New Results on the 3-Loop Heavy Flavour Wilson Coefficients for $F_2(x, Q^2)$

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- Introduction
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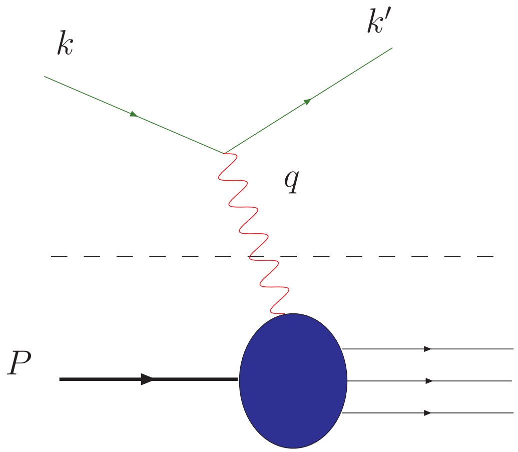
- All–N Results: N_f terms and Ladder Graphs
- Conclusions

1. Introduction

Deep–Inelastic Scattering:



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq}$$ Bjorken–x

$$\nu := \frac{Pq}{M} \,,$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

unpol. $\left\{ \; = \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)\right.$

Structure Functions: $F_{2,L}$ contain light and heavy quark contributions:

$O(\alpha_s^2)$: Heavy flavor contributions [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]

$O(\alpha_s^3)$: Light flavor Wilson coefficients: [Moch, Vermaseren, Vogt, 2005.]

Need for the Calculation:

- Heavy flavor (charm) contributions to DIS structure functions are rather large
  $\Longrightarrow$ Precision understanding of structure functions is required

- Precision determination of $\alpha_s(M_Z^2)$:

  $\alpha_s(M_Z^2) = 0.1141^{+0.0020}_{-0.0022}$ N^{3}LO NS-analysis [1]

  $\alpha_s(M_Z^2) = 0.1135^{+0.0014}_{-0.0014}$ N^{2}LO fit based on DIS, DY, di-muon data [2]

  Aim : $\delta\alpha_s/\alpha_s < 1\,\%$.

  [1] J.B. , Böttcher, Guffanti, 2007; [2] Alekhin, J.B., Klein, Moch, 2009.

- Precise determination of the gluon and sea quark distributions.

Goal:

- Calculation of the heavy flavor Wilson coefficients to higher orders for $Q^2 \geq 25\,\mathrm{GeV}^2$ [sufficient in many applications].
- Application to the polarized (NLO) and transversity (NLO,NNLO) case.

2. The Method

- In Bjorken limit: factorization of the structure functions:

$$\underbrace{F_i(x,Q^2)}_{\text{structure functions}} = \sum_j \underbrace{\mathcal{C}_{i,j}\left(x,\frac{Q^2}{\mu^2},\frac{m_k^2}{\mu^2}\right)}_{\text{Wilson coefficients, perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{parton densities, non-perturbative}},$$

- Process dependent Wilson coefficients contain both light and heavy flavor contributions:

$$\mathcal{C}_{i,j}\left(x,\frac{Q^2}{\mu^2},\frac{m_k^2}{\mu^2}\right) = C_{i,j}^{\text{light}}\left(x,\frac{Q^2}{\mu^2}\right) + \mathsf{H}_{i,j}\left(x,\frac{Q^2}{\mu^2},\frac{m_k^2}{\mu^2}\right), k = c, b\ .$$

- Heavy quark contributions given by heavy quark Wilson coefficients

$$\mathsf{H}_{(2,L),i}^{\text{S, NS}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right) = \underbrace{H_{(2,L),i}^{\text{S}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)}_{\gamma + q_{heavy} \to X} + \underbrace{L_{(2,L),i}^{\text{S,NS}}\left(\frac{Q^2}{\mu^2},\frac{m^2}{Q^2}\right)}_{\gamma + q_{light} \to X}$$

- Consider only one species of heavy quarks

- Factorization for $F_2^{Q\overline{Q}}(x,Q^2)$ at the level of twist $\tau = 2$:

$$
\begin{aligned}
F_2^{Q\overline{Q}}(n_f,x,Q^2,m^2) = \sum_{k=1}^{n_f} e_k^2 \Biggl\{ & \quad L_{2,q}^{\mathsf{NS}}\left(n_f,x,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) \otimes \left[f_k(n_f,x,\mu^2) + f_{\overline{k}}(n_f,x,\mu^2)\right] \\
& + \tilde{L}_{2,q}^{\mathsf{PS}}\left(n_f,x,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) \otimes \Sigma(n_f,x,\mu^2) \\
& + \tilde{L}_{2,g}^{\mathsf{S}}\left(n_f,x,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) \otimes G(n_f,x,\mu^2) \Biggr\} \\
+ e_Q^2 \Biggl\{ & \quad H_{2,q}^{\mathsf{PS}}\left(n_f,x,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) \otimes \Sigma(n_f,x,\mu^2) \\
& + H_{2,g}^{\mathsf{S}}\left(n_f,x,\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) \otimes G(n_f,x,\mu^2) \Biggr\} .
\end{aligned}
$$

- In the limit $Q^2 \gg m^2$ [$Q^2 \approx 10\ m^2$ for F_2]: all terms but power corrections calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$:

$$H^{\rm S}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A^{\rm S}_{k,i}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C^{\rm S}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–parton–Wilson coefficients}} + O\left(\frac{m^2}{Q^2}\right).$$

- Formula holds for completely inclusive quantities $\Longrightarrow$ Virtual heavy quarks are included.
- OMEs obey expansion

$$A^{\rm S,NS}_{k,i}\left(\frac{m^2}{\mu^2}\right) = \langle i|O^{\rm S,NS}_k|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A^{\rm S,NS,(l)}_{k,i}\left(\frac{m^2}{\mu^2}\right), \qquad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

- NNLO light parton Wilson coefficients are known
  $\Longrightarrow$ 3–loop OMEs needed to calculate NNLO HQ corrections in the asymptotic limit.
- OMEs occur as well in the definition of a variable flavor number scheme starting from a fixed flavor number scheme.

[Buza, Matiounine, Smith, van Neerven, 1998; Chuvakin, Smith, van Neerven, 1998.]

- Expansion up to $O(a_s^3)$ of L_2 and H_2 in the $\overline{\rm MS}$–scheme reads

$$
\begin{aligned}
L_{2,q}^{\sf NS}(n_f) &= a_s^2\left[A_{qq,Q}^{{\sf NS},(2)}(n_f)+\hat{C}_{2,q}^{{\sf NS},(2)}(n_f)\right]+a_s^3\left[A_{qq,Q}^{{\sf NS},(3)}(n_f)+A_{qq,Q}^{{\sf NS},(2)}(n_f)C_{2,q}^{{\sf NS},(1)}(n_f)+\hat{C}_{2,q}^{{\sf NS},(3)}(n_f)\right]\\
\tilde{L}_{2,q}^{\sf PS}(n_f) &= a_s^3\left[\ \tilde{A}_{qq,Q}^{{\sf PS},(3)}(n_f)+A_{gq,Q}^{(2)}(n_f)\ \ \tilde{C}_{2,g}^{(1)}(n_f+1)+\hat{\tilde{C}}_{2,q}^{{\sf PS},(3)}(n_f)\right]\\
\tilde{L}_{2,g}^{\sf S}(n_f) &= a_s^2A_{gg,Q}^{(1)}(n_f)\tilde{C}_{2,g}^{(1)}(n_f+1)+a_s^3\left[\ \tilde{A}_{qg,Q}^{(3)}(n_f)+A_{gg,Q}^{(1)}(n_f)\ \ \tilde{C}_{2,g}^{(2)}(n_f+1)\right.\\
&\qquad\left.+A_{gg,Q}^{(2)}(n_f)\ \ \tilde{C}_{2,g}^{(1)}(n_f+1)+\ A_{Qg}^{(1)}(n_f)\ \ \tilde{C}_{2,q}^{{\sf PS},(2)}(n_f+1)+\hat{\tilde{C}}_{2,g}^{(3)}(n_f)\right]\\
H_{2,q}^{\sf PS}(n_f) &= a_s^2\left[\ A_{Qq}^{{\sf PS},(2)}+\ \tilde{C}_{2,q}^{{\sf PS},(2)}(n_f+1)\right]+a_s^3\left[\ A_{Qq}^{{\sf PS},(3)}+\ \tilde{C}_{2,q}^{{\sf PS},(3)}(n_f+1)+A_{gq,Q}^{(2)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)+A_{Qq}^{{\sf PS},(2)}\ C_{2,q}^{{\sf NS},(1)}(n_f+1)\right]\\
H_{2,g}^{\sf S}(n_f) &= a_s\left[\ A_{Qg}^{(1)}+\ \tilde{C}_{2,g}^{(1)}(n_f+1)\right]+a_s^2\left[\ A_{Qg}^{(2)}+\ A_{Qg}^{(1)}\ C_{2,q}^{{\sf NS},(1)}(n_f+1)+A_{gg,Q}^{(1)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)+\ \tilde{C}_{2,g}^{(2)}(n_f+1)\right]\\
&+\ a_s^3\left[\ A_{Qg}^{(3)}+\ A_{Qg}^{(2)}\ C_{2,q}^{{\sf NS},(1)}(n_f+1)+A_{gg,Q}^{(2)}\ \tilde{C}_{2,g}^{(1)}(n_f+1)+A_{gg,Q}^{(1)}\ \tilde{C}_{2,g}^{(2)}(n_f+1)\right.\\
&\qquad\left.+\ A_{Qg}^{(1)}\left[C_{2,q}^{{\sf NS},(2)}(n_f+1)+\ \tilde{C}_{2,q}^{{\sf PS},(2)}(n_f+1)\right]+\ \tilde{C}_{2,g}^{(3)}(n_f+1)\right]\ .
\end{aligned}
$$

