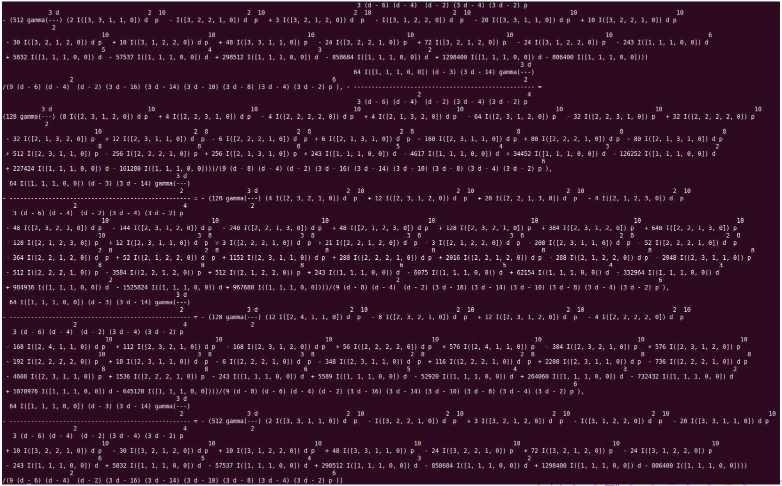
$$I([3,1,3,0,0]) = \frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{4(d-6)\,p^8},\quad I([3,3,1,0,0]) = \frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{4(d-6)\,p^8},$$

$$I([2,2,3,0,0]) = \frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-4)\,p^8},\quad I([2,3,2,0,0]) = \frac{3\,I([1,1,1,0,0])\,(d-5)(d-3)(3d-14)(3d-10)(3d-8)}{2(d-4)\,p^8},$$

Figure: Sunset sector's reductions over Maxima CAS environment

In the moon sector, any integral of of the form $I(\nu_1, \nu_2, \nu_3, \nu_4, 0)$ can be reduced into a linear combination of a MI, $I(1,1,1,0,0)$. In particular, the scalar integral $I(1,1,1,1,0)$ is reducible:

$$I\,(1,1,1,1,0) = \frac{(3d-8)}{(d-4)\,p^2} I\,(1,1,1,0,0)$$



Combining the reduction of glasses sector with another MI, $I(1,0,1,1,1)$, we can finally build a full reduction chain for the kite; for any integral with five indices is reduced into a linear combination of two MI, e.g.,

