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The Infrared Structure of QCD Scattering Amplitudes

Thomas Becher
University of Bern

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Outline

- structure of infrared singularities of massless four-loop amplitudes   TB, Neubert, 1908.11379

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- Infrared singularities and low-energy effective field theory
    - Factorization constraints
    - Renormalization and resummation of large logarithms in cross sections
- structure of infrared singularities of massless four-loop amplitudes TB, Neubert, 1908.11379

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- Infrared singularities and low-energy effective field theory
  - Factorization constraints
  - Renormalization and resummation of large logarithms in cross sections
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- Application: resummation at N^3LL
    - Event shapes, transverse momentum spectra, …

Infrared singularities

Scattering amplitudes in theories with massless particles, such as QED or QCD suffer from infrared divergences.

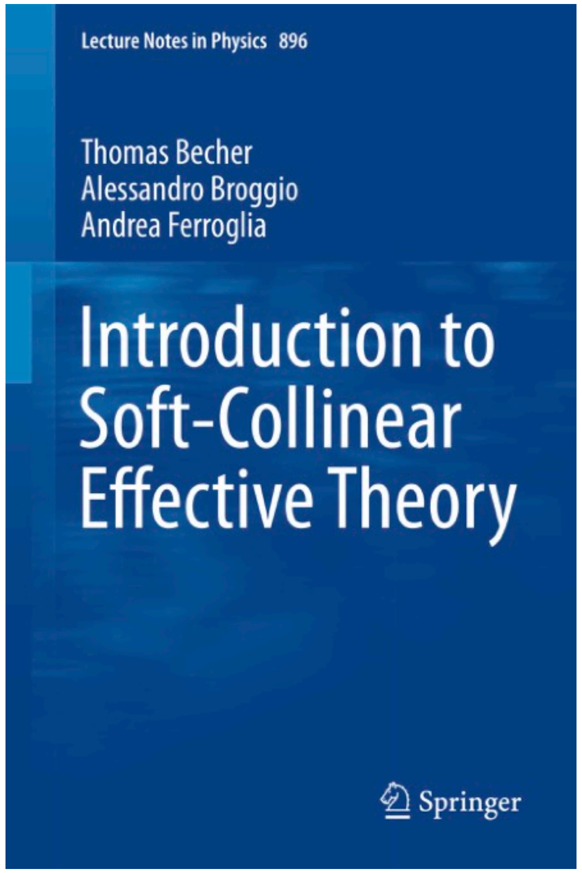
Bloch, Nordsieck 1937
Kinoshita 1962; Lee, Nauenberg 1964

- Exclusive cross sections are unphysical, need to allow for soft and collinear radiation!

A nuisance for cross section calculations.

- Regularize scattering amplitudes and phase-space integrals.
- Isolate and cancel divergences before obtaining numerical predictions.

Textbook material?

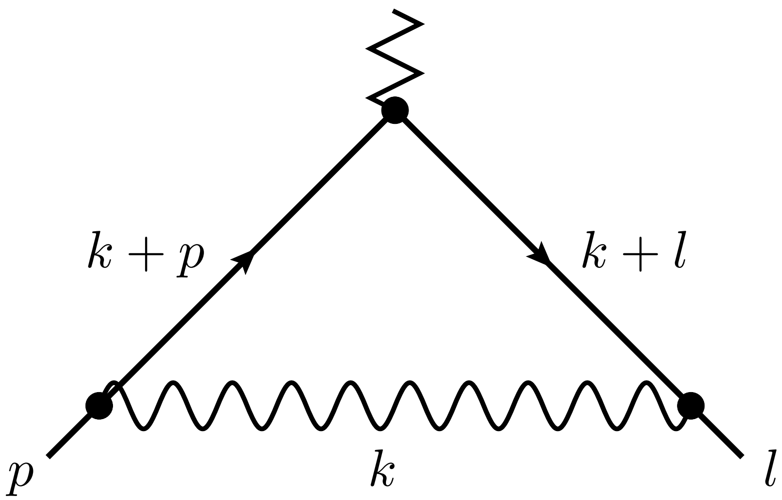


Series of papers on IR structure of amplitudes about years ago TB, Neubert ’09, Gardi, Magnea ’09, … (see SCET book from ‘15 for a review) but several open questions remained:

- Dipole conjecture, Casimir scaling of γ_{cusp}, non-abelian exponentiation for n legs

These have been answered in the meantime!

Example: form factor integral



$$p^2 = l^2 = m^2 = 0$$

$$Q^2 = (p-l)^2$$

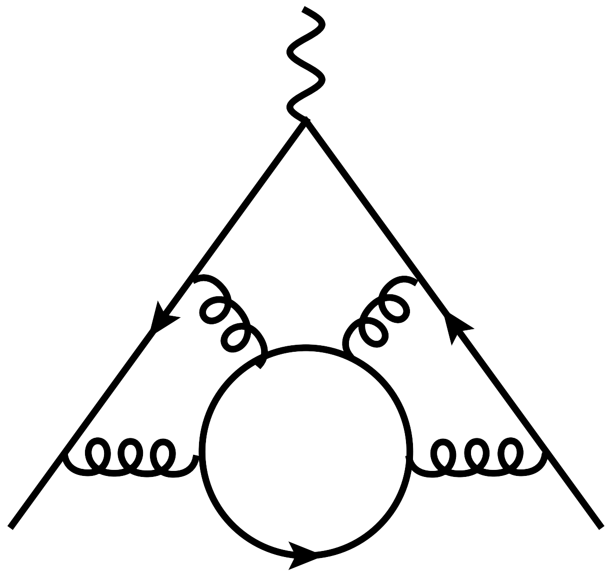
$$\boldsymbol{T}^a \boldsymbol{T}^a = C_F$$

$$F(Q^2) = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left(-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \frac{\pi^2}{6} - 8 + \mathcal{O}(\varepsilon) \right) \left(\frac{Q^2}{\mu^2} \right)^{-\varepsilon}$$

Use dimensional regularization d=4-2ε

- Two divergent integrations: energy and angle. Soft and collinear divergences.
- Massive case: only single, soft divergence.

Form factor at 4 loops



Interesting color structures!

$$d_R^{a_1 \dots a_n} = \mathrm{Tr}_R \big(\boldsymbol{T}^{a_1} \dots \boldsymbol{T}^{a_n} \big)_+$$ ← symm.

→ higher Casimir invariants.

Two powers of 1/ε per loop. At four loops

$$\Delta F(Q^2) = \left(\frac{\alpha_s(\mu)}{4\pi} \right)^4 \left[\frac{c_8}{\epsilon^8} + \frac{c_7}{\epsilon^7} + \dots \frac{c_2}{\epsilon^2} + \frac{c_1}{\epsilon} + c_0 \right] \left(\frac{Q^2}{\mu^2} \right)^{4\epsilon}$$

The analytical calculation of the coefficient c_2 of the $1/\varepsilon^2$ pole ("cusp anomalous dimension") was finished very recently: Henn, Korchemsky and Mistlberger 1911.10174. Numerical result Moch, Ruijl, Ueda, Vermaseren and Vogt '18 and many color structures were known earlier.

Color-space formalism

- Represent amplitudes as vectors in color space:

$$|c_1, c_2, \ldots, c_n\rangle$$

Catani, Seymour 1996

color index of first parton

- Color generator for i^{th} parton $\boldsymbol{T}_i^a \, |c_1, c_2, \ldots, c_n\rangle$ acts like a matrix:
  - t^a for quarks, f^{abc} for gluons
  - product $\boldsymbol{T}_i \cdot \boldsymbol{T}_j = \sum_a \boldsymbol{T}_i^a \boldsymbol{T}_j^a$ (commutative)
  - charge conservation $\sum_i \boldsymbol{T}_i^a = 0$ implies:

$$\sum_{(i,j)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j = -\sum_i \boldsymbol{T}_i^2 = -\sum_i C_i$$

$i \neq j$

C_F or C_A

Catani's two-loop formula '98

- Specifies IR singularities of dimensionally regularized n-parton amplitudes at two loops:

$$\left[1 - \frac{\alpha_s}{2\pi}\, \boldsymbol{I}^{(1)}(\epsilon) - \left(\frac{\alpha_s}{2\pi}\right)^2 \boldsymbol{I}^{(2)}(\epsilon) + \ldots\right] |\mathcal{M}_n(\epsilon, \{\underline{p}\})\rangle = \text{finite}$$

amplitude is vector in color space

with

$$\boldsymbol{I}^{(1)}(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{g_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right) \sum_{j\neq i} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} \left(\frac{\mu^2}{-s_{ij}}\right)^\epsilon$$

$$\boldsymbol{I}^{(2)}(\epsilon) = \frac{e^{-\epsilon\gamma_E}\,\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(K + \frac{\beta_0}{2\epsilon}\right) \boldsymbol{I}^{(1)}(2\epsilon)$$

$$- \frac{1}{2}\, \boldsymbol{I}^{(1)}(\epsilon) \left(\boldsymbol{I}^{(1)}(\epsilon) + \frac{\beta_0}{\epsilon}\right) + \boldsymbol{H}^{(2)}_{\text{R.S.}}(\epsilon)$$

$(p_i + p_j)^2$

unspecified

- Later derivation using factorization properties and IR evolution equation for form factor

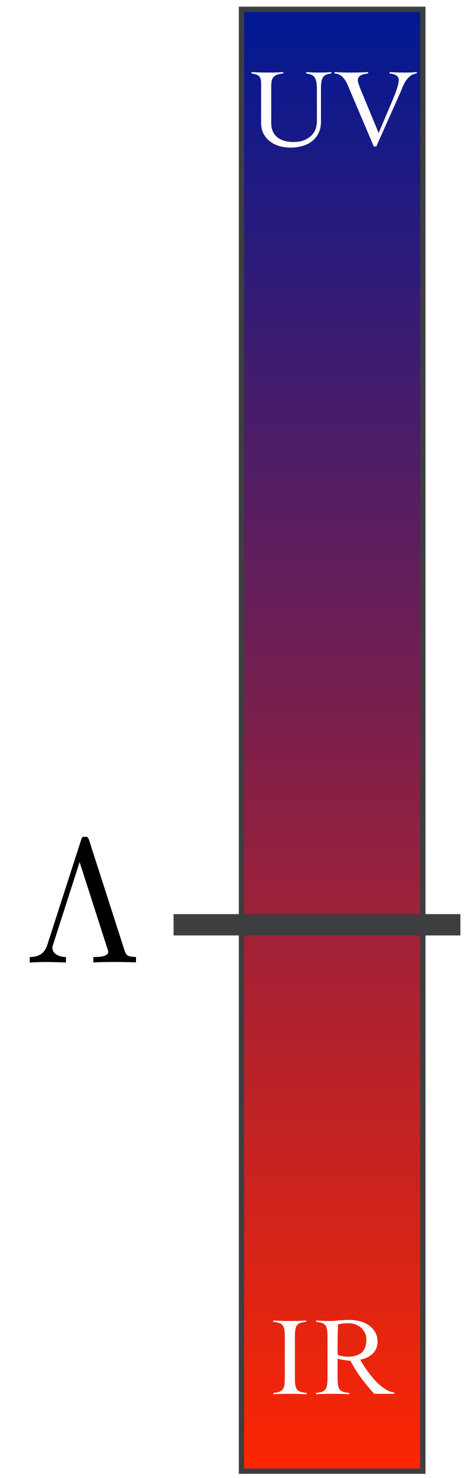
Sterman, Tejeda-Yeomans '03

Misconception

Conventional thinking is that UV and IR divergences are of totally different nature:

- UV divergences are absorbed into renormalization of parameters of theory; structure constrained by RG equations
- IR divergences arise in unphysical calculations; cancel between virtual corrections and real emissions