- n_f–dependence non–trivial: $\hat{f}(n_f)\equiv f(n_f+1)-f(n_f)$, $\tilde{f}(n_f)\equiv f(n_f)/n_f$.
- Comparison to exact order $O(a_s^2)$ result: asymptotic formulae valid for $Q^2\gtrsim 20\ ({\rm GeV}/c)^2$ in case of $F_2^{c\overline{c}}(x,Q^2)$ and $Q^2\gtrsim 800\ ({\rm GeV}/c)^2$ for $F_L^{c\overline{c}}(x,Q^2)$

3. Fixed moments at 3–Loop

Contributing OMEs:

Singlet	A_{Qg}	A_{Qg}	$A_{gg,Q}$	$A_{gq,Q}$	} mixing
Pure–Singlet		$A_{Qq}^{\rm PS}$	$A_{qq,Q}^{\rm PS}$		
Non–Singlet	$A_{qq,Q}^{\rm NS,+}$	$A_{qq,Q}^{\rm NS,-}$	$A_{qq,Q}^{\rm NS,v}$		

- The renormalized OMEs at 3–loop order are of the general structure

$$A_{ij}^{(3)} = a_{ij}^{(3),3} \ln^3\left(\frac{m^2}{\mu^2}\right) + a_{ij}^{(3),2} \ln^2\left(\frac{m^2}{\mu^2}\right) + a_{ij}^{(3),1} \ln\left(\frac{m^2}{\mu^2}\right) + a_{ij}^{(3),0} \ .$$

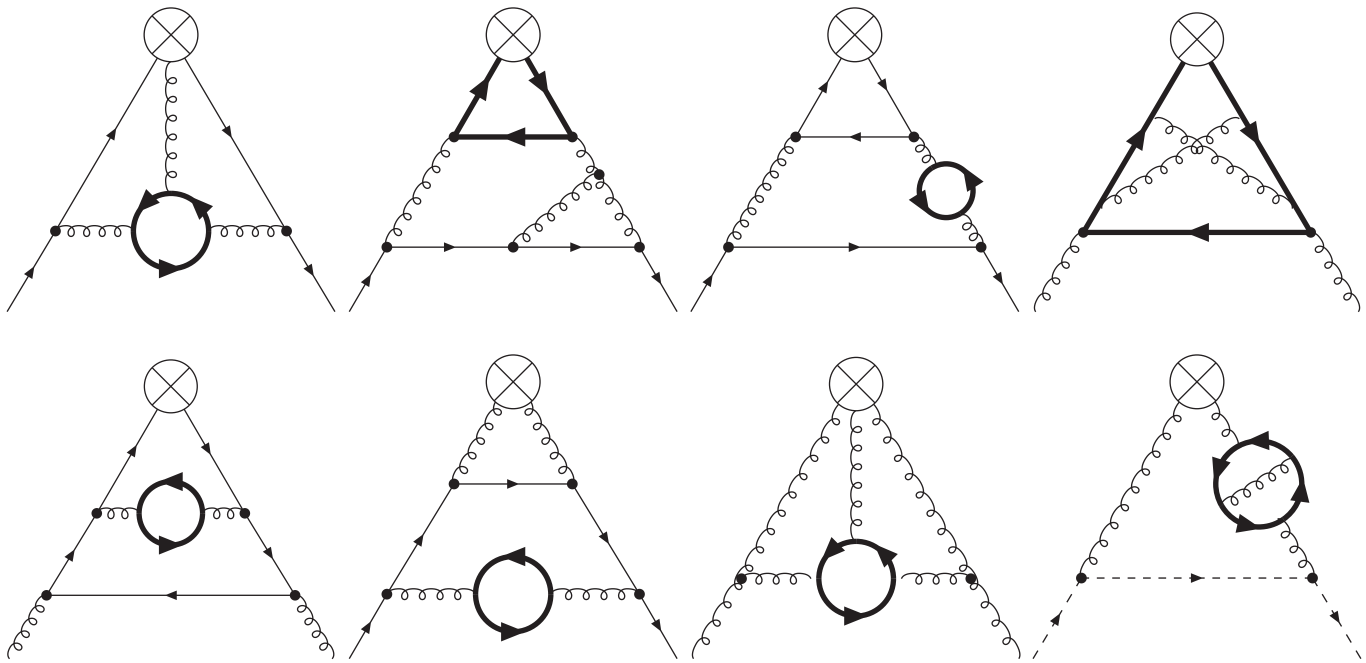
- Logarithmic terms are completely determined by renormalization and given in terms of anomalous dimensions and lower order OMEs.

- For fixed N: Calculation using MATAD [Steinhauser, 2001] and FORM [Vermaseren, 2000]:

$A_{Qq}^{(3),\mathsf{PS}}$: $(2, .., 12)$; $A_{qq,Q}^{(3),\mathsf{PS}}, A_{gq,Q}^{(3)}, A_{qq,Q}^{(3),\mathsf{NS}\pm}$: $(2, .., 14)$; $A_{Q(q)g}^{(3)}, A_{gg,Q}^{(3)}$: $(2, .., 10)$;

- Agreement for the terms $\propto T_F$ of the anomalous dimensions $\gamma_{ij}^{(2),\mathsf{NS}^\pm,\ \mathsf{S},\ \mathsf{PS}}$ with [Larin, Nogueira, Ritbergen, Vermaseren, 1997; Moch, Vermaseren, Vogt, 2004.]

Example-diagrams contributing to the different channels



$\sim$ 2800 diagrams contribute.