$$I(1,1,1,1,1) = \frac{2\,(3\,d-10)\,(3\,d-8)}{(d-4)^2\;p^4} I(1,1,1,0,0) + \frac{-2\,(d-3)}{(d-4)\;p^2} I(1,0,1,1,1).$$

```
                   3 I([1, 1, 1, 0, 0]) (d - 5) (d - 3) (3 d - 10) (3 d - 8)
I([1, 2, 0, 2, 1]) = ---------------------------------------------------------, I([2, 2, 0, 2, 1]) = I([2, 2, 1, 2, 0]), I([1, 1, 0, 1,
                                                                    6
                                       (d - 6) (d - 4) p
                     3 I([1, 1, 1, 0, 0]) (d - 3) (3 d - 10) (3 d - 8)                          3 I([1, 1, 1, 0, 0]) (d - 3) (3 d - 10) (
I([2, 1, 0, 1, 2]) = - -------------------------------------------------, I([1, 2, 0, 1, 2]) = -----------------------------------------
                                                                  6                                                                6
                                     (d - 8) (d - 6) p                                                                    (d - 8) p
                   3 I([1, 1, 1, 0, 0]) (d - 5) (d - 3) (3 d - 10) (3 d - 8)
I([1, 1, 0, 2, 2]) = ---------------------------------------------------------, I([2, 1, 0, 2, 2]) = I([2, 1, 2, 2, 0]), I([1, 2, 0, 2,
                                                                    6
                                       (d - 6) (d - 4) p
                   I([1, 1, 1, 0, 0]) (3 d - 8)                        3 I([1, 1, 1, 0, 0]) (d - 3) (3 d - 10) (3 d - 8)
I([0, 1, 1, 1, 1]) = ---------------------------, I([0, 2, 1, 1, 1]) = - -------------------------------------------------, I([0, 1, 2,
                                         2                                                                   6
                              (d - 4) p                                                 (d - 8) (d - 6) p
                     3 I([1, 1, 1, 0, 0]) (d - 3) (3 d - 10) (3 d - 8)                            I([1, 1, 1, 0, 0]) (d - 3) (3 d - 10) (
I([0, 2, 2, 1, 1]) = - -------------------------------------------------, I([0, 1, 1, 2, 1]) = - ---------------------------------------
                                                                  6                                                              4
                                     (d - 8) (d - 6) p                                                           (d - 6) (d - 4) p
                     I([1, 1, 1, 0, 0]) (d - 3) (3 d - 10) (3 d - 8)
I([0, 1, 2, 2, 1]) = - -----------------------------------------------, I([0, 2, 2, 2, 1]) = I([1, 2, 2, 2, 0]), I([0, 1, 1, 1, 2]) = -
                                                                4
                                    (d - 6) (d - 4) p
                     I([1, 1, 1, 0, 0]) (3 d - 10) (3 d - 8)                                                                   3 I([1, 1,
I([0, 1, 2, 1, 2]) = - ---------------------------------------, I([0, 2, 2, 1, 2]) = I([2, 2, 1, 2, 0]), I([0, 1, 1, 2, 2]) = ----------
                                                  4
                                    (d - 4) p
                                                3 I([1, 1, 1, 0, 0]) (d - 5) (d - 3) (3 d - 10) (3 d - 8)
I([0, 2, 1, 2, 2]) = I([2, 2, 2, 2, 0]), I([0, 1, 2, 2, 2]) = ---------------------------------------------------------, I([0, 2, 2, 2,
                                                                                                  6
                                                                                    (d - 6) (d - 4) p
                        2             2                                    2
                     (2 d  - 14 d + 24) p  I([1, 0, 1, 1, 1]) + ((- 18 d ) + 108 d - 160) I([1, 1, 1, 0, 0])
I([1, 1, 1, 1, 1]) = - -------------------------------------------------------------------------------------,
                                                             2          4
                                                          (d  - 8 d + 16) p
                        3       2                  2                        3         2
                     (2 d  - 26 d  + 108 d - 144) p  I([1, 0, 1, 1, 1]) + ((- 27 d ) + 297 d  - 1050 d + 1200) I([1, 1, 1, 0, 0])
I([1, 1, 1, 1, 2]) = -------------------------------------------------------------------------------------------------------------,
                                                                       2             6
                                                                     (d  - 10 d + 24) p
                        3       2                  2                        3         2
                     (2 d  - 26 d  + 108 d - 144) p  I([1, 0, 1, 1, 1]) + ((- 27 d ) + 297 d  - 1050 d + 1200) I([1, 1, 1, 0, 0])
I([2, 1, 1, 1, 1]) = -------------------------------------------------------------------------------------------------------------]
                                                                       2             6
                                                                     (d  - 10 d + 24) p
```

Conclusion

We showed an alternative way of constructing linear equations, IBP-like equations over Schwinger-Feynman parameters.

Setting no-dimension shift as an additional constrain, we built syzygies, and constructed a set of linear equations by looping over monomials in the parameter space.

We showed that an example of full reductions; the results are cross-checked with Kira (Maierhöfer et al. 2018).

The next steps will be

- more complicated diagrams (more scales, masses, loops etc)
- automation

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Non-positive indices and CIF

The edge contraction "pinching" implies the following natural interpretation of the non-positive indices:

$$\int \frac{dx_a}{\Gamma(-n) x_a^{n+1}} \mapsto (-)^n \frac{n!}{2\pi i} \oint_{\circlearrowleft} \frac{dx_a}{x_a^{n+1}} = (-)^n \frac{\partial^n}{\partial x_a^n}\bigg|_{x_a=0} .$$

This is nothing but Cauchy's integral formula.