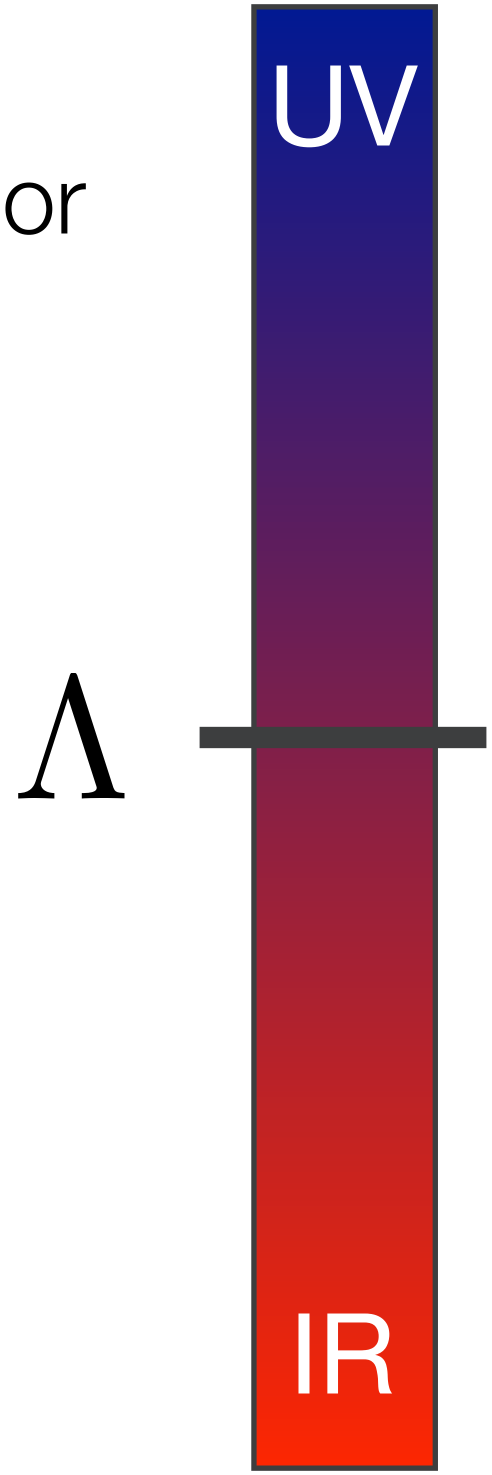
In fact, IR divergences can be mapped onto UV divergences of operators in effective field theory!



High-energy perspective: Λ is infrared regulator

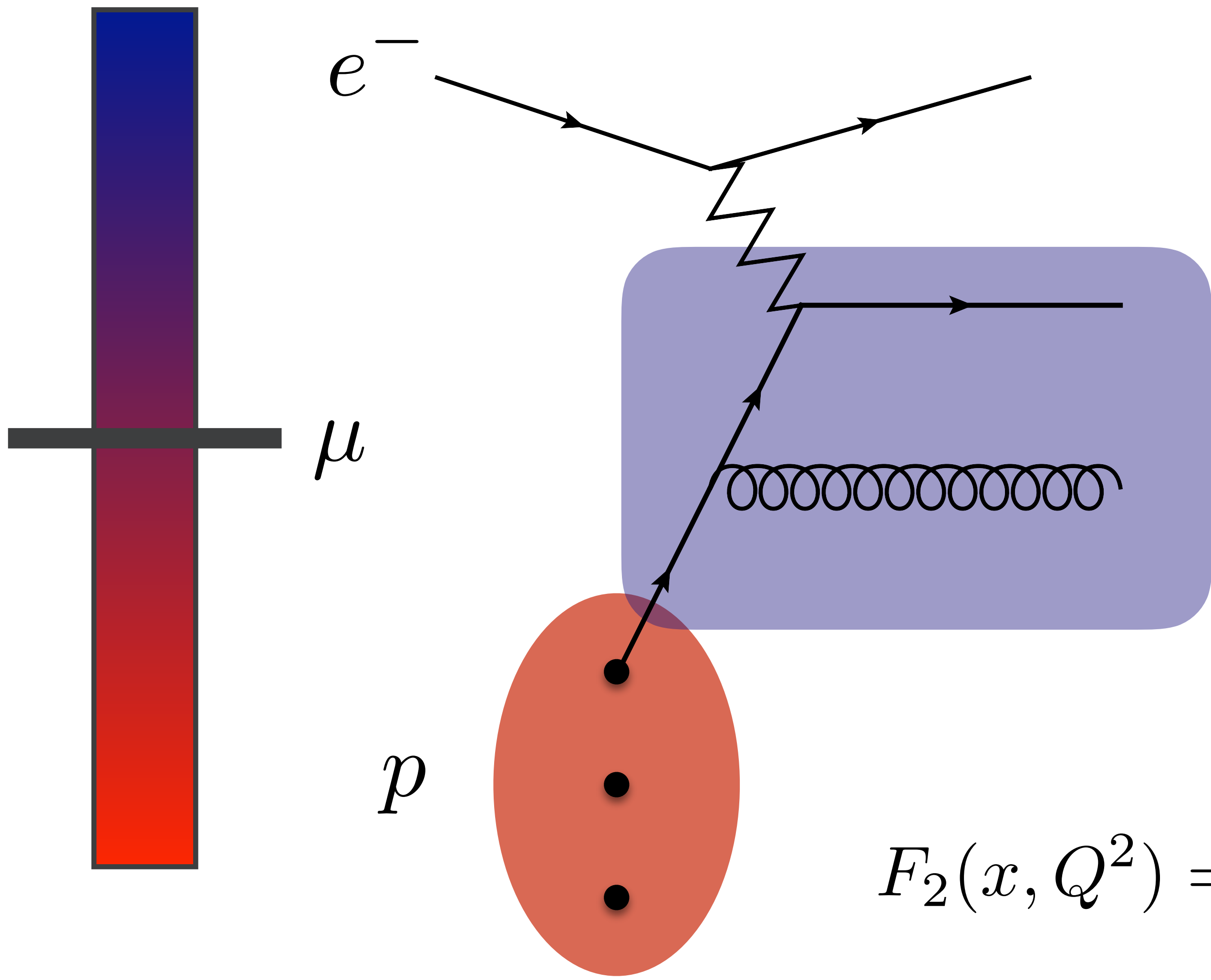
Low-energy perspective: Λ is ultraviolet regulator

- Effective Field Theory (EFT)
- Renormalization, RG evolution



Physics example: DIS

$$e^- + p \to e^- + X$$



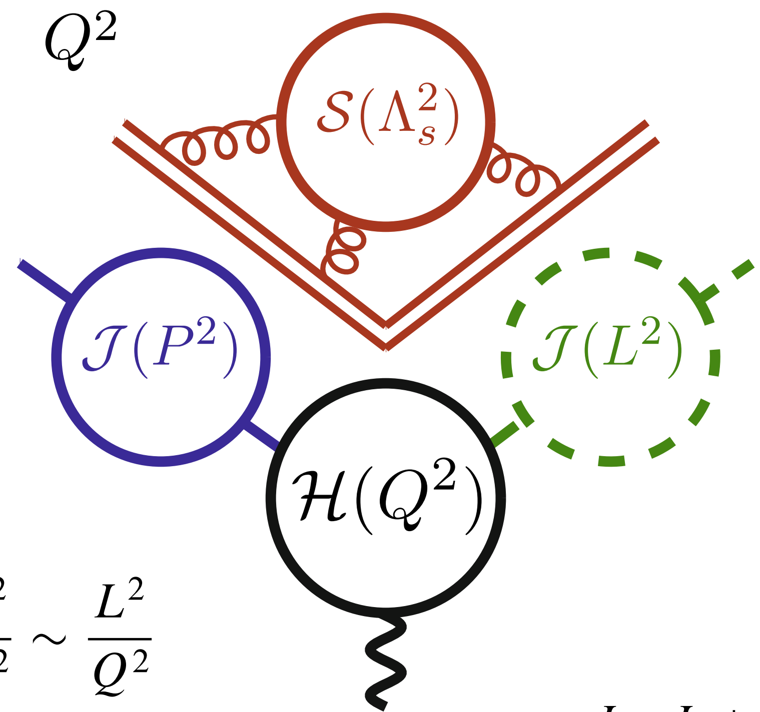
PDF
operator matrix element
needs renormalization

$$F_2(x, Q^2) = \sum_i \int_x^1 d\xi \, H_i(\frac{x}{\xi}, Q, \mu) \, f_i(\xi, \mu)$$

One-to-one correspondence between UV divergences in PDFs and IR-div's in H_i !

Unphysical example: off-shell form factor

$$\Lambda_s^2 = \frac{P^2 L^2}{Q^2}$$



$$\lambda^2 \sim \frac{P^2}{Q^2} \sim \frac{L^2}{Q^2}$$

*

$$I_h = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_c = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_{\bar{c}} = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2} - \frac{1}{2}\ln^2\frac{\mu^2}{L^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

$$I_s = \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2 Q^2}{L^2 P^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{L^2 P^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$

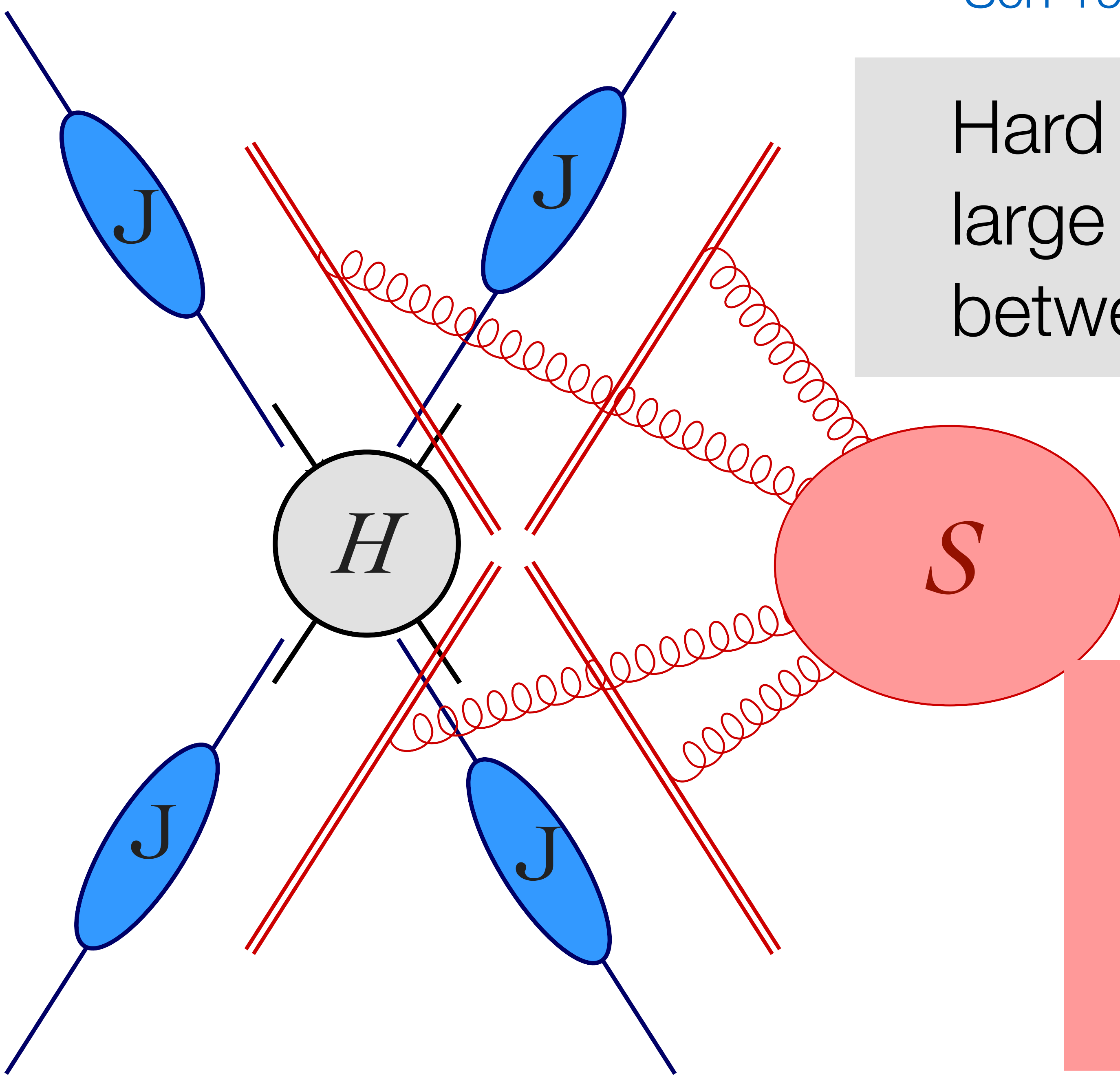
$$I \equiv I_h + I_c + I_{\bar{c}} + I_s = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right).$$