4. Numeric Implementation for the Logarithmic Terms

$$
\begin{aligned}
A_{Qq}^{(3),\mathsf{PS},\overline{\mathrm{MS}}} \quad = \quad & \frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}}{48}\Biggl\{\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 16\beta_{0,Q}\Biggr\}\ln^3\left(\frac{m^2}{\mu^2}\right) \\
& + \frac{1}{8}\Biggl\{-4\hat{\gamma}_{qq}^{(1),\mathsf{PS}}\left(\beta_0 + \beta_{0,Q}\right) + \hat{\gamma}_{qg}^{(0)}\left(\hat{\gamma}_{gq}^{(1)} - \gamma_{gq}^{(1)}\right) - \gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}\Biggr\}\ln^2\left(\frac{m^2}{\mu^2}\right) \\
& + \frac{1}{16}\Biggl\{8\hat{\gamma}_{qq}^{(2),\mathsf{PS}} - 8n_f\hat{\tilde{\gamma}}_{qq}^{(2),\mathsf{PS}} - 32a_{Qq}^{(2),\mathsf{PS}}(\beta_0 + \beta_{0,Q}) + 8\hat{\gamma}_{qg}^{(0)}a_{gq,Q}^{(2)} - 8\gamma_{gq}^{(0)}a_{Qg}^{(2)} \\
& -\zeta_2\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}\left(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 8\beta_{0,Q}\right)\Biggr\}\ln\left(\frac{m^2}{\mu^2}\right) \\
& + 4(\beta_0 + \beta_{0,Q})\overline{a}_{Qq}^{(2),\mathsf{PS}} + \gamma_{gq}^{(0)}\overline{a}_{Qg}^{(2)} - \hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)} + \zeta_3\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}}{48}\left(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0\right) \\
& + \frac{\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(1)}\zeta_2}{16} + C_F\left(-(4 + \frac{3}{4}\zeta_2)\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)} - 4\hat{\gamma}_{qq}^{(1),\mathsf{PS}} + 12a_{Qq}^{(2),\mathsf{PS}}\right) + a_{Qq}^{(3),\mathsf{PS}} \quad .
\end{aligned}
$$

All terms but $a_{Qq}^{(3),\mathsf{PS}}$ known for all N.

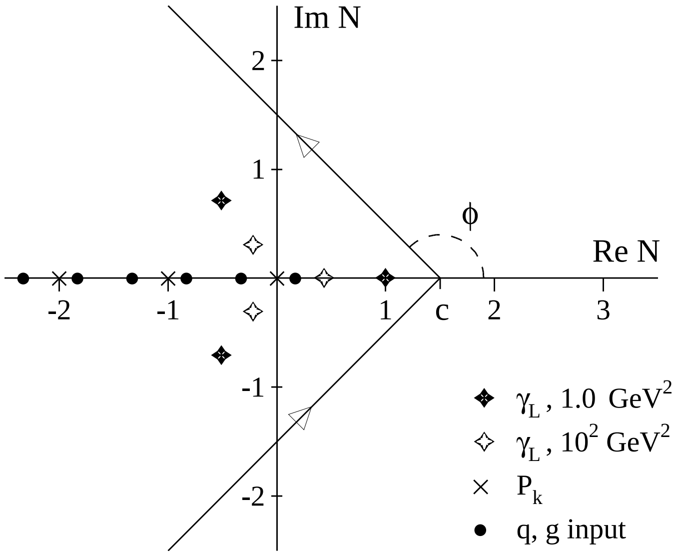
- There are similar formulas for the remaining OMEs.
- Explicit relations can be given for all logaritmnic contributions.

- Results can be represented in terms of harmonic sums and possible generalizations thereof for an analysis in moment–space [J.B., Kurth, 1999; Vermaseren, 1999.]

$$S_{a_1,\ldots,a_m}(N) = \sum_{n_1=1}^{N}\sum_{n_2=1}^{n_1}\cdots\sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}}\frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}}\cdots\frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}} .$$

- Inversion from Mellin-space to z-space:

$$xF_2^{Q\bar{Q}}(x,Q^2) = \int_0^\infty dz\,\mathrm{Im}\left[e^{i\Phi}x^{-c(z)}\,F_2^{Q\bar{Q}}(c(z),Q^2)\right], \qquad c(z) = c_0 + ze^{i\Phi}$$



Analytic continuation of harmonic sums

$$S_1(N) = \Psi(N+1) + \gamma_E, \quad \text{etc} .$$

For nested harmonic sums e.g. using ANCONT [J.B., 2000.]

- Work on all–N results for the constant terms is in progress.
- We will provide the 3–loop OMEs in N–space, which can be combined with the light–flavor Wilson coefficients.
- Representations in terms of HPLs will be given as well for a direct analysis in x–space [Remiddi, Vermaseren, 2000; J. Ablinger, J.B., C Schneider, 2010].

$$A_{ij}^{(3)} = a_{ij}^{(3),3} \ln^3\left(\frac{m^2}{\mu^2}\right) + a_{ij}^{(3),2} \ln^2\left(\frac{m^2}{\mu^2}\right) + a_{ij}^{(3),1} \ln\left(\frac{m^2}{\mu^2}\right) + a_{ij}^{(3),0} \ .$$

$$\begin{aligned}
a_{Qg}^{(3),3} = & \Bigg(C_F\Bigg(\frac{2(N^2+N+2)P_1}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{8(N^2+N+2)S_1}{9N(N+1)(N+2)}\Bigg) + C_A\Bigg(\frac{8(N^2+N+2)S_1}{9N(N+1)(N+2)} - \frac{16(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2}\Bigg)\Bigg)n_f \\
& + C_A{}^2\Bigg(-\frac{16(N^2+N+2)S_1{}^2}{3N(N+1)(N+2)} - \frac{4(N^2+N+2)(11N^4+22N^3-59N^2-70N-48)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{8(N^2+N+1)(N^2+N+2)(11N^4+22N^3-35N^2-46N-24)}{9(N-1)^2N^3(N+1)^3(N+2)^3}\Bigg) + C_A\Bigg(\frac{56(N^2+N+2)S_1}{9N(N+1)(N+2)} - \frac{112(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2}\Bigg) \\
& - \frac{8(N^2+N+2)}{9N(N+1)(N+2)} + C_F{}^2\Bigg(-\frac{(N^2+N+2)(3N^2+3N+2)^2}{3N^3(N+1)^3(N+2)} + \frac{8(N^2+N+2)S_1(3N^2+3N+2)}{3N^2(N+1)^2(N+2)} - \frac{16(N^2+N+2)S_1{}^2}{3N(N+1)(N+2)}\Bigg) \\
& + C_F\Bigg(\frac{4(N^2+N+2)P_2}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{32(N^2+N+2)S_1}{9N(N+1)(N+2)} + C_A\Bigg(\frac{32(N^2+N+2)S_1{}^2}{3N(N+1)(N+2)} - \frac{4(N^2+N+2)(N^2+N+6)(7N^2+7N+4)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{(N^2+N+2)(3N^2+3N+2)(11N^4+22N^3-59N^2-70N-48)}{9(N-1)N^3(N+1)^3(N+2)^2}\Bigg)\Bigg) \ .
\end{aligned}$$