For massive kite, consider $I(1,1,1,-1,0)$:

$$I^d(1,1,1,-1,0) = \frac{\Gamma(\frac{d}{2})}{(4\pi)^d} \frac{1}{\Gamma\left(\frac{d}{2}(2+1)-(3-1)\right)} \int dx_1 dx_2 dx_3 \int \frac{dx_4 x_4^{-1-1}}{\Gamma(-1)} \, \mathcal{G}^{-\frac{d}{2}}\bigg|_{x_5=0}$$

$$= \frac{\Gamma(\frac{d}{2})}{(4\pi)^d \, \Gamma\left(\frac{3d}{2}-2\right)} \int dx_1 dx_2 dx_3 \int \frac{dx_4 x_4^{-1-1}}{\Gamma(-1)} \, \mathcal{G}^{-\frac{d}{2}}\bigg|_{x_5=0}$$

Replacing x_4-integration by the partial derivative with respect to it, we have:

$$I^d(1,1,1,-1,0) = \frac{\Gamma(\frac{d}{2})}{(4\pi)^d\,\Gamma\left(\frac{3d}{2}-2\right)} \int dx_1 dx_2 dx_3 \frac{d}{2} \left.\frac{\partial \mathcal{G}}{\partial x_4}\right|_{\substack{x_5=0\\x_4=0}} \left.\mathcal{G}^{-\frac{d}{2}-1}\right|_{\substack{x_5=0\\x_4=0}}$$

$$= \frac{\Gamma(\frac{d+2}{2})}{(4\pi)^d\,\Gamma\left(\frac{3(d+2)}{2}-5\right)} \int dx_1 dx_2 dx_3 \left.\frac{\partial \mathcal{G}}{\partial x_4}\right|_{\substack{x_5=0\\x_4=0}} \left.\mathcal{G}^{-\frac{d+2}{2}}\right|_{\substack{x_5=0\\x_4=0}}$$

Note that, since here we consider massive kite, a tadpole becomes a scale-full integral and indeed has a non-zero contribution.

Since $\left.\frac{\partial \mathcal{G}}{\partial x_4}\right|_{\substack{x_5=0\\x_4=0}}$ becomes

$$\begin{aligned}\left.\frac{\partial \mathcal{G}}{\partial x_4}\right|_{\substack{x_5=0\\x_4=0}} &= m^2x_3^2 + 3m^2x_2x_3 + p^2x_1x_3 + 2m^2x_1x_3 + x_3 + m^2x_2^2 + p^2x_1x_2 + 2m^2x_1x_2 + x_2 \\ &= m^2\left(x_3^2 + x_2^2\right) + 3m^2x_2x_3 + \left(p^2 + 2m^2\right)(x_1x_3 + x_1x_2) + x_1 + x_2,\end{aligned}$$

the right hand side of the above becomes the following integrals in $(d+2)$-dimension:

$$\begin{aligned}(4\pi)^2 \Big[& m^2\Gamma(3)\left(I^{d+2}(1,1,3,0,0) + I^{d+2}(1,3,1,0,0)\right) + 3m^2 I^{d+2}(1,2,2,0,0) \\ & + \left(p^2 + 2m^2\right)\left(I^{d+2}(2,1,2,0,0) + I^{d+2}(2,2,1,0,0)\right) \\ & + \frac{3d-5}{2}\left(I^{d+2}(2,1,1,0,0) + I^{d+2}(1,2,1,0,0)\right)\Big]\end{aligned}$$

When we plug the above expression using `LoweringDRR` function in LiteRed (R. N. Lee 2012), we obtain the consistent reduction with Kira; LiteRed can also gives us the same reduction directory from $I^d(1,1,1,-1,0)$:

$$I(1,1,1,-1,0) = \frac{p^2}{3} I(1,1,1,0,0) + I(1,1,0,0,0)$$