- Cancellations of divergences implies remarkable relations among H, J and S
- Factorization can be obtained in Soft-Collinear Effective Theory (SCET)
- Soft function is given by Wilson line matrix element

* from TB, Broggio Ferroglia '15; result is for scalar loop integral instead of form factor

Soft-collinear factorization: *n* jet case

Sen 1983; Kidonakis, Oderda, Sterman 1998



Hard function H depends on large momentum transfers s_{ij} between jets

Soft function S depends on scales $\Lambda_{ij}^2 = \frac{p_i^2 p_j^2}{s_{ij}}$

Jet functions $J_i = J_i\,(p_i{}^2)$

Factorization

Off-shell Green's function factorize as

on-shell amplitude

$$\boldsymbol{S}(\{\underline{\beta}\},\epsilon)\prod_i J(L_i^2,\epsilon)\,|\mathcal{M}(\{\underline{s}\},\epsilon)\rangle = \text{finite}$$

$$s_{ij} \equiv \pm 2 p_i \cdot p_j$$

$$L_i \equiv \ln \frac{\mu^2}{-p_i^2}$$

$$\beta_{ij} = \ln \frac{(-s_{ij})\,\mu^2}{(-p_i^2)(-p_j^2)}$$

Soft function $\boldsymbol{S}$ and on-shell amplitude $\mathcal{M}$ depend on colors of all particles!

Factorization

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$$\beta_{ij} = \ln\frac{(-s_{ij})\,\mu^2}{(-p_i^2)(-p_j^2)} = L_i + L_j - \ln\frac{\mu^2}{-s_{ij}}$$

Soft function $\boldsymbol{S}$ and on-shell amplitude $\mathcal{M}$ depend on colors of all particles!

Renormalization

Soft and jet functions are operators in SCET. Renormalize:

$$\boldsymbol{S}(\{\underline{\beta}\},\mu) \prod_i J(L_i^2,\mu)\,|\mathcal{M}(\{\underline{s}\},\mu\rangle = \text{finite}$$

Renormalized, finite amplitude

$$|\mathcal{M}_n(\{\underline{s}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{s}\},\mu)\,|\mathcal{M}_n(\epsilon,\{\underline{s}\})\rangle$$

TB, Neubert '09

This renormalized amplitude defines a finite *S*-matrix for massless theories. Corresponds to subtracting asymptotic soft+collinear int's.

Hannesdottir and Schwartz '19

Renormalization

Renormalization Group (RG) equation

$$\frac{d}{d\ln\mu}\,|\mathcal{M}_n(\{\underline{s}\},\mu)\rangle = \mathbf{\Gamma}(\{\underline{s}\},\mu)\,|\mathcal{M}_n(\{\underline{p}\},\mu)\rangle$$

Anomalous dimension $\mathbf{\Gamma}$ determines IR singularities.
Independence of μ imposes constraint

$$\mathbf{\Gamma}(\{\underline{s}\},\mu) = \mathbf{\Gamma}_s(\{\underline{\beta}\},\mu) + \sum_{i=1}^{n} \Gamma_c^i(L_i,\mu)\,\mathbf{1}\,,$$

Note:

TB, Neubert '09; Gardi, Magnea '09

- $\boldsymbol{\Gamma}_x$ contains logarithms of associated scales
- $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_s$ are matrices in color space

Dipole form

The following form is consistent with factorization

$$\boldsymbol{\Gamma}(\{\underline{s}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i^a \boldsymbol{T}_j^a}{2}\, \gamma_{\rm cusp}(\alpha_s)\, \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\, \mathbf{1}$$

Using color conservation

$$\sum_j \boldsymbol{T}_j^a = 0 \quad \to \quad \sum_{(ij)} \boldsymbol{T}_i^a \boldsymbol{T}_j^a = -\sum_i \boldsymbol{T}_i^a \boldsymbol{T}_i^a = -\sum_i C_i$$

one can rewrite the hard logarithms as soft+jet using

$$\beta_{ij} = \ln\frac{(-s_{ij})\,\mu^2}{(-p_i^2)(-p_j^2)} = L_i + L_j - \ln\frac{\mu^2}{-s_{ij}}$$

Up to 2 loops above dipole form is correct. IR singularities agree with Catani '98 and gives $H^{(2)}{}_{\rm RS}$.

Additional terms beyond 2 loops?

1.) Extra terms must be the same when expressed in ln(s_{ij}) or β_{ij} to be compatible with factorization.

$\longrightarrow$ functions of conformal cross ratios

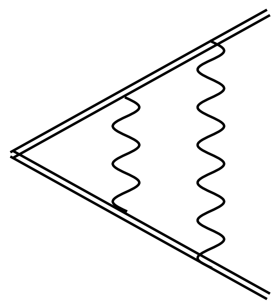
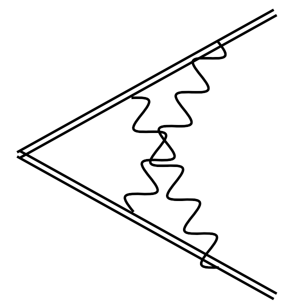
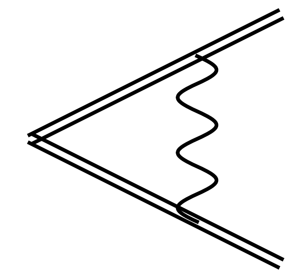
$$\beta_{ijkl} = \beta_{ij} + \beta_{kl} - \beta_{ik} - \beta_{jl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

Gardi, Magnea '09

independent of collinear scales.

2.) Non-abelian exponentiation: only connected color structures.

Non-abelian exponentiation

 +  $= \frac{1}{2}\Big($  $\Big)^2$

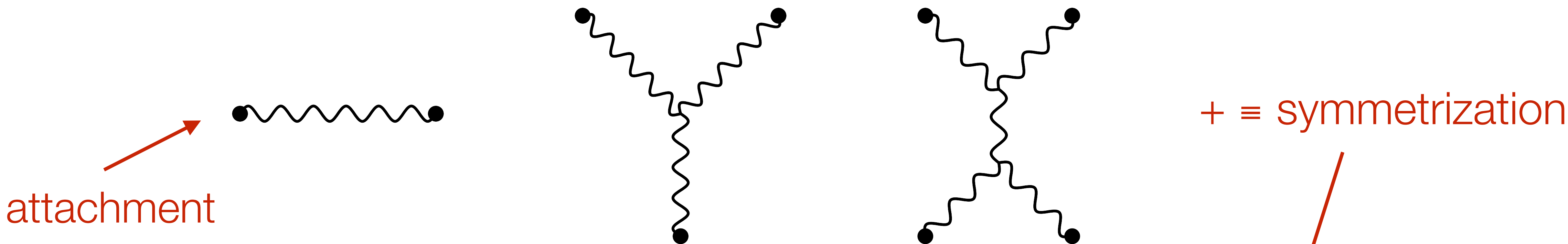
In massive QED, the soft function exponentiates

$$S = \exp\big(\widetilde{S}\big) = \exp\Big(\frac{\alpha}{4\pi} S^{(1)}\Big)$$

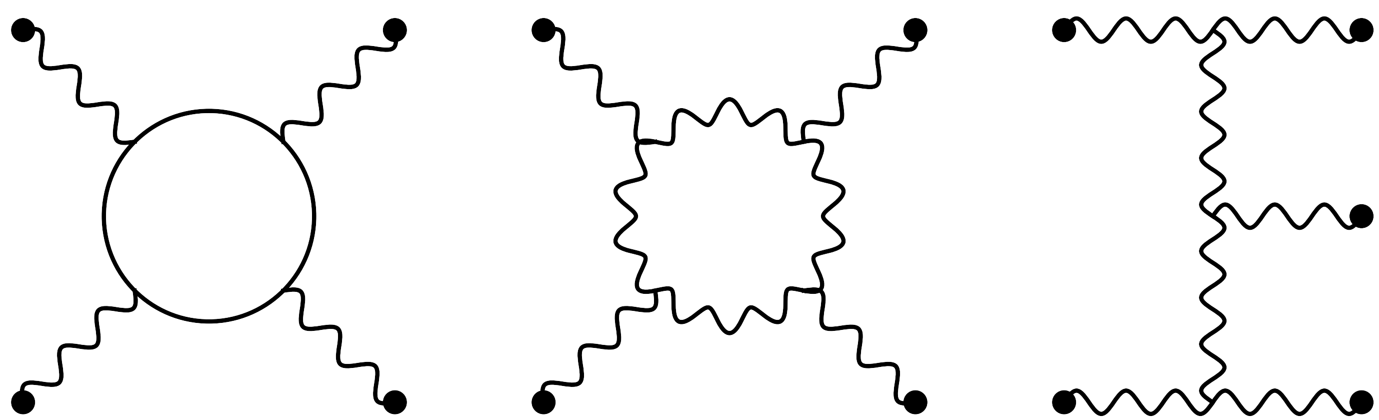
In QCD, simple exponentiation does not hold, but only connected webs contribute to the anomalous dimension. (2 legs: Gatheral '83, Frenkel and Taylor '84. *n* legs: Gardi, Smillie and White '11, '13)

Connected webs up to 4 loops

Show that we only need color connected webs that are *symmetrized* in their attachments to legs i,j,k…



$$\mathcal{D}_{ij} = \boldsymbol{T}_i^a \boldsymbol{T}_j^a \qquad \mathcal{T}_{ijk} = i f^{abc} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \right)_+ \qquad \mathcal{T}_{ijkl} = f^{ade} f^{bce} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \right)_+$$



$$\mathcal{D}_{ijkl}^R = d_R^{abcd}\, \boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \qquad \mathcal{T}_{ijklm} = i f^{adf} f^{bcg} f^{efg} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \boldsymbol{T}_m^e \right)_+$$

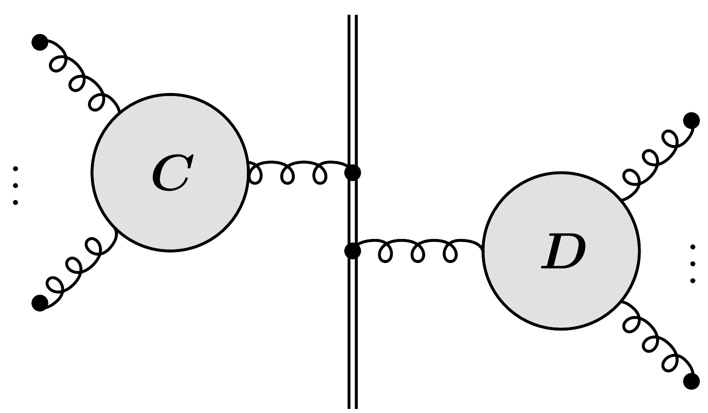
Replica trick

Stat. phys. see Mezard, Parisi, Vorasoro ’87; Wilson lines: Laenen et al. ’08 Gardi et al. ‘10

Compute color structure of the soft exponent

$$\widetilde{S} = \ln S = \lim_{N \to 0} \frac{S^N - 1}{N}$$

by working with *N* copies of QCD and extracting the terms which scale as the first power of *N*. after replica ordering.



$$I = J: \qquad N F\, \boldsymbol{C}^a \boldsymbol{D}^b\, \boldsymbol{T}_i^a \boldsymbol{T}_i^b\,,$$

$$I < J: \qquad \frac{N(N-1)}{2} F\, \boldsymbol{C}^a \boldsymbol{D}^b\, \boldsymbol{T}_i^a \boldsymbol{T}_i^b\,,$$

$$I > J: \qquad \frac{N(N-1)}{2} F\, \boldsymbol{C}^a \boldsymbol{D}^b\, \boldsymbol{T}_i^b \boldsymbol{T}_i^a\,.$$

Replika ordering along Wilson line!