$$
\begin{aligned}
a_{Qg}^{(3),2} = {} & n_f \Biggl(C_F \Biggl(-\frac{4(N^2+N+2)S_1{}^2}{3N(N+1)(N+2)} + \frac{8(5N^3+8N^2+19N+6)S_1}{9N^2(N+1)(N+2)} - \frac{P_3}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{4(N^2+N+2)S_2}{3N(N+1)(N+2)} \Biggr) \\
& + C_A \Biggl(\frac{4(N^2+N+2)S_1{}^2}{3N(N+1)(N+2)} - \frac{8(5N^4+20N^3+47N^2+58N+20)S_1}{9N(N+1)^2(N+2)^2} - \frac{2P_4}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{8(N^2+N+2)S_{-2}}{3N(N+1)(N+2)} \\
& + \frac{4(N^2+N+2)S_2}{3N(N+1)(N+2)} \Biggr) \Biggr) + \Biggl(-\frac{8(N^2+N+2)S_1{}^3}{N(N+1)(N+2)} - \frac{2P_5 S_1{}^2}{3(N-1)N^2(N+1)^2(N+2)^2} + \Biggl[-\frac{24(N^2+N+2)S_2}{N(N+1)(N+2)} \\
& + \frac{4P_6}{9(N-1)^2N^3(N+1)^3(N+2)^3} \Biggr] S_1 + \frac{8P_7}{9(N-1)^2N^4(N+1)^4(N+2)^4} - \frac{8(N^2+N+2)S_{-3}}{N(N+1)(N+2)} + S_{-2} \Biggl[-\frac{32(N^2+N+2)S_1}{N(N+1)(N+2)} \\
& - \frac{4(N^2+N+2)(11N^4+22N^3-59N^2-70N-48)}{3(N-1)N^2(N+1)^2(N+2)^2} \Biggr] - \frac{2(N^2+N+2)(11N^4+22N^3-83N^2-94N-72)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{8(N^2+N+2)S_3}{N(N+1)(N+2)} \\
& + \frac{16(N^2+N+2)S_{-2,1}}{N(N+1)(N+2)} \Biggr) C_A{}^2 + \Biggl(\frac{4(N^2+N+2)S_1{}^2}{N(N+1)(N+2)} + \frac{8(5N^4+20N^3-N^2-14N+20)S_1}{9N(N+1)^2(N+2)^2} - \frac{2P_8}{9(N-1)N^3(N+1)^3(N+2)^3} \\
& + \frac{8(N^2+N+2)S_{-2}}{N(N+1)(N+2)} + \frac{4(N^2+N+2)S_2}{N(N+1)(N+2)} \Biggr) C_A + C_F{}^2 \Biggl(-\frac{8(N^2+N+2)S_1{}^3}{N(N+1)(N+2)} + \frac{2(3N^4+14N^3+43N^2+48N+20)S_1{}^2}{N^2(N+1)^2(N+2)} \\
& + \Biggl[\frac{24(N^2+N+2)S_2}{N(N+1)(N+2)} - \frac{4P_9}{N^3(N+1)^3(N+2)} \Biggr] S_1 + \frac{P_{10}}{2N^4(N+1)^4(N+2)} + \frac{16(N^2+N+2)S_{-3}}{N(N+1)(N+2)} + S_{-2} \Biggl[\frac{32(N^2+N+2)S_1}{N(N+1)(N+2)} \\
& - \frac{16(N^2+N+2)}{N^2(N+1)^2(N+2)} \Biggr] - \frac{6(N^2+N+2)(3N^2+3N+2)S_2}{N^2(N+1)^2(N+2)} + \frac{16(N^2+N+2)S_3}{N(N+1)(N+2)} - \frac{32(N^2+N+2)S_{-2,1}}{N(N+1)(N+2)} \Biggr) + C_F \Biggl(-\frac{4(N^2+N+2)S_1{}^2}{N(N+1)(N+2)} \\
& + \frac{8(5N^3+14N^2+37N+18)S_1}{9N^2(N+1)(N+2)} - \frac{P_{11}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{4(N^2+N+2)S_2}{3N(N+1)(N+2)} + C_A \Biggl(\frac{16(N^2+N+2)S_1{}^3}{N(N+1)(N+2)} \\
& + \frac{4P_{12}S_1{}^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{4P_{13}S_1}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{P_{14}}{18(N-1)N^3(N+1)^3(N+2)^3} - \frac{8(N^2+N+2)S_{-3}}{N(N+1)(N+2)} \\
& - \frac{12(N^2+N+2)S_{-2}}{N(N+1)(N+2)} + \frac{4(N^2+N+2)(N^4+2N^3+8N^2+7N+18)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{8(N^2+N+2)S_3}{N(N+1)(N+2)} + \frac{16(N^2+N+2)S_{-2,1}}{N(N+1)(N+2)} \Biggr) \Biggr) .
\end{aligned}
$$

$$
\begin{aligned}
a_{Qg}^{(3),1} \quad = \quad & n_f \Bigg(-\frac{1}{2}\hat{\tilde{\gamma}}_{qg}^{(2)} + C_F \Bigg(-\frac{4(N^2+N+2)S_1{}^3}{9N(N+1)(N+2)} + \frac{4(3N+2)S_1{}^2}{3N^2(N+2)} + \Bigg[\frac{4(N^4-N^3-20N^2-10N-4)}{3N^2(N+1)^2(N+2)} - \frac{4(N^2+N+2)S_2}{3N(N+1)(N+2)} \Bigg] S_1 \\
& + \frac{2P_{15}}{3(N-1)N^5(N+1)^5(N+2)^4} + \frac{4P_{16}S_2}{3(N-1)N^3(N+1)^3(N+2)^2} + \frac{16(N^2+N+2)S_3}{9N(N+1)(N+2)} \Bigg) + C_A \Bigg(\frac{4(N^2+N+2)S_1{}^3}{9N(N+1)(N+2)} \\
& - \frac{4(N^3+8N^2+11N+2)S_1{}^2}{3N(N+1)^2(N+2)^2} + \Bigg[\frac{4(N^2+N+2)S_2}{N(N+1)(N+2)} - \frac{4P_{17}}{3N(N+1)^3(N+2)^3} \Bigg] S_1 + \frac{4P_{18}}{3(N-1)N^4(N+1)^4(N+2)^4} + \frac{8(N^2+N+2)S_{-3}}{3N(N+1)(N+2)} \\
& + S_{-2} \Bigg[\frac{16(N^2-N-4)}{3(N+1)^2(N+2)^2} + \frac{16(N^2+N+2)S_1}{3N(N+1)(N+2)} \Bigg] - \frac{4P_{19}S_2}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{32(N^2+N+2)S_3}{9N(N+1)(N+2)} - \frac{16(N^2+N+2)S_{-2,1}}{3N(N+1)(N+2)} \Bigg) \Bigg) \\
& + \frac{1}{2}\hat{\tilde{\gamma}}_{qg}^{(2)} + \Bigg(-\frac{8(N^2+N+2)S_1{}^4}{3N(N+1)(N+2)} - \frac{2P_{20}S_1{}^3}{9(N-1)N^2(N+1)^2(N+2)^2} + \Bigg[\frac{2P_{21}}{3(N-1)N^2(N+1)^3(N+2)^3} - \frac{24(N^2+N+2)S_2}{N(N+1)(N+2)} \Bigg] S_1{}^2 \\
& + \Bigg[-\frac{2P_{22}}{3(N-1)N^4(N+1)^4(N+2)^4} - \frac{2P_{23}S_2}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{64(N^2+N+2)S_3}{3N(N+1)(N+2)} + \frac{32(N^2+N+2)S_{-2,1}}{N(N+1)(N+2)} \Bigg] S_1 \\
& - \frac{2P_{24}}{3(N-1)^2N^5(N+1)^5(N+2)^5} + S_{-3} \Bigg[-\frac{4(N^2+N+2)(11N^4+22N^3-35N^2-46N-24)}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{16(N^2+N+2)S_1}{N(N+1)(N+2)} \Bigg] \\
& + S_{-2} \Bigg[-\frac{32(N^2+N+2)S_1{}^2}{N(N+1)(N+2)} - \frac{8P_{25}S_1}{3(N-1)N^2(N+1)(N+2)^2} - \frac{8(N^2-N-4)(11N^4+22N^3-35N^2-46N-24)}{3(N-1)N(N+1)^3(N+2)^3} \Bigg] \\
& + \frac{2(11N^4+22N^3-35N^2-46N-24)P_{19}S_2}{3(N-1)^2N^3(N+1)^3(N+2)^3} - \frac{16(N^2+N+2)(11N^4+22N^3-35N^2-46N-24)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{8(N^2+N+2)(11N^4+22N^3-35N^2-46N-24)S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \Bigg) C_A{}^2 + \Bigg(\frac{8(N^2+N+2)S_1{}^3}{9N(N+1)(N+2)} - \frac{8(N^3+8N^2+11N+2)S_1{}^2}{3N(N+1)^2(N+2)^2} \\
& + \Bigg[\frac{4P_{26}}{27N(N+1)^3(N+2)^3} + \frac{8(N^2+N+2)S_2}{N(N+1)(N+2)} \Bigg] S_1 + \frac{2P_{27}}{27(N-1)N^4(N+1)^4(N+2)^4} + \frac{16(N^2+N+2)S_{-3}}{3N(N+1)(N+2)} + S_{-2} \Bigg[\frac{32(N^2-N-4)}{3(N+1)^2(N+2)^2} \\
& + \frac{32(N^2+N+2)S_1}{3N(N+1)(N+2)} \Bigg] - \frac{8P_{19}S_2}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{64(N^2+N+2)S_3}{9N(N+1)(N+2)} - \frac{32(N^2+N+2)S_{-2,1}}{3N(N+1)(N+2)} \Bigg) C_A + C_F{}^2 \Bigg(-\frac{8(N^2+N+2)S_1{}^4}{3N(N+1)(N+2)} \\
& + \frac{2(3N^4+42N^3+107N^2+92N+28)S_1{}^3}{3N^2(N+1)^2(N+2)} + \Bigg[\frac{2P_{28}}{N^3(N+1)^2(N+2)} - \frac{8(N^2+N+2)S_2}{N(N+1)(N+2)} \Bigg] S_1{}^2 + \Bigg[\frac{2P_{29}}{N^4(N+1)^4(N+2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2(7N^4+74N^3+79N^2-12N-4)S_2}{N^2(N+1)^2(N+2)}+\frac{32(N^2+N+2)S_3}{3N(N+1)(N+2)}\Bigg]S_1-\frac{P_{30}}{N^5(N+1)^5(N+2)}-\frac{8(N^2+N+2)(3N^2+3N+2)S_3}{3N^2(N+1)^2(N+2)}\\
&-\frac{2(3N^2+3N+2)(N^4+17N^3+17N^2-5N-2)S_2}{N^3(N+1)^3(N+2)}\Bigg)+C_F\Bigg(-\frac{8(N^2+N+2){S_1}^3}{9N(N+1)(N+2)}+\frac{8(3N+2){S_1}^2}{3N^2(N+2)}+\Bigg[\frac{8(N^4-N^3-20N^2-10N-4)}{3N^2(N+1)^2(N+2)}\\
&-\frac{8(N^2+N+2)S_2}{3N(N+1)(N+2)}\Bigg]S_1+\frac{P_{31}}{3(N-1)N^5(N+1)^5(N+2)^2}+\frac{8(N^4+17N^3+17N^2-5N-2)S_2}{3N^2(N+1)^2(N+2)}+\frac{32(N^2+N+2)S_3}{9N(N+1)(N+2)}+C_A\Bigg(\frac{16(N^2+N+2){S_1}^4}{3N(N+1)(N+2)}\\
&+\frac{4P_{32}{S_1}^3}{9(N-1)N^2(N+1)^2(N+2)^2}+\Bigg[\frac{32(N^2+N+2)S_2}{N(N+1)(N+2)}-\frac{4P_{33}}{3(N-1)N^3(N+1)^3(N+2)^3}\Bigg]{S_1}^2+\Bigg[-\frac{4P_{34}}{3(N-1)N^4(N+1)^4(N+2)^4}\\
&-\frac{4P_{35}S_2}{3(N-1)N^2(N+1)^2(N+2)^2}+\frac{32(N^2+N+2)S_3}{3N(N+1)(N+2)}-\frac{32(N^2+N+2)S_{-2,1}}{N(N+1)(N+2)}\Bigg]S_1-\frac{P_{36}}{3(N-1)N^4(N+1)^5(N+2)^4}\\
&+S_{-3}\Bigg[\frac{16(N^2+N+2)S_1}{N(N+1)(N+2)}-\frac{4(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)}\Bigg]+S_{-2}\Bigg[\frac{32(N^2+N+2){S_1}^2}{N(N+1)(N+2)}-\frac{8P_{37}S_1}{N^2(N+1)^2(N+2)^2}-\frac{8(N^2-N-4)(3N^2+3N+2)}{N(N+1)^3(N+2)^2}\Bigg]\\
&-\frac{2P_{38}S_2}{3(N-1)N^3(N+1)^3(N+2)^2}-\frac{8(N^2+N+2)(29N^4+58N^3-41N^2-70N-48)S_3}{9(N-1)N^2(N+1)^2(N+2)^2}+\frac{8(N^2+N+2)(3N^2+3N+2)S_{-2,1}}{N^2(N+1)^2(N+2)}\Bigg)\Bigg)\,.
\end{aligned}
$$