Replica trick

Proof that only connected structures arise: Gardi, Smillie and White ’13, see also Vladimirov ’14, ’15

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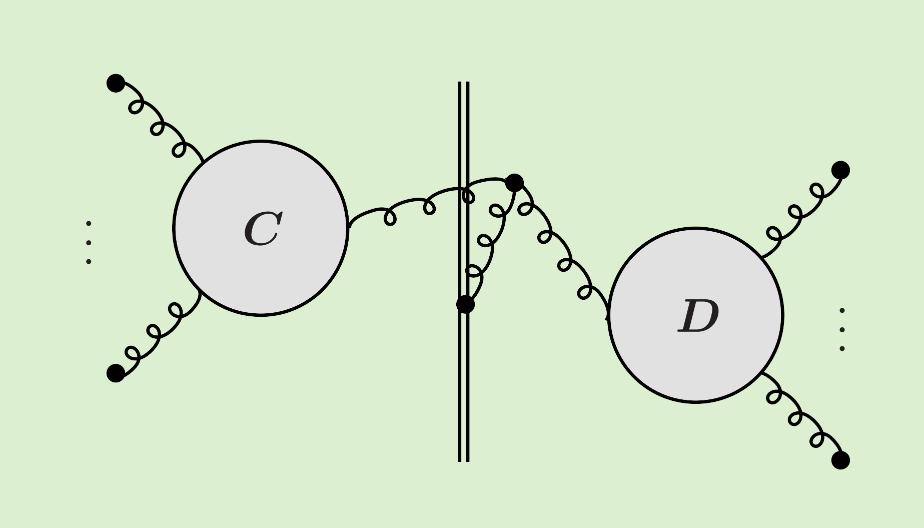


coefficient of N^1

$$I = J: \qquad F\, \boldsymbol{C}^a \boldsymbol{D}^b\, \boldsymbol{T}_i^a \boldsymbol{T}_i^b\,,$$

$$I < J: \qquad -\frac{1}{2}\, F\, \boldsymbol{C}^a \boldsymbol{D}^b\, \boldsymbol{T}_i^a \boldsymbol{T}_i^b\,,$$

$$I > J: \qquad -\frac{1}{2}\, F\, \boldsymbol{C}^a \boldsymbol{D}^b\, \boldsymbol{T}_i^b \boldsymbol{T}_i^a\,.$$

$$\widetilde{D} = \frac{1}{2}\, F\, \boldsymbol{C}^a \boldsymbol{D}^b\, [\boldsymbol{T}_i^a, \boldsymbol{T}_i^b] = \frac{i}{2}\, F\, f^{abc}\, \boldsymbol{C}^a \boldsymbol{D}^b\, \boldsymbol{T}_i^c$$

Symmetrization of lines

We want to symmetrize the attachments, e.g.

$$T_{ijkl} = f^{ade} f^{bce} \boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \quad \longrightarrow \quad \mathcal{T}_{ijkl} = f^{ade} f^{bce} \left(\boldsymbol{T}_i^a \boldsymbol{T}_j^b \boldsymbol{T}_k^c \boldsymbol{T}_l^d \right)_+$$

Can eliminate antisymmetric parts using group identity $[\boldsymbol{T}_i^a, \boldsymbol{T}_i^b] = i\, f^{abc}\, \boldsymbol{T}_i^c$. Leads to identities

$$T_{iijk} = -T_{ijik} = -T_{jiki} = T_{jkii} = \mathcal{T}_{iijk} - \frac{C_A}{4} \mathcal{T}_{ijk} \,, \qquad T_{ijki} = T_{jiik} = \frac{C_A}{2} \mathcal{T}_{ijk}$$

$$T_{iijj} = -T_{ijij} = \mathcal{T}_{iijj} + \frac{C_A^2}{8} \mathcal{D}_{ij} \,, \qquad T_{ijji} = -\frac{C_A^2}{4} \mathcal{D}_{ij} \,, \qquad T_{iiii} = \frac{C_A^2}{4} C_{R_i} \mathbf{1}$$

$$T_{iiij} = T_{jiii} = \frac{C_A^2}{4} \mathcal{D}_{ij} \,, \qquad T_{iiji} = -T_{ijii} = 0$$

Construction of $\boldsymbol{\Gamma}$

- Write down all possible terms with connected webs, attaching to different numbers of legs.
- Coefficient functions are functions of cusp logs or conformal cross ratios
    - Two independent cross ratios for 4 legs
    - Five independent conformal cross ratios
    - Cusp terms must obey soft-collinear factorization constraint!
- Later: additional constraints on coefficient functions from collinear limit.

4-loop anomalous dimension

$$\begin{aligned}\mathbf{\Gamma}(\{\underline{s}\},\mu) = &\sum_{(i,j)} \frac{\boldsymbol{T}_i^a \boldsymbol{T}_j^a}{2}\, \gamma_{\text{cusp}}(\alpha_s)\, \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\, \mathbf{1} \\
&+ f(\alpha_s) \sum_{(i,j,k)} \mathcal{T}_{iijk} + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl}\, F(\beta_{ijlk}, \beta_{iklj}; \alpha_s) \\
&+ \sum_R g^R(\alpha_s) \Big[\sum_{(i,j)} \big(\mathcal{D}^R_{iijj} + 2\mathcal{D}^R_{iiij}\big) \ln\frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)} \mathcal{D}^R_{ijkk} \ln\frac{\mu^2}{-s_{ij}} \Big] \\
&+ \sum_R \sum_{(i,j,k,l)} \mathcal{D}^R_{ijkl}\, G^R(\beta_{ijlk}, \beta_{iklj}; \alpha_s) + \sum_{(i,j,k,l)} \mathcal{T}_{ijkli}\, H_1(\beta_{ijlk}, \beta_{iklj}; \alpha_s) \\
&+ \sum_{(i,j,k,l,m)} \mathcal{T}_{ijklm}\, H_2(\beta_{ijkl}, \beta_{ijmk}, \beta_{ikmj}, \beta_{jiml}, \beta_{jlmi}; \alpha_s) + \mathcal{O}(\alpha_s^5)\,.\end{aligned}$$

Simplified compared to TB Neubert '09, Ahrens, Neubert and Vernazza '12. Earlier papers concluded that higher Casimir cusp terms were excluded by factorization in collinear limit — true individually, but certain linear combinations are allowed!

Ingredients

$$\begin{aligned}\boldsymbol{\Gamma}(\{\underline{s}\},\mu) = & \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\text{cusp}}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\,\mathbf{1} \\ & + f(\alpha_s)\sum_{(i,j,k)}\mathcal{T}_{iijk} + \sum_{(i,j,k,l)}\mathcal{T}_{ijkl}\,F(\beta_{ijlk},\beta_{iklj};\alpha_s) \\ & + \sum_R g^R(\alpha_s)\left[\sum_{(i,j)}\left(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\right)\ln\frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\,\ln\frac{\mu^2}{-s_{ij}}\right] \\ & + \sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\,G^R(\beta_{ijlk},\beta_{iklj};\alpha_s) + \sum_{(i,j,k,l)}\mathcal{T}_{ijkli}\,H_1(\beta_{ijlk},\beta_{iklj};\alpha_s) \\ & + \sum_{(i,j,k,l,m)}\mathcal{T}_{ijklm}\,H_2(\beta_{ijkl},\beta_{ijmk},\beta_{ikmj},\beta_{jiml},\beta_{jlmi};\alpha_s) + \mathcal{O}(\alpha_s^5)\,.\end{aligned}$$

known to 4 loops

Henn, Smirnov, Smirnov, Steinhauser '16, Henn, Smirnov, Smirnov, Steinhauser, Lee '16, Davies, Vogt, B. Ruijl, T. Ueda and Vermaseren '16; Moch, Ruijl, Ueda, Vermaseren and Vogt '17 '18; Grozin '18; Lee, Smirnov, Smirnov and Steinhauser '17 '19; Henn, Peraro, Stahlhofen and Wasser '19, von Manteuffel and Schabinger '19; Brüser, Grozin, Henn and Stahlhofen '19, Henn, Korchemsky and Mistlberger 19

known to 3 loops

f, F : Almelid, Duhr and Gardi '16

unknown, 4 loops

Vladimirov '17 claims only even structures should arise: H_1 and H_2 zero?

Three-loop coefficients

Computed by Almelid, Duhr, Gardi ’16. Can also be *bootstrapped* Almelid, Duhr, Gardi, McLeod, White ‘17

$$F(x_1, x_2; \alpha_s) = 2\, \mathcal{F}(e^{x_1}, e^{x_2}) \left(\frac{\alpha_s}{4\pi}\right)^3 + \mathcal{O}(\alpha_s^4)\,,$$

$$f(\alpha_s) = 16\, (\zeta_5 + 2\zeta_2\zeta_3) \left(\frac{\alpha_s}{4\pi}\right)^3 + \mathcal{O}(\alpha_s^4)$$

$$\mathcal{F}(x, y) = \mathcal{L}(1-z) - \mathcal{L}(z) \qquad z\bar{z} = x \quad (1-z)(1-\bar{z}) = y$$

$$\mathcal{L}(z) = \mathcal{L}_{10101}(z) + 2\zeta_2 \left[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)\right]$$

Brown’s single-valued harmonic polylogarithms

Four-loop cusp terms

- Dipole four-loop coefficient

$$\gamma_3^{\rm cusp} = C_A^3\left(-16\zeta_3^2 - \frac{176\pi^2\zeta_3}{9} + \frac{20944\zeta_3}{27} - \frac{3608\zeta_5}{9} - \frac{2504\pi^6}{2835} + \frac{902\pi^4}{45} - \frac{44200\pi^2}{243} + \frac{84278}{81}\right)$$
$$+ n_f T_F\left[C_A^2\left(\frac{448\pi^2\zeta_3}{9} - \frac{46208\zeta_3}{27} + \frac{4192\zeta_5}{9} - \frac{176\pi^4}{135} + \frac{20320\pi^2}{243} - \frac{48274}{81}\right)\right.$$
$$\left. + C_A C_F\left(-\frac{128}{3}\pi^2\zeta_3 + \frac{7424\zeta_3}{9} + 320\zeta_5 - \frac{176\pi^4}{45} + \frac{440\pi^2}{9} - \frac{68132}{81}\right) + \left(\frac{1184\zeta_3}{3} - 640\zeta_5 + \frac{1144}{9}\right)C_F^2\right]$$
$$+ n_f^2 T_F^2\left[C_A\left(\frac{8960\zeta_3}{27} - \frac{224\pi^4}{135} - \frac{1216\pi^2}{243} + \frac{3692}{81}\right) + C_F\left(-\frac{2560\zeta_3}{9} + \frac{64\pi^4}{45} + \frac{9568}{81}\right)\right] + \left(\frac{512\zeta_3}{27} - \frac{256}{81}\right)n_f^3 T_F^3$$

- Higher casimir terms

$$g^F(\alpha_s) = T_F n_f\left(\frac{128\pi^2}{3} - \frac{256\zeta_3}{3} - \frac{1280\zeta_5}{3}\right)\left(\frac{\alpha_s}{4\pi}\right)^4 + \mathcal{O}(\alpha_s^5)\,,$$

$$g^A(\alpha_s) = \left(-\frac{32\pi^2}{3} + \frac{64\zeta_3}{3} + \frac{1760\zeta_5}{3} - \frac{496\pi^6}{945} - 192\zeta_3^2\right)\left(\frac{\alpha_s}{4\pi}\right)^4 + \mathcal{O}(\alpha_s^5)$$

Henn, Smirnov, Smirnov, Steinhauser ’16, Henn, Smirnov, Smirnov, Steinhauser, Lee ’16, Davies, Vogt, B. Ruijl, T. Ueda and Vermaseren ’16; Moch, Ruijl, Ueda, Vermaseren and Vogt ’17 ’18; Grozin ’18; Lee, Smirnov, Smirnov and Steinhauser ’17 ’19; Henn, Peraro, Stahlhofen and Wasser ’19, von Manteuffel and Schabinger ’19; Brüser, Grozin, Henn and Stahlhofen ‘19, Henn, Korchemsky and Mistlberger ’19; Huber, Manteuffel, Panzer, Schabinger, Yang ‘19

Ingredients

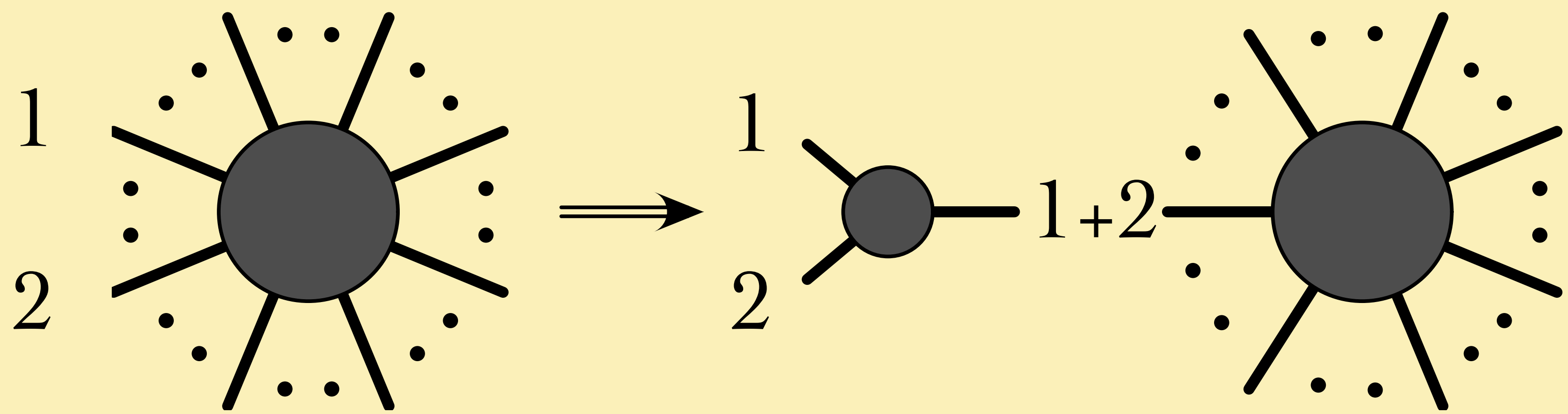
$$\boldsymbol{\Gamma}(\{\underline{s}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\rm cusp}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\,\boldsymbol{1}$$
$$+ f(\alpha_s)\sum_{(i,j,k)}\mathcal{T}_{iijk} + \sum_{(i,j,k,l)}\mathcal{T}_{ijkl}\,F(\beta_{ijlk},\beta_{iklj};\alpha_s)$$
$$+ \sum_R g^R(\alpha_s)\left[\sum_{(i,j)}\left(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\right)\ln\frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\,\ln\frac{\mu^2}{-s_{ij}}\right]$$
$$+ \sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\,G^R(\beta_{ijlk},\beta_{iklj};\alpha_s) + \sum_{(i,j,k,l)}\mathcal{T}_{ijkli}\,H_1(\beta_{ijlk},\beta_{iklj};\alpha_s)$$
$$+ \sum_{(i,j,k,l,m)}\mathcal{T}_{ijklm}\,H_2(\beta_{ijkl},\beta_{ijmk},\beta_{ikmj},\beta_{jiml},\beta_{jlmi};\alpha_s) + \mathcal{O}(\alpha_s^5)\,.$$

- The full three-loop result is known
    - IR singularities of all 3-loop amplitudes are known
- All logarithmic pieces are known to four loops
    - All IR singularities at 4-loops, except 1/ε are known
    - Resummation to N^3LL for *n*-jet processes

Consistency with collinear limits

- When two partons become collinear, an *n*-point amplitude M_n reduces to an (*n*-1)-parton amplitude times a splitting function: Berends, Giele ’89; Mangano, Parke ’91 Kosower ’99; Catani, de Florian, Rodrigo ‘03

$$|\mathcal{M}_n(\{p_1, p_2, p_3, \ldots, p_n\})\rangle = \mathbf{Sp}(\{p_1, p_2\})\, |\mathcal{M}_{n-1}(\{P, p_3, \ldots, p_n\})\rangle + \ldots$$



$$\mathbf{\Gamma}_{\text{Sp}}(\{p_1, p_2\}, \mu) = \mathbf{\Gamma}(\{p_1, \ldots, p_n\}, \mu) - \mathbf{\Gamma}(\{P, p_3 \ldots, p_n\}, \mu)\big|_{\boldsymbol{T}_P \to \boldsymbol{T}_1 + \boldsymbol{T}_2}$$

TB, Neubert ‘09

- $\mathbf{\Gamma}_{\text{Sp}}$ must be independent of momenta and colors of partons 3, ..., n

Consistency with collinear limits

- The fact that $\mathbf{\Gamma}_{\text{Sp}}$ must be independent of the colors and momenta of the remaining particles imposes strong constraint on $\mathbf{\Gamma}$.

- ’09, ‘12 papers concluded that the coefficients of the higher-multiplicity terms should vanish in the collinear limit.

- Deriving the 3-loop result Almelid, Duhr and Gardi ‘16 realized that this is not true: different terms can conspire in the limit to be compatible!

$$\lim_{\omega\to-\infty} F(\omega, 0; \alpha_s) = \frac{f(\alpha_s)}{2}$$

- Similarly, the higher Casimir coefficients must obey

$$\lim_{\omega\to-\infty} G^R(\omega, 0; \alpha_s) = -\frac{g^R(\alpha_s)}{6}\,\omega$$

Result for $\mathbf{\Gamma}_{\text{Sp}}$

Evaluating *

$$\mathbf{\Gamma}_{\text{Sp}}(\{p_1,p_2\},\mu) = \mathbf{\Gamma}(\{p_1,\dots,p_n\},\mu) - \mathbf{\Gamma}(\{P,p_3\dots,p_n\},\mu)\big|_{\boldsymbol{T}_P\to\boldsymbol{T}_1+\boldsymbol{T}_2}$$

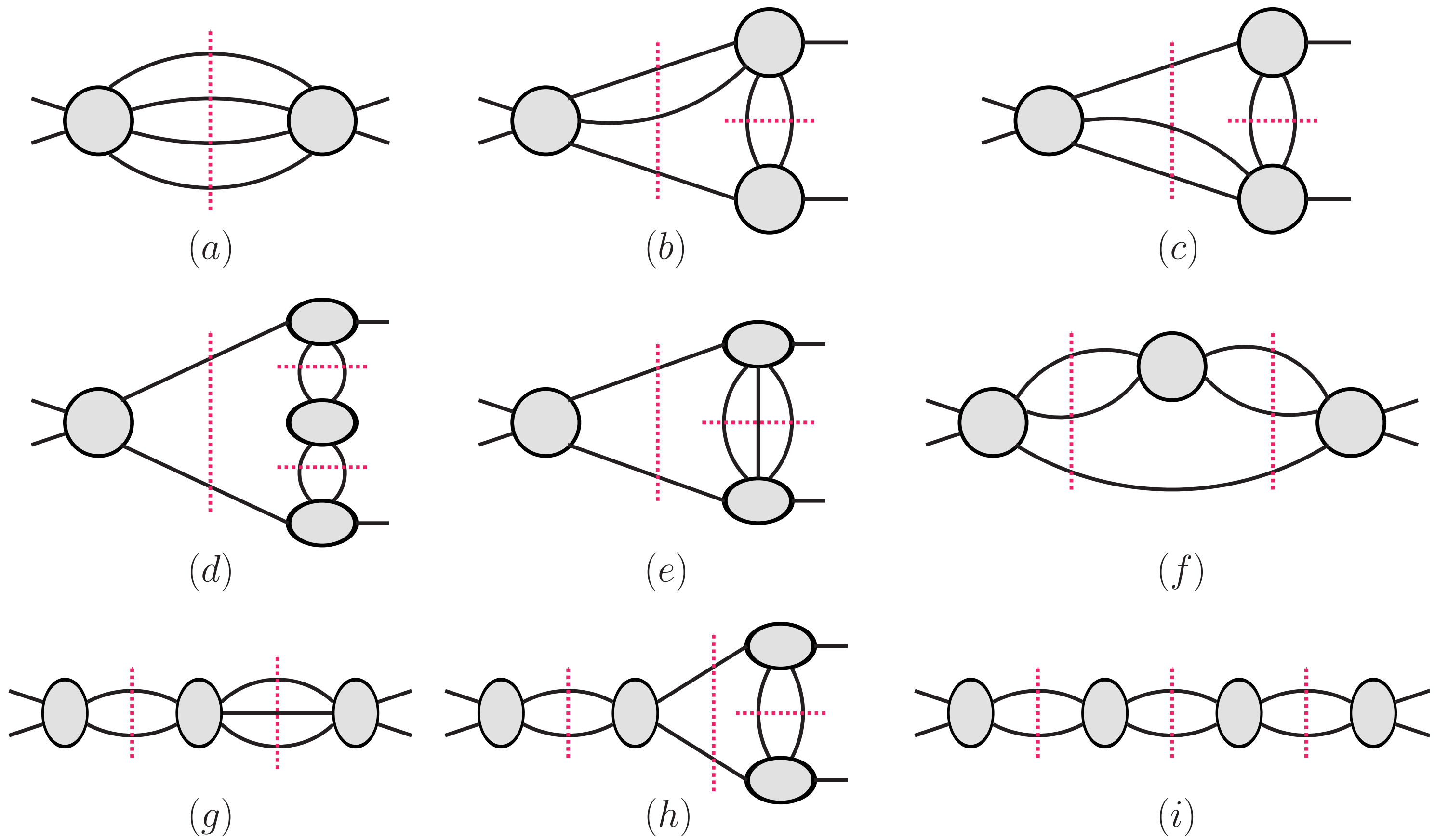
in the collinear limit, one obtains

$$\begin{aligned}
&\mathbf{\Gamma}_{\text{Sp}}(\{p_1,p_2\},\mu) \\
&= \Big\{\gamma_{\text{cusp}}(\alpha_s)\,\boldsymbol{T}_1\cdot\boldsymbol{T}_2 + \sum_R 2g^R(\alpha_s)\Big[3\mathcal{D}^R_{1122} + 2\big(\mathcal{D}^R_{1112}+\mathcal{D}^R_{1222}\big)\Big]\Big\}\Big[\ln\frac{\mu^2}{-s_{12}} + \ln z(1-z)\Big] \\
&\quad + \gamma_{\text{cusp}}(\alpha_s)\Big[C_{R_1}\ln z + C_{R_2}\ln(1-z)\Big] + \gamma^1(\alpha_s) + \gamma^2(\alpha_s) - \gamma^P(\alpha_s) \\
&\quad - 6f(\alpha_s)\Big(\mathcal{T}_{1122} + \frac{C_A^2}{8}\boldsymbol{T}_1\cdot\boldsymbol{T}_2\Big) + \sum_R 2g^R(\alpha_s)\Big[\mathcal{D}^R_{1111}\ln z + \mathcal{D}^R_{2222}\ln(1-z)\Big] + \mathcal{O}(\alpha_s^5)\,.
\end{aligned}$$

Log terms known to 4 loops! (f, γ^i only to 3 loops)

* a painful exercise in color algebra!!