- Using MATAD, we calculated the OMEs ($\approx$ 250 days of computer time)

  $A_{Qq}^{(3),\mathsf{PS}}$: $(2,..,12)$; $A_{qq,Q}^{(3),\mathsf{PS}}$, $A_{gq,Q}^{(3)}$, $A_{qq,Q}^{(3),\mathsf{NS}\pm}$: $(2,..,14)$; $A_{Q(q)g}^{(3)}$, $A_{gg,Q}^{(3)}$: $(2,..,10)$;

  and find agreement with the predictions obtained from renormalization.
- Additional checks are provided by sums rules for $N=2$, which are fullfilled by our result.
- All terms proportional to ζ_2 cancel in the renormalized result in the $\overline{\text{MS}}$–scheme.
- We observe the number

$$\mathsf{B4} \quad = \quad -4\zeta_2 \ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{13}{2}\zeta_4 + 16\mathsf{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2}\zeta_4 \ .$$

Example: non–logarithmic term of $A_{Qg}^{(3)}$ for $N=2$

$$A_{Qg}^{(3),\overline{\text{MS}}}(\mu^2=m^2, N=2) = T_F {C_A}^2 \left(\frac{174055}{4374} - \frac{88}{9}\mathsf{B}_4 + 72\zeta_4 - \frac{29431}{324}\zeta_3\right)$$

$$+T_F C_F C_A \left(-\frac{18002}{729} + \frac{208}{9}\mathsf{B}_4 - 104\zeta_4 + \frac{2186}{9}\zeta_3 - \frac{64}{3}\zeta_2 + 64\zeta_2\ln(2)\right)$$

$$+T_F {C_F}^2 \left(-\frac{8879}{729} - \frac{64}{9}\mathsf{B}_4 + 32\zeta_4 - \frac{701}{81}\zeta_3 + 80\zeta_2 - 128\zeta_2\ln(2)\right) + {T_F}^2 C_A \left(-\frac{21586}{2187} + \frac{3605}{162}\zeta_3\right)$$

$$+{T_F}^2 C_F \left(-\frac{55672}{729} + \frac{889}{81}\zeta_3 + \frac{128}{3}\zeta_2\right) + n_f {T_F}^2 C_A \left(-\frac{7054}{2187} - \frac{704}{81}\zeta_3\right) + n_f {T_F}^2 C_F \left(-\frac{22526}{729} + \frac{1024}{81}\zeta_3 - \frac{64}{3}\zeta_2\right) .$$

5. 3-Loop All N-Results: the N_f Contributions

- First complete all–N results [J.B., S. Klein, C. Schneider, F. Wißbrock, 2010]

- Flavor non-singlet contributions $\propto N_f$: [also for transversity]

$$a_{qq,Q}^{(3),\mathrm{NS}} = C_F T_F^2 N_F \Bigg\{ -\frac{55552}{729} S_1 + \frac{448}{27} \zeta_3 S_1 - \frac{160}{27} \zeta_2 S_1 + \frac{640}{27} S_2 + \frac{32}{9} \zeta_2 S_2 - \frac{320}{81} S_3 + \frac{64}{27} S_4$$
$$+ \frac{2}{729} \frac{P_1}{N^4 (1+N)^4} - \frac{112}{27} \frac{\left(3N^2 + 3N + 2\right)}{N(1+N)} \zeta_3 + \frac{4}{27} \frac{\left(3N^4 + 6N^3 + 47N^2 + 20N - 12\right)}{N^2 (1+N)^2} \zeta_2 \Bigg\}$$