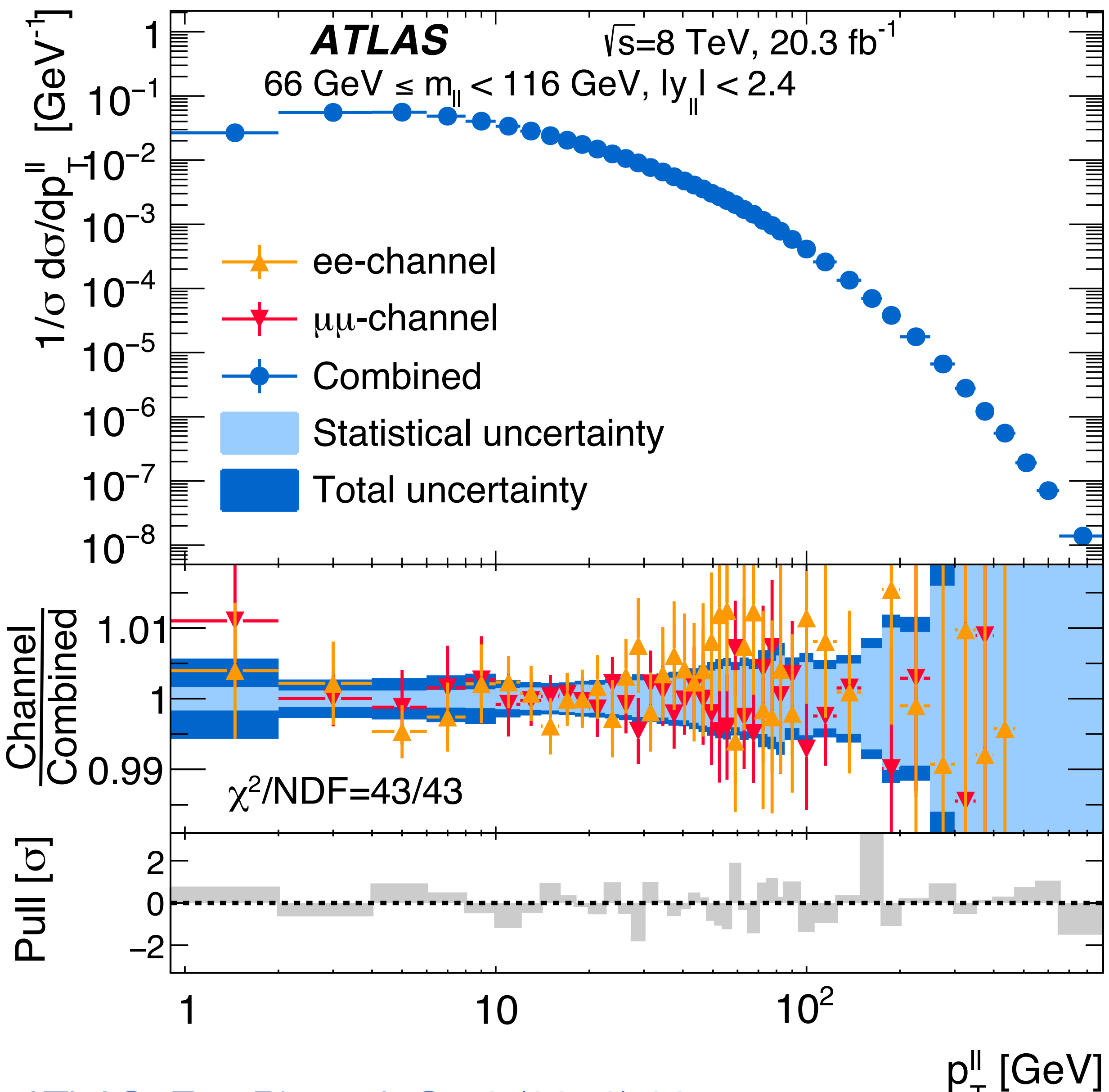
Does it work?

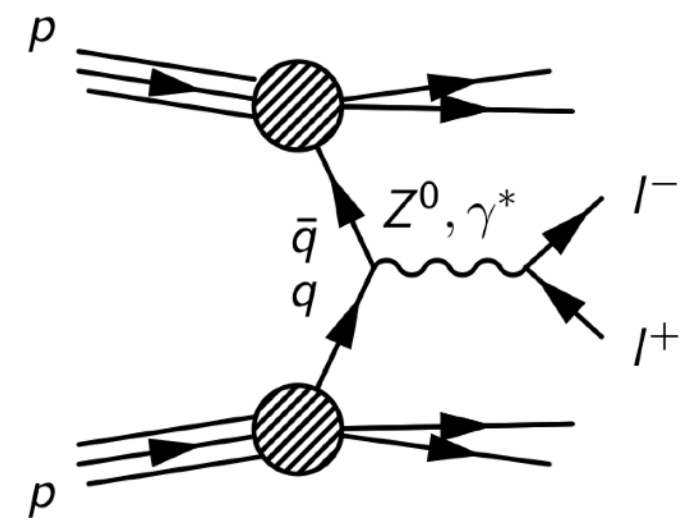


Yes! Recent computation of 3-loop four-gluon amplitude in pure YM theory verified that IR singularities agree with general result. Jin, Luo '19

Resummation at N^{3}LL

Precision measurements at the LHC





ATLAS, Eur. Phys. J. C 76 (2016) 291

← transverse momentum of the lepton pair

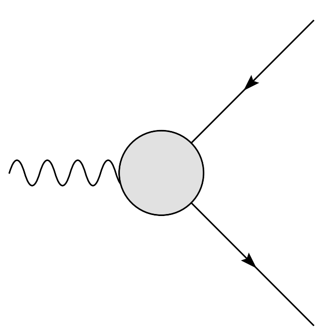
Sub-percent accuracy
over large range of energies and many orders of cross section!

A huge challenge for theory!

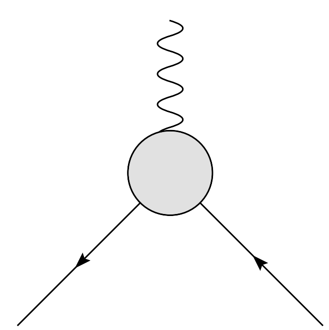
We have derived our factorization formula using off-shell Green's functions, but the factorization

$$d\sigma = \mathrm{tr}\left[\boldsymbol{H}_n \cdot \prod_{i=1}^{n} J \otimes \boldsymbol{S}_n\right]$$

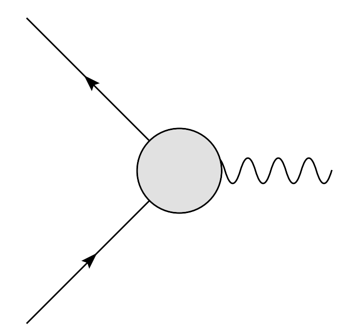
arises for many physical cross sections. $\boldsymbol{J}$ and $\boldsymbol{S}$ are observable dependent, but H is square of on-shell amplitudes.