- Flavor pure singlet contributions $\propto N_f$:

$$a_{qq,Q}^{(3),\mathrm{PS}} = C_F T_F^2 n_f \Bigg\{ \frac{128}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} S_1^3 - \frac{64}{27} \frac{\left(266 N^4 + 181 N^5 + 269 N^3 + 230 N^2 + 74 N^6 + 16 N^7 + 44 N - 24\right)}{N^3 (-1+N)(2+N)^2 (1+N)^3} S_1^2$$
$$+ \frac{128}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} S_1 S_2 + \frac{64}{81} \frac{P_3}{(-1+N) N^4 (1+N)^4 (2+N)^3} + \frac{32}{3} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} \zeta_2 S_1$$
$$- \frac{64}{27} \frac{\left(266 N^4 + 181 N^5 + 269 N^3 + 230 N^2 + 74 N^6 + 16 N^7 + 44 N - 24\right)}{N^3 (-1+N)(2+N)^2 (1+N)^3} S_2 + \frac{256}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} S_3$$
$$- \frac{32}{243} \frac{P_4}{N^5 (-1+N)(2+N)^4 (1+N)^5} - \frac{16}{9} \frac{P_5}{N^3 (-1+N)(2+N)^2 (1+N)^3} \zeta_2 + \frac{224}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} \zeta_3 \Bigg\} .$$

$$a_{Qq}^{(3),\mathrm{PS}} = C_F T_F^2 n_f \Bigg\{ -\frac{16}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} S_1^3 + \frac{16}{27} \frac{\left(68 N^5 + 37 N^6 + 8 N^7 - 11 N^4 - 86 N^3 - 56 N^2 - 104 N - 48\right)}{N^3 (1+N)^3 (2+N)^2 (-1+N)} S_1^2$$
$$- \frac{208}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} S_1 S_2 - \frac{32}{81} \frac{P_6}{(-1+N) N^4 (1+N)^4 (2+N)^3} S_1 - \frac{16}{3} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} \zeta_2 S_1$$
$$+ \frac{208}{27} \frac{68 N^5 + 37 N^6 + 8 N^7 - 11 N^4 - 86 N^3 - 56 N^2 - 104 N - 48}{N^3 (1+N)^3 (2+N)^2 (-1+N)} S_2 - \frac{1760}{27} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} S_3$$
$$+ \frac{32}{243} \frac{P_7}{N^5 (1+N)^5 (2+N)^4 (-1+N)} + \frac{224}{9} \frac{\left(N^2+N+2\right)^2}{(-1+N) N^2 (1+N)^2 (2+N)} \zeta_3$$
$$+ \frac{16}{9} \frac{\left(68 N^5 + 37 N^6 + 8 N^7 - 11 N^4 - 86 N^3 - 56 N^2 - 104 N - 48\right)}{N^3 (1+N)^3 (2+N)^2 (-1+N)} \zeta_2 \Bigg\} .$$

- All contributions to the anomalous dimensions agree with: [Moch, Vermaseren, Vogt, 2004; Gracey, 2003, resp.]

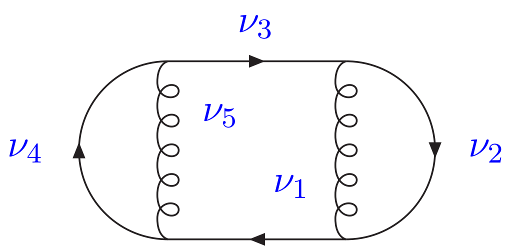
$T_F^2 N_F^2$ contribution to $\gamma_{qg}^{(2)}$

$$
\begin{aligned}
\gamma_{qg}^{(2)} \quad = \quad & T_F^2 N_F^2 \Biggl\{ C_A \Biggl[\frac{32}{9} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_1^3 - \frac{64}{9} \frac{\left(20+49N+41N^2+20N^3+5N^4\right)}{N(1+N)^2(2+N)^2} S_1^2 - \frac{32}{3} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_1 S_2 \\
& + \frac{64}{27} \frac{\left(152+712N+1362N^2+1153N^3+492N^4+124N^5+19N^6\right)}{N(1+N)^3(2+N)^3} S_1 + \frac{128}{3} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_{-3} \\
& - \frac{128}{9} \frac{\left(10+8N+5N^2\right)}{N(1+N)(2+N)} S_{-2} + \frac{64}{9} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_3 + \frac{128}{3} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_{2,1} \\
& - \frac{64}{9} \frac{\left(20+43N+47N^2+26N^3+5N^4\right)}{N(1+N)^2(2+N)^2} S_2 - \frac{128}{9} (-1)^N \frac{\left(16+107N+166N^2+117N^3+43N^4+7N^5\right)}{(1+N)^4(2+N)^4} \\
& + \frac{16}{27} \frac{(1152+7296N+19904N^2+30864N^3+25512N^4+4616N^5-11780N^6)}{(-1+N)N^4(1+N)^4(2+N)^4} \\
& + \frac{16}{27} \frac{(-7723N^7+5333N^8+9398N^9+5362N^{10}+1485N^{11}+165N^{12})}{(-1+N)N^4(1+N)^4(2+N)^4} \Biggr] \\
& + C_F \Biggl[- \frac{32}{9} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_1^3 + \frac{32}{9} \frac{\left(6+29N+13N^2+10N^3\right)}{N^2(1+N)(2+N)} S_1^2 - \frac{32}{3} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_1 S_2 \\
& - \frac{32}{27} \frac{\left(120+412N+426N^2+145N^3+47N^4\right)}{N^2(1+N)^2(2+N)} S_1 + \frac{320}{9} \frac{\left(2+N+N^2\right)}{N(1+N)(2+N)} S_3 + \frac{32}{3} \frac{\left(2+3N+5N^2\right)}{N^2(1+N)(2+N)} S_2 \\
& + \frac{4}{27} \frac{(-27648-122112N-269760N^2-455168N^3-586928N^4-479472N^5-267524N^6)}{(-1+N)N^5(1+N)^5(2+N)^4} \\
& + \frac{4}{27} \frac{(-95592N^7+32283N^8+72446N^9+46649N^{10}+17916N^{11}+4925N^{12}+990N^{13}+99N^{14})}{(-1+N)N^5(1+N)^5(2+N)^4} \Biggr] \Biggr\}
\end{aligned}
$$

- First independent confirmation. Both ladder- and bubble-topologies contribute.
- $a_{Qg}^{(3)}(N)$ for the same color factors will be available in a weeks time. $\Longrightarrow$ LL2010.

5. All N-Results: 3-Loop Ladder Graphs

Next complex topology is based on the
3–loop tadpole diagram $\Longrightarrow$
[J.B., A. Hasselhuhn, S. Klein, C. Schneider, 2010]



It can be represented in terms of an Appell–function of the first kind, F_1 :

$$F_1\left[a; b, b'; c; x_3, x_4\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_n(b')_m}{(1)_m(1)_n(c)_{m+n}} x_3^n x_4^m$$
$$= \iint_0^1 dx_1 dx_2\, \theta(1-x_1-x_2)\, \frac{x_1^{b-1}\, x_2^{b'-1}\, (1-x_1-x_2)^{c-b-b'-1}}{(1-x_1x_3-x_2x_4)^a} ,$$

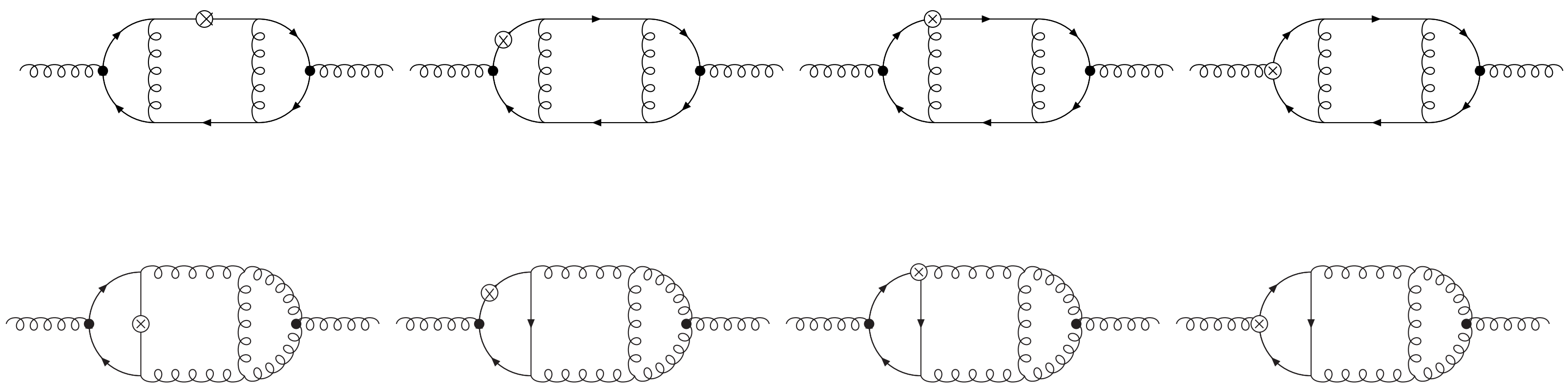
leading to a double infinite sums (with $\nu_{ij} = \nu_i + \nu_j$, etc.)

$$I = C \sum_{m,n=0}^{\infty} \frac{(\nu_{345}-2-\varepsilon/2)_{n+m}(\nu_{12345}-6-3/2\varepsilon)_m(2+\varepsilon/2-\nu_1)_m(2+\varepsilon/2-\nu_5)_n(\nu_{45}-2-\varepsilon/2)_n}{m!n!(\nu_{12345}-4-\varepsilon)_{n+m}(\nu_{345}-2-\varepsilon/2)_m(\nu_{345}-2-\varepsilon/2)_n} .$$

$\Longrightarrow$ F_1 occurs due to the diagram's topology and mass distribution, and its form is independent of the operator insertion.