$e^+e^- \to 2$ jets

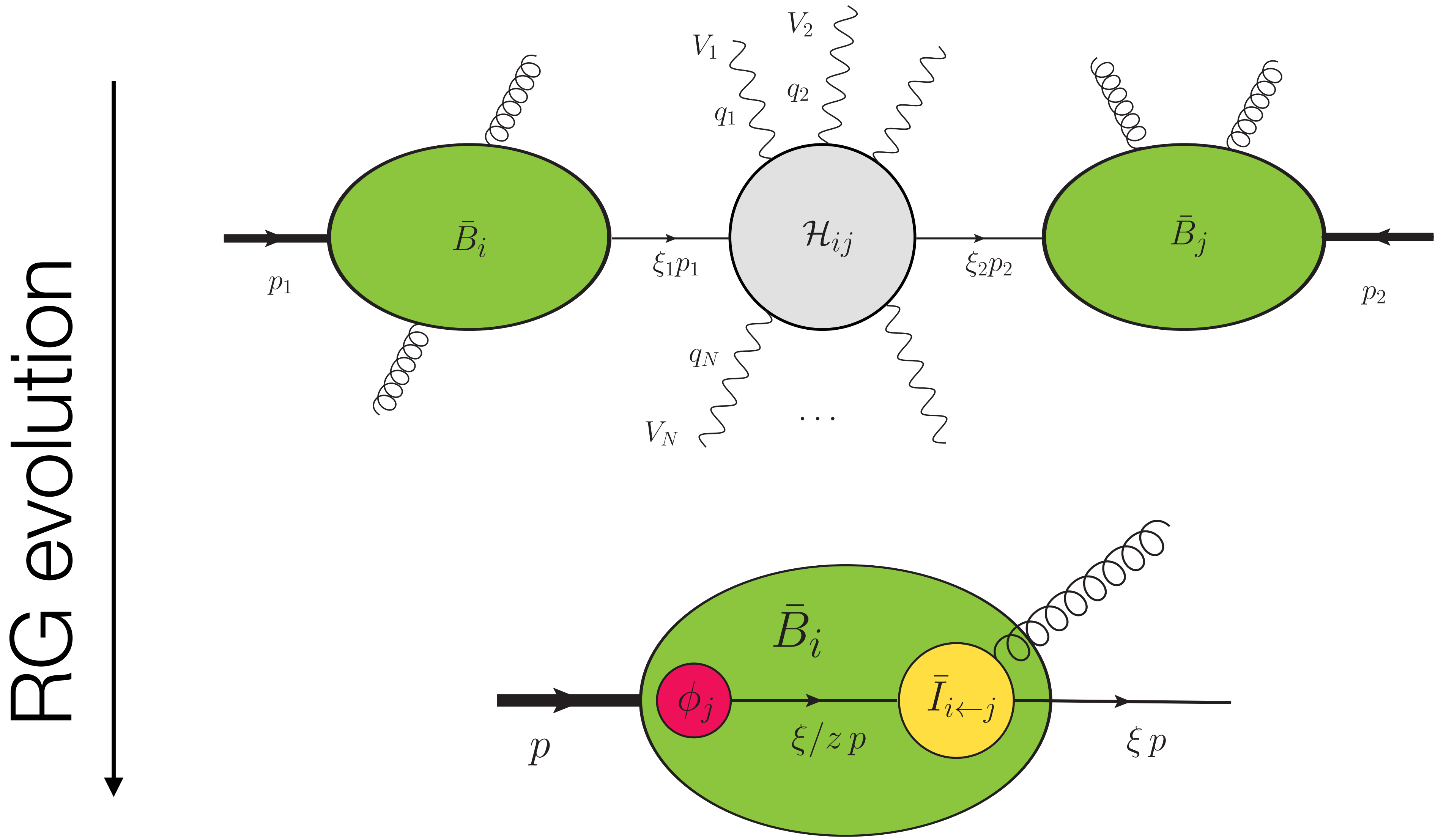


DIS



Z,W,H production

EW boson production at small q_T



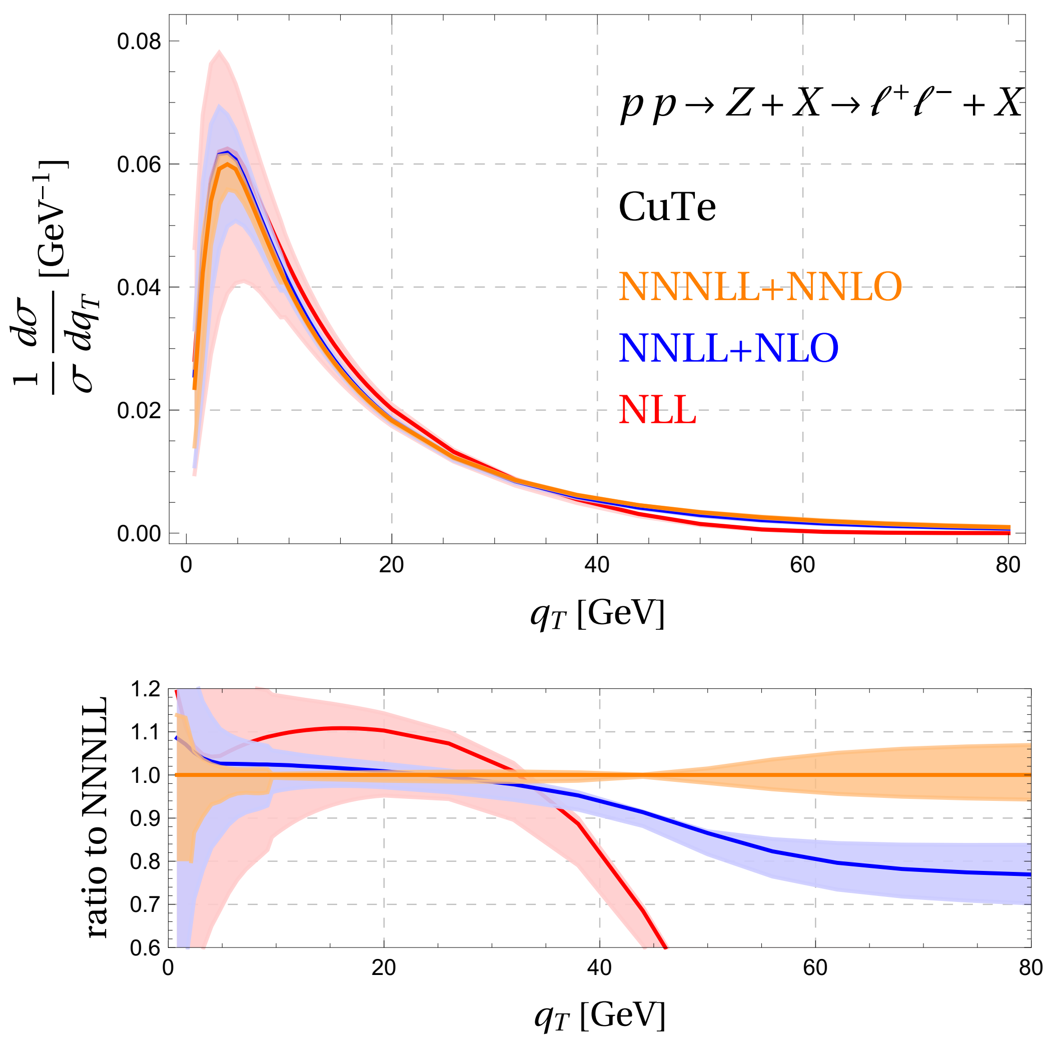
RG can be used to resum large logarithms $\alpha_s{}^n \ln^m(q_T/Q)$.

Ingredients for resummation

Log. approx.	γ_{cusp}	γ^i	H, J, S
LL	1-loop	tree-level	tree-level
NLL	2-loop	1-loop	tree-level
NNLL	3-loop	2-loop	1-loop
NNNLL	4-loop	3-loop	2-loop

- NNNLL has parametrically the same accuracy as NNLO fixed order!
- NNNLL resummations have been performed in the past, but were missing 4-loop γ_{cusp}.
  - now in place, also for *n*-jet processes

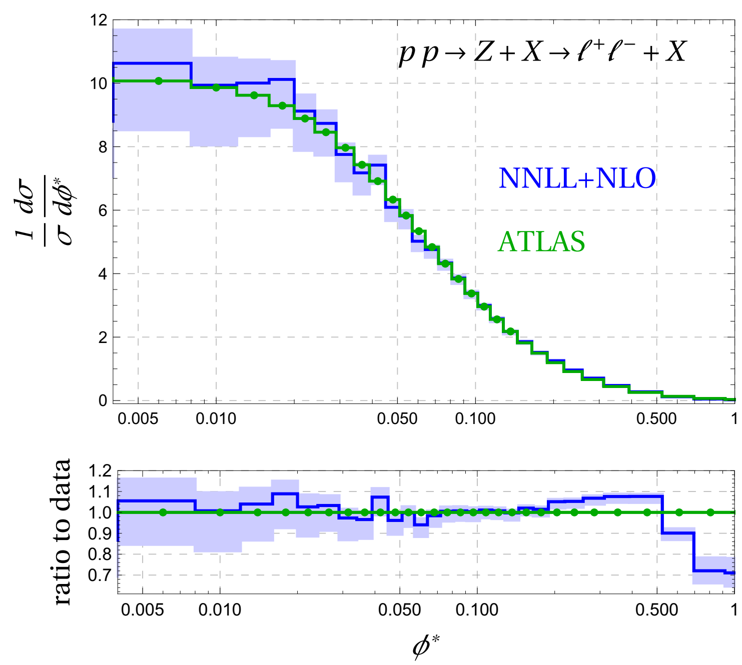
Transverse momentum spectrum

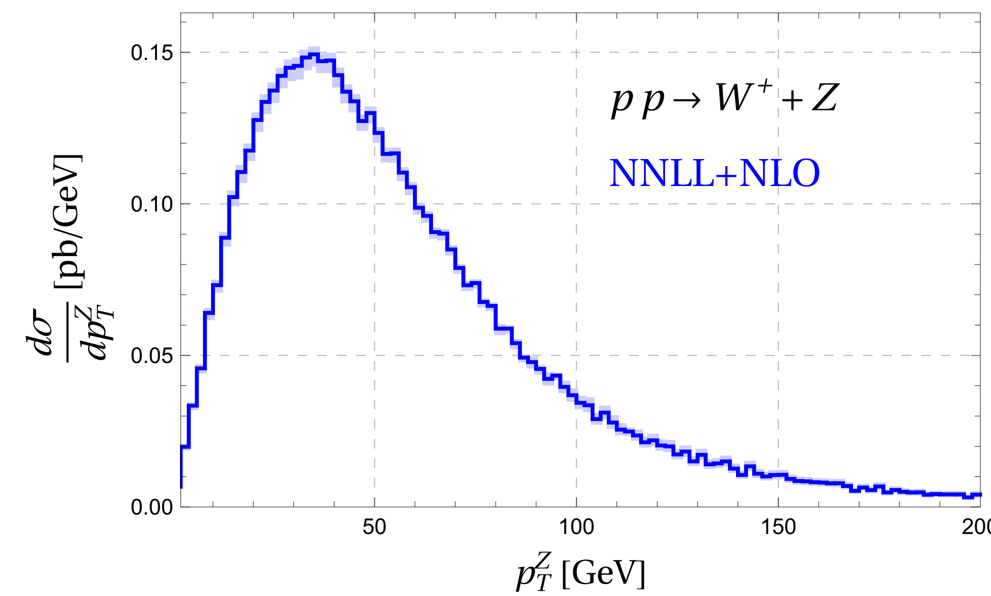


CuTe
TB, Neubert,Wilhelm '12,
+ Lübbert, '16

- At NNNLL, one reaches an accuracy of a few per cent
- 4-loop cusp has numerically only very small effect
- At higher q_T one matches to fixed-order result.
- Here: NNLO = $O(\alpha_s^2)$, but $O(\alpha_s^3)$ is known.
- CuTe only produces inclusive spectrum.