Ladder diagrams

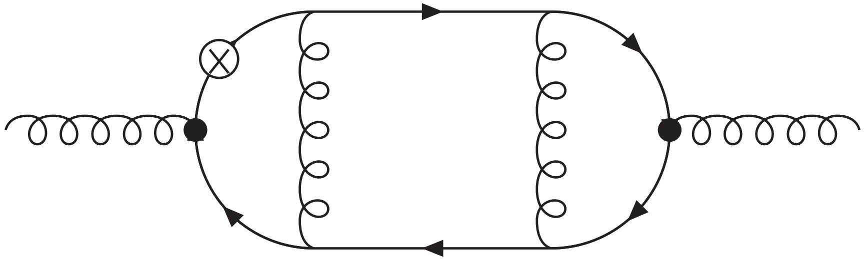
The following diagrams contribute.



.

Further, the external gluons can be linked in all possible other ways.

Example 1



One obtains the following parameter integral

$$
\begin{aligned}
I \;=\; & C\int_0^1 dx_i\; x_3^{\frac{\varepsilon}{2}-1}(1-x_3)^{\frac{\varepsilon}{2}}(1-x_5)x_4^{\frac{\varepsilon}{2}-1}(1-x_4)^{\frac{\varepsilon}{2}} \\
& \times\left(x_5(1-x_4)+x_4x_6(1-x_1-x_2)+x_4x_1x_7+x_4x_2x_5\right)^{N-1} \\
& \times x_1^{-\frac{\varepsilon}{2}}x_2^{-\frac{\varepsilon}{2}}\theta(1-x_1-x_2)(1-x_1-x_2)\left(1-x_1\frac{x_3-1}{x_3}-x_2\frac{x_4-1}{x_4}\right)^{-2+3\varepsilon/2}
\end{aligned}
$$

- The operator insertion contributes as an integer power of a polynomial linear in each Feynman parameter.
- Application of the binomial theorem leads to additional binomial sums.

Thus one obtains:

$$
\begin{aligned}
I &= C\frac{1}{(N+1)(N+2)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=2}^{N+2}\binom{N+2}{l}\sum_{j=2}^{l}\binom{l}{j}\Bigg\{ \\
&\quad\times\sum_{k=1}^{j}\binom{j}{k}\sum_{r=0}^{l-k}\binom{l-k}{r}(-1)^{l+j+k+r}B\left(k,m+1+\frac{\varepsilon}{2}\right) \\
&\quad\times\Gamma\left[\begin{array}{c} k+r+j+m+n+\frac{\varepsilon}{2} \\ m+1,n+1,k+r+\frac{\varepsilon}{2}\end{array}\right]\frac{B\left(k+m-\frac{\varepsilon}{2},r+1+n-\frac{\varepsilon}{2}\right)B\left(r+l-1,n+1+\frac{\varepsilon}{2}\right)}{(k+r+1+m+n-\varepsilon)(N+3-j)} \\
&\quad+\sum_{r=0}^{l-j}\binom{l-j}{r}(-1)^{l+j+r}B\left(j,m+1+\frac{\varepsilon}{2}\right) \\
&\quad\times\Gamma\left[\begin{array}{c} j+r+m+n+\frac{\varepsilon}{2} \\ m+1,n+1,j+r+\frac{\varepsilon}{2}\end{array}\right]\frac{B\left(j+m-\frac{\varepsilon}{2},r+1+n-\frac{\varepsilon}{2}\right)B\left(r+l-1,n+1-\frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(N+3-j)}\Bigg\}
\end{aligned}
$$

- Sums over hypergeometric expressions are solved using the package Sigma by C.Schneider applying symbolic summation techniques.
- At 3–loops, generalizations of harmonic sums appear:

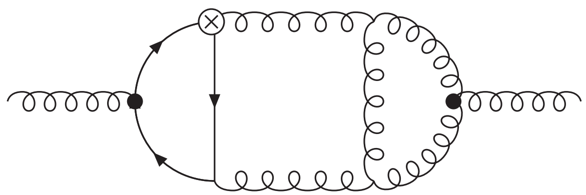
$$\begin{aligned} I &= \frac{C}{(N+1)(N+2)(N+3)}\Bigg\{\frac{1}{6}S_1^3 + \frac{N^2+12N+16}{2(N+1)(N+2)}S_1^2 + \frac{4(2N+3)}{(N+1)^2(N+2)}S_1 \\ &+\frac{8(2N+3)}{(N+1)^3(N+2)} + 2\left[-2^{N+3}+3-(-1)^N\right]\zeta_3 - (-1)^N S_{-3} + \left[\frac{3N^2+40N+56}{2(N+1)(N+2)} - 2S_1\right]S_2 \\ &-\frac{3N+17}{3}S_3 - 2(-1)^N S_{-2,1} - (N+3)S_{2,1} + 2^{N+4}S_{1,2}\left(\frac{1}{2}\right) + 2^{N+3}S_{1,1,1}\left(\frac{1}{2}\right)\Bigg\} + O(\varepsilon)\ , \end{aligned}$$

$$S_{b,\vec{a}}(\xi;N) \ := \sum_{k=1}^{N}\frac{\mathrm{sign}(b)^k\xi^k}{k^{|b|}}S_{\vec{a}}(k) \quad \to S_{b,\vec{a}}(\xi)\ , \quad \xi \in]0,1] \quad \text{[Moch, Uwer, Weinzierl 2002]}\ .$$

[J. Ablinger, J.B., C. Schneider, 2010]

- The powers 2^N lead to divergences as $N \to \infty$ and are therefore expected to cancel in the full expression.
- Complete solution for the 3–loop case might be found by studying generalized hypergeometric functions and their relations to Feynman–integrals combined with advanced summation techniques.

Example 2



The distribution of momenta is the same as before. So the momentum integral reads:

$$\iiint \hat{d}k\hat{d}r\hat{d}s \sum_{j=0}^{N} \frac{(\Delta.k)^j(\Delta.k-\Delta.r)^{N-j}}{((k-p)^2-m^2)(k-r)^2(k-s)^2s^2r^2(k^2-m^2)((k-r)^2-m^2)(s-r)^2}$$

The Feynman parametrization becomes simpler, due to the more local mass distribution:

$$\Gamma\left(2-\frac{3\varepsilon}{2}\right)\sum_{j=0}^{N}\int_0^1 dx\,dz\,du\,dw\,da\,ds\,dt\; z^{\frac{\varepsilon}{2}-1}(1-z)^{\frac{\varepsilon}{2}}w^{1-\varepsilon}(1-w)^{2-\varepsilon}\theta(1-s-t)(1-s-t)s^{-\frac{\varepsilon}{2}}t^{\varepsilon-2}\times$$

$$\times((1-w)u+w((1-s-t)a+sx+tu))^j(1-w)^{N-j}(u-a(1-s-t)-sx-tu)^{N-j}$$

Having introduced binomial sums, the integrals represent rational, Gamma and Beta functions:

$$\begin{aligned} I_9 \;=\; & -iS_\varepsilon^3\Gamma(\varepsilon-1)\Gamma\left(2-\frac{3}{2}\varepsilon\right)B\left(\frac{\varepsilon}{2},1+\frac{\varepsilon}{2}\right)\sum_{i=0}^{N}\sum_{j=0}^{N-i}\binom{N-i}{j}(-1)^j\sum_{l=0}^{i+j}\binom{i+j}{l}(-1)^l \\ & \times\Gamma\left[\begin{matrix}3+j+i-\frac{\varepsilon}{2},3+i-\varepsilon,2+j-\varepsilon\\ 5+i+j-2\varepsilon,2+i+j+\frac{\varepsilon}{2}\end{matrix}\right]\frac{1}{(l+1)(l+2)(N+1-l)} \\ & \times\left[B\left(-\frac{\varepsilon}{2},1\right)-B\left(l+2-\frac{\varepsilon}{2},1\right)-B\left(-\frac{\varepsilon}{2},l+3\right)\right]\end{aligned}$$