CuTeR

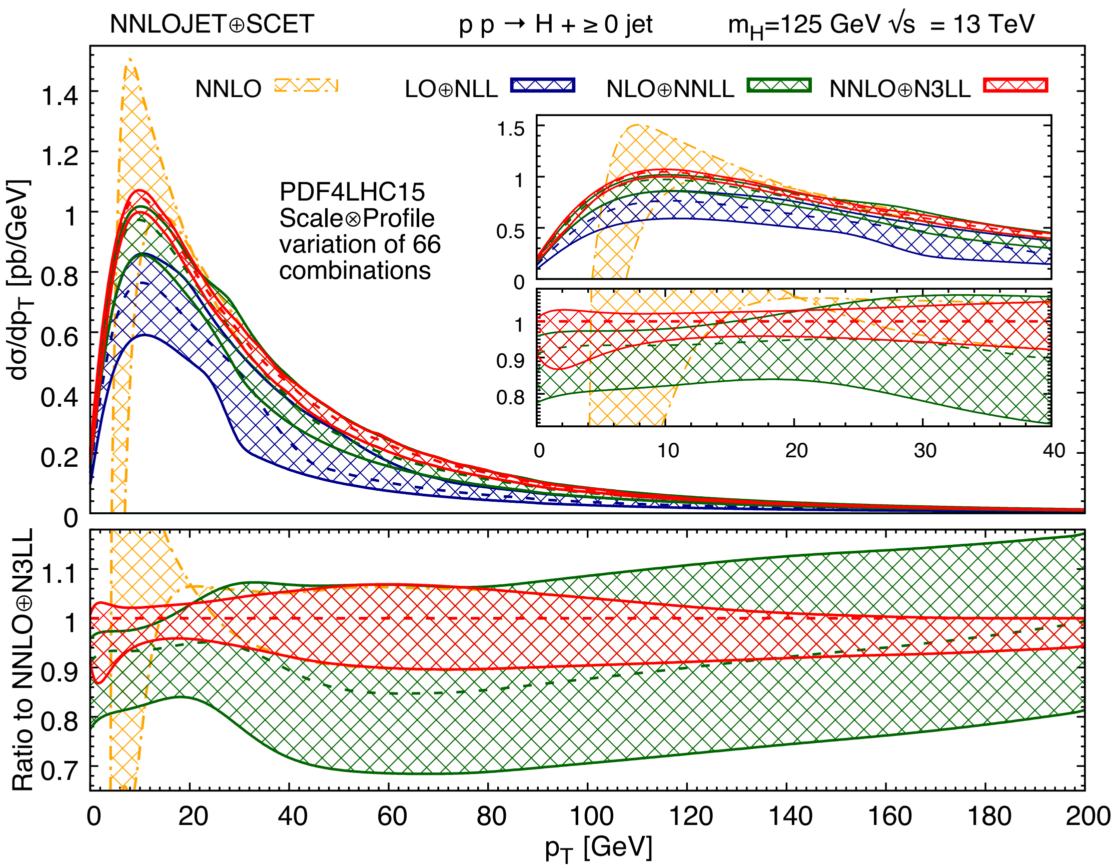


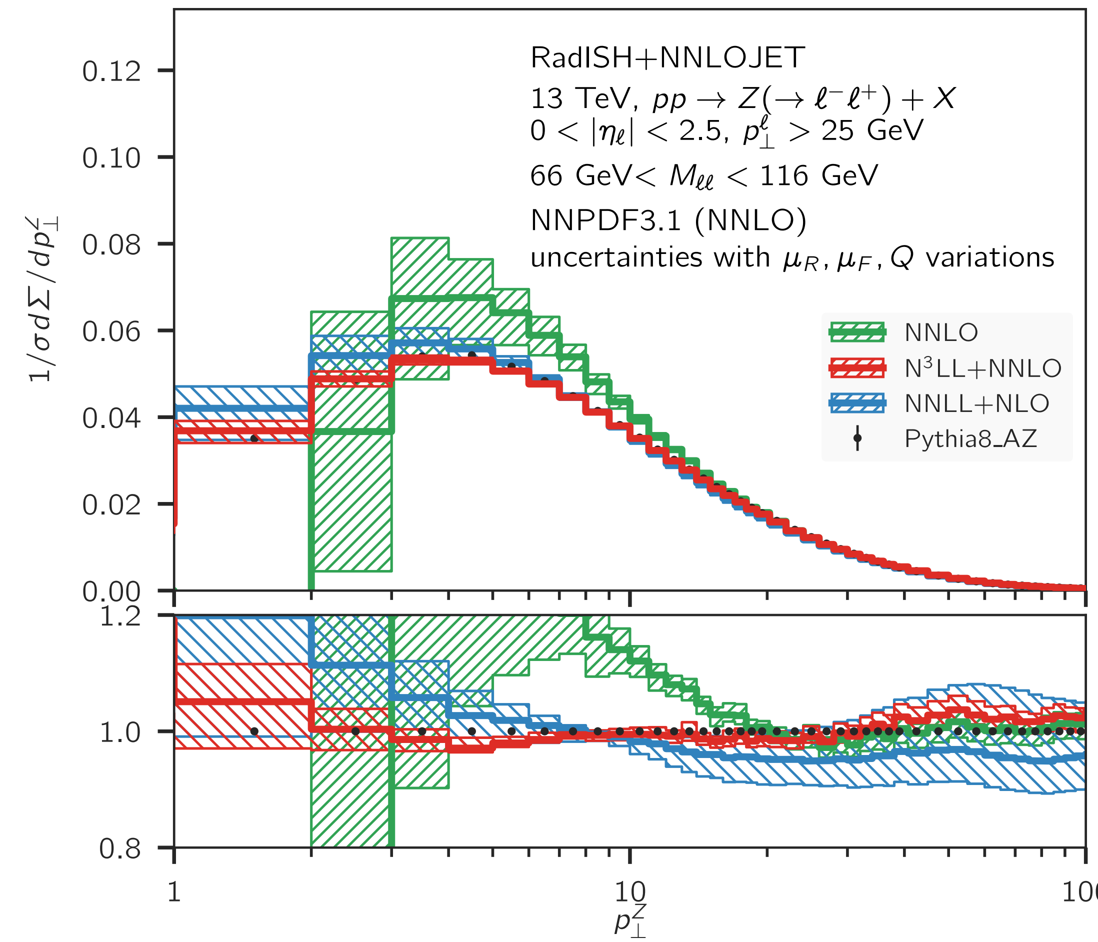


Have implemented q_T resummation in an event-based framework TB, Hager 1904.08325.

- Reweight tree-level event files from MG5_aMC@NLO
- Arbitrary (quark-induced) electroweak boson processes (W,Z, WZ, ZZ, …) at NNLL + O(α_s)
- Can impose experimental cuts on leptonic final states and compute related variables such as ϕ^*

1805.00736 1905.05171

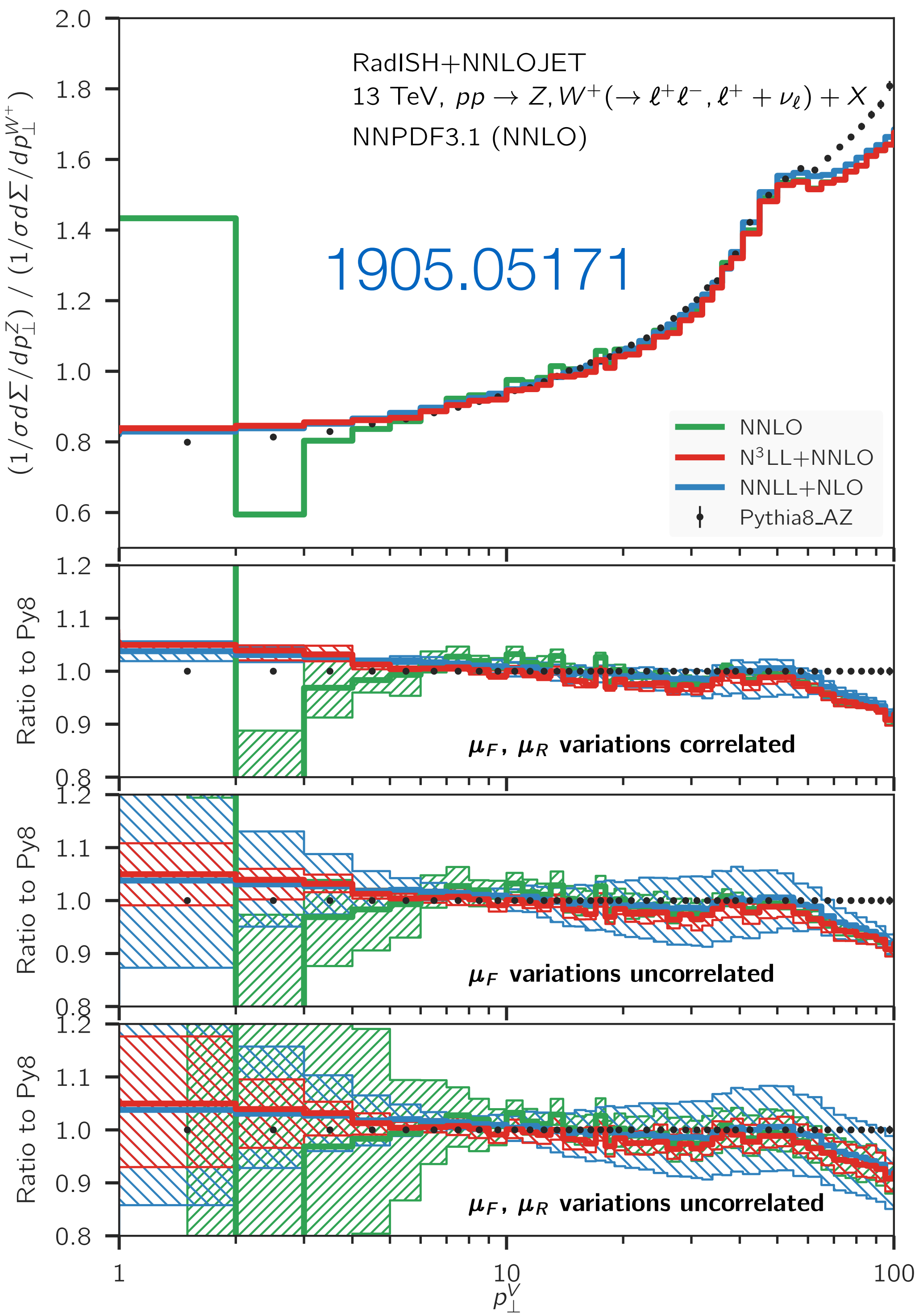




State of the art is now N^{3}LL + O(α_s^3) (here called NNLO) matching

- *W*, *Z*, *H* using RadISH Bizon, Chen, Gehrmann-De Ridder, Glover, Huss, Monni, Re, Rottoli, Torrielli Walker '18 '19
- *H* using SCET Chen, Gehrmann, Glover, Alexander Huss, Li, Neill, Schulze, Stewart, Zhu '18

Ratio of Z and W spectrum



W-spectrum is important for M_W measurement. Analysis needs extremely precise predictions

- Experiments use measured *Z*-spectrum to tune Pythia
- Pythia is then used to predict *W/Z* ratio

A better understanding of the uncertainties would be important

- Ongoing effort to compare and benchmark results of different resummation codes.

Towards NNNNLL

By now even some ingredients for resummation beyond N^3LL have become available

- 3-loop dijet hard functions Baikov, Chetyrkin, Smirnov, Smirnov Steinhauser ’10, Lee, Smirnov, Smirnov ’10, Gehrmann, Glover, Huber, Ikizlerli, Studerus ’10, …
- 3-loop jet functions: quark Brüser, Liu, Stahlhofen ’18; gluon Banerjee, Dhani, Ravindran ’18
- 3-loop soft function for q_T Li and Zhu for EEC, Moult, Zhu ’18, for heavy-to-light decays Brüser, Liu, Stahlhofen ‘19
- double-real for 3-loop quark beam function Melnikov, Rietkerk, Tancredi, Wever ‘18

Summary

Have discussed the structure of IR singularities of amplitudes with massless particles

- heavily constrained by
  - soft-collinear factorization, collinear limits, non-abelian exponentiation
  - regge limit Del Duca, Claude Duhr, Einan Gardi, Lorenzo Magnea, White ’11; Caron-Huot, Gardi, Reichel, Vernazza ’17
- determined by an anomalous dimension $\boldsymbol{\Gamma}$
  - known to three loops, logarithmic part to 4 loops
- $\boldsymbol{\Gamma}$ is an important ingredient to resummation of *n*-jet processes
  - N^3LL + NNLO for weak boson q_T spectra!

$$
\begin{aligned}
\boldsymbol{\Gamma}(\{\underline{s}\},\mu) &= \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\,\gamma_{\rm cusp}(\alpha_s)\,\ln\frac{\mu^2}{-s_{ij}} \\
&\quad + \sum_R g^R(\alpha_s)\left[\sum_{(i,j)}\big(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\big)\ln\frac{\mu^2}{-s_{ij}} + \sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\,\ln\frac{\mu^2}{-s_{ij}}\right] \\
&\quad + \sum_i \gamma^i(\alpha_s) + f(\alpha_s)\sum_{(i,j,k)}\mathcal{T}_{iijk} + \sum_{(i,j,k,l)}\mathcal{T}_{ijkl}\,F(\beta_{ijlk},\beta_{iklj};\alpha_s) \\
&\quad + \mathcal{O}\left(\alpha_s^4,\alpha_s^5\ln\frac{\mu^2}{-s_{ij}}\right).
\end{aligned}
$$

Thank you!

Extra slides

4-loop Z-factor

$$\ln \mathbf{Z} = \frac{\alpha_s}{4\pi}\left(\frac{\Gamma'_0}{4\epsilon^2}+\frac{\mathbf{\Gamma}_0}{2\epsilon}\right)+\left(\frac{\alpha_s}{4\pi}\right)^2\left(-\frac{3\beta_0\Gamma'_0}{16\epsilon^3}+\frac{\Gamma'_1-4\beta_0\mathbf{\Gamma}_0}{16\epsilon^2}+\frac{\mathbf{\Gamma}_1}{4\epsilon}\right)$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^3\left(\frac{11\beta_0^2\Gamma'_0}{72\epsilon^4}-\frac{5\beta_0\Gamma'_1+8\beta_1\Gamma'_0-12\beta_0^2\mathbf{\Gamma}_0}{72\epsilon^3}+\frac{\Gamma'_2-6\beta_0\mathbf{\Gamma}_1-6\beta_1\mathbf{\Gamma}_0}{36\epsilon^2}+\frac{\mathbf{\Gamma}_2}{6\epsilon}\right)$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^4\left(-\frac{25\beta_0^3\Gamma'_0}{192\epsilon^5}+\frac{13\beta_0^2\Gamma'_1+40\beta_0\beta_1\Gamma'_0-24\beta_0^3\mathbf{\Gamma}_0}{192\epsilon^4}\right.$$
$$-\frac{7\beta_0\Gamma'_2+9\beta_1\Gamma'_1+15\beta_2\Gamma'_0-24\beta_0^2\mathbf{\Gamma}_1-48\beta_0\beta_1\mathbf{\Gamma}_0}{192\epsilon^3}$$
$$\left.+\frac{\Gamma'_3-8\beta_0\mathbf{\Gamma}_2-8\beta_1\mathbf{\Gamma}_1-8\beta_2\mathbf{\Gamma}_0}{64\epsilon^2}+\frac{\mathbf{\Gamma}_3}{8\epsilon}\right)+\mathcal{O}(\alpha_s^5)\,,$$

$$\mathbf{\Gamma}(\alpha_s)=\sum_{n=0}^{\infty}\mathbf{\Gamma}_n\left(\frac{\alpha_s}{4\pi}\right)^{n+1}$$

$$\Gamma'(\alpha_s)=\frac{\partial}{\partial\ln\mu}\mathbf{\Gamma}(\{\underline{s}\},\mu)=-\sum_i\Gamma^i_{\text{cusp}}(\alpha_s)$$