Example 2

After expanding in ε the sums are again performed using Sigma:

$$
\begin{aligned}
I_9 = {}& \frac{iS_\varepsilon^3}{(2+N)(4+N)(5+N)}\Biggl\{\Biggl[S_1^2+3S_2+\Biggl(\frac{2(-1)^N\left(N^2+5N+7\right)}{(N+2)(N+3)^2}\\
&+\frac{2\left(2N^3+13N^2+27N+20\right)}{(N+1)(N+3)^2}\Biggr)S_1+\frac{2(-1)^N\left(2N^3+13N^2+29N+21\right)}{(N+1)(N+2)^2(N+3)^2}\\
&-\frac{2\left(2N^6+18N^5+57N^4+60N^3-53N^2-163N-99\right)}{(N+1)^2(N+2)^2(N+3)^2}\Biggr]\frac{1}{\varepsilon^2}\\
&+\frac{1}{N+5}\Biggl[\frac{(N+3)}{2}S_1^3+\Biggl((-1)^N\frac{\left(N^2+5N+7\right)}{2(N+2)(N+3)}\\
&+\frac{2N^6+43N^5+360N^4+1529N^3+3524N^2+4218N+2048}{2(N+1)(N+2)(N+3)(N+4)(N+5)}\Biggr)S_1^2\\
&+\Biggl(\frac{P_{12}(N)}{(N+1)^2(N+2)(N+3)^2(N+4)(N+5)}+\frac{(-1)^N P_{13}}{(N+1)^2(N+2)^2(N+3)^2(N+4)(N+5)}\\
&+4S_{-2}(N)\Biggr)S_1+(-1)^N\frac{P_{11}}{(N+1)^3(N+2)^3(N+3)^2(N+4)(N+5)}\\
&+\frac{P_{10}}{(N+1)^2(N+2)^3(N+3)^2(N+4)(N+5)}+\frac{4(2N+3)}{(N+1)(N+2)}S_{-2}\\
&+\Biggl((-1)^N\frac{\left(N^2+5N+7\right)}{2(N+2)(N+3)}+\frac{-10N^6-133N^5-612N^4-915N^3+1052N^2+4246N+3104}{2(N+1)(N+2)(N+3)(N+4)(N+5)}\\
&+\frac{7}{2}(N+3)S_1\Biggr)S_2+2(N+5)S_3-4(N+3)S_{2,1}\Biggr]\frac{1}{\varepsilon}\Biggr\}\\
&+\frac{1}{(N+2)(N+3)(N+4)(N+5)}\Biggl\{\frac{7(N+3)}{48}S_1^4+\Biggl[\frac{(-1)^N\left(N^2+5N+7\right)}{12(N+2)(N+3)}\\
&+\frac{2N^6+59N^5+588N^4+2805N^3+7040N^2+8974N+4544}{12(N+1)(N+2)(N+3)(N+4)(N+5)}\Biggr]S_1^3\\
&+\Biggl[\frac{(-1)^N P_{15}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)(N+5)}\\
&+\frac{P_{14}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2}+7S_{-2}\Biggr]S_1^2\\
&+\Biggl[\frac{(-1)^N P_{16}(N)}{2(N+1)^3(N+2)^3(N+3)^3(N+4)^2(N+5)^2}\\
&+\frac{P_{17}(N)}{2(N+1)^3(N+2)^2(N+3)^3(N+4)^2(N+5)^2}+5S_{-3}\\
&-\frac{2\left(5N^5+49N^4+104N^3-285N^2-1213N-1036\right)}{(N+1)(N+2)(N+3)(N+4)(N+5)}S_{-2}\Biggr]S_1+\frac{(55N+141)}{16}S_2^2\\
&+\frac{(-1)^N P_{18}(N)}{2(N+1)^4(N+2)^4(N+3)^3(N+4)^2(N+5)^2}\\
&+\frac{P_{19}(N)}{2(N+1)^4(N+2)^4(N+3)^3(N+4)^2(N+5)^2}\\
&+\frac{5(2N+3)}{(N+1)(N+2)}S_{-3}-\frac{4\left(5N^6+63N^5+275N^4+425N^3-160N^2-1004N-684\right)}{(N+1)^2(N+2)^2(N+3)(N+4)(N+5)}S_{-2}\\
&+\Biggl(\frac{3}{8}(9N+31)S_1^2+\Biggl(\frac{13(-1)^N\left(N^2+5N+7\right)}{4(N+2)(N+3)}\\
&+\frac{-10N^6-65N^5+420N^4+5213N^3+18860N^2+29514N+16976}{4(N+1)(N+2)(N+3)(N+4)(N+5)}\Biggr)S_1\\
&+\frac{(-1)^N P_{20}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)(N+5)}\\
&+\frac{P_{21}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2}+S_{-2}\Biggr)S_2\\
&+\zeta_2\Biggl[\frac{3}{8}(N+3)S_1^2+\Biggl(\frac{3(-1)^N\left(N^2+5N+7\right)}{4(N+2)(N+3)}+\frac{3\left(2N^3+13N^2+27N+20\right)}{4(N+1)(N+3)}\Biggr)S_1\\
&+\frac{3(-1)^N\left(2N^3+13N^2+29N+21\right)}{4(N+1)(N+2)^2(N+3)}-\frac{3\left(2N^6+18N^5+57N^4+60N^3-53N^2-163N-99\right)}{4(N+1)^2(N+2)^2(N+3)}\\
&+\frac{9}{8}(N+3)S_2\Biggr]+\Biggl(\frac{(-1)^N\left(N^2+5N+7\right)}{6(N+2)(N+3)}+\frac{-34N^5-383N^4-1379N^3-1280N^2+1830N+2632}{6(N+1)(N+2)(N+3)(N+4)}\\
&+\frac{(13N+105)}{6}S_1\Biggr)S_3+\frac{(53-N)}{8}S_4+\Biggl(-\frac{6(2N+3)}{(N+1)(N+2)}-6S_1\Biggr)S_{-2,1}\\
&+\Biggl(\frac{12N^5+140N^4+546N^3+725N^2-93N-532}{(N+1)(N+2)(N+4)(N+5)}\\
&-(4+N15)S_1\Biggr)S_{2,1}+(N-11)S_{3,1}+(N+9)S_{2,1,1}\Biggr\}+O(\varepsilon)\ .
\end{aligned}
$$

6. Conclusions

- QCD precision analyses require the description of the heavy quark contributions to 3–loops.
- The 3-loop heavy flavor contributions to $F_2(x, Q^2)$ in the region $Q^2/m^2 \geq 10$ can be computed using a light-cone expansion technique. A large number of Mellin moments has been calculated.
- The remaining problem consists in calculating the massive Wilson coefficients for general values of the Mellin variable N. To achieve this, quite different technologies have to be developed and applied.
- The methods are based on direct Feynman diagram calculation identifying (generalized) hypergeometric structures and include new and advanced summation technologies.
- All logarithmic contributions $\propto \ln^k(\mu^2/m^2), k = 1, 2, 3$ were calculated in Mellin space and will be made available in x-space soon to be implemented into the various 3-loop analysis codes.
- The computation of the 3-loop contributions $\propto T_F^2 N_f C_{A,F}$ to the massive Wilson coefficients for $F_2(x, Q^2)$ will be finished in a few days.
- Further progress has been made in computing massive 3-loop ladder topologies. Here, generalizations of harmonic sums occur. Mathematical tools to deal with these functions were developed.
- We hope to report more at DIS 2011.