Antenna Subtraction for Two Hadronic Initial States at NNLO

Radja Boughezal

ITP, University of Zurich

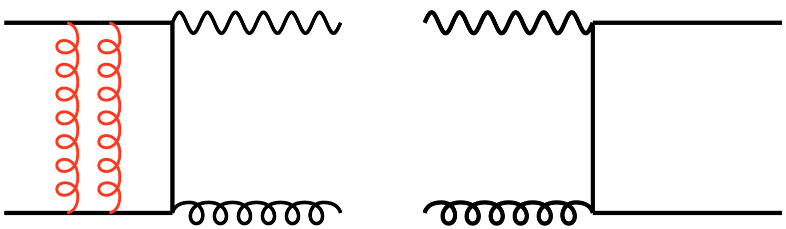
DIS 2010, 20th April, Florence, Italy

In collaboration with:

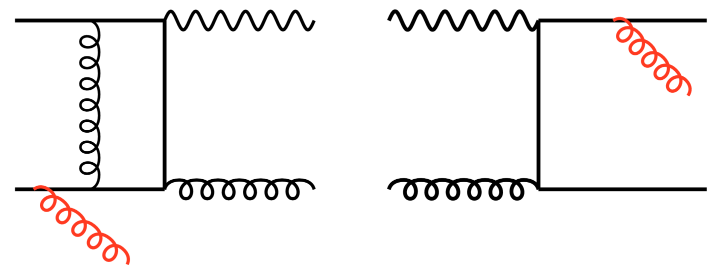
A. Gehrmann-De Ridder & M. Ritzmann

Structure of NNLO Calculations

2-loop matrix elements, m partons



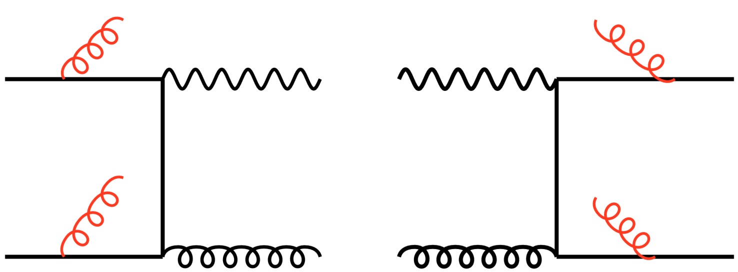
1-loop matrix elements, m+1 partons



- Explicit IR poles from loop integrals

- Explicit IR poles from loops
- Implicit IR poles from single unresolved radiation

Tree level matrix elements, m+2 partons



Implicit IR poles from double unresolved radiation

IR Singularities cancel in the sum of real and virtual contributions and mass factorization counterterm but only after phase space integration for real radiations

NNLO Real Corrections and the IR Singularities Problem

Analytic calculation of phase space is either not possible (jets) or not appropriate (differential cross sections) ⇒ do it numerically but first remove the singularities

Possible Approaches

- Phase space slicing (Giele, Glover, Kosower)
- Sector decomposition → many NNLO results:

  ee → 2 jets (Anastasiou, Melnikov, Petriello)
  ee → 3 jets (Heinrich)
  fully differential Higgs production xsection (Anastasiou, Melnikov, Petriello)
  fully differential W production xsection (Melnikov, Petriello)
  NNLO QED correction to the electron energy spectrum in muon decay (Anastasiou, Melnikov, Petriello)

- Subtraction based methods

Subtraction Methods for IR Singularities

For m-jet cross section @ NLO (Kunszt, Soper)

$$\mathrm{d}\hat{\sigma}_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\hat{\sigma}^{R}_{NLO} - \mathrm{d}\hat{\sigma}^{S}_{NLO}\right) + \int_{\mathrm{d}\Phi_{m}} \left(\int_{1} \mathrm{d}\hat{\sigma}^{S}_{NLO} + \mathrm{d}\hat{\sigma}^{V}_{NLO} + \mathrm{d}\hat{\sigma}^{MF}_{NLO}\right)$$

Finite, can be integrated numerically

Integrated analytically

Subtraction Methods for IR Singularities

For m-jet cross section @ NLO (Kunszt, Soper)

$$d\hat{\sigma}_{NLO} = \int_{d\Phi_{m+1}} \left(d\hat{\sigma}^R_{NLO} - d\hat{\sigma}^S_{NLO}\right) + \int_{d\Phi_m} \left(\int_1 d\hat{\sigma}^S_{NLO} + d\hat{\sigma}^V_{NLO} + d\hat{\sigma}^{MF}_{NLO}\right)$$

Finite, can be integrated numerically

Integrated analytically

No preferred method so far, but fast development ...

- Dipole subtraction (NLO: Catani, Seymour; NNLO: Weinzierl)
- ϵ-prescription (NLO: Frixione, Kunszt, Signer ;
  NNLO: Frixione, Grazzini; Del Duca, Somogyi, Trocsanyi)
- Antenna Subtraction (NLO: D. Kosower, J. Campbell, M. Cullen, N. Glover, A. Daleo, D: Maitre, T. Gehrmann
  NNLO: A. Gehrmann, T. Gehrmann, N. Glover)

Subtraction Methods for IR Singularities

For m-jet cross section @ NLO (Kunszt, Soper)

$$\mathrm{d}\hat{\sigma}_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\hat{\sigma}^{R}_{NLO} - \mathrm{d}\hat{\sigma}^{S}_{NLO}\right) + \int_{\mathrm{d}\Phi_{m}} \left(\int_1 \mathrm{d}\hat{\sigma}^{S}_{NLO} + \mathrm{d}\hat{\sigma}^{V}_{NLO} + \mathrm{d}\hat{\sigma}^{MF}_{NLO}\right)$$

Finite, can be integrated numerically

Integrated analytically

No preferred method so far, but fast development ...

- Dipole subtraction (NLO: Catani, Seymour; NNLO: Weinzierl)
- ϵ-prescription (NLO: Frixione, Kunszt, Signer ; NNLO: Frixione, Grazzini; Del Duca, Somogyi, Trocsanyi)

Antenna Subtraction (NLO: D. Kosower, J. Campbell, Cullen, Glover, Daleo, Maitre, T.Gehrmann NNLO: A. Gehrmann, T. Gehrmann, N. Glover)

First results: ee $\rightarrow$ 2 jets @ NNLO (A. Gehrmann, T. Gehrmann, N. Glover ; S: Weinzierl)

ee $\rightarrow$ 3 jets @ NNLO (A. Gehrmann, T. Gehrmann, N. Glover, G. Heinrich ; S. Weinzierl)

Subtraction Methods

All based on the factorization properties of phase space and matrix elements in soft and collinear limits

$$|M(\ldots,a,b,c,\ldots)|^2 \to P_{abc\to X}|M(\ldots,X,\ldots)|^2 + ang \quad \text{for} \quad a\|b\|c$$

$$|M(\ldots,a,b,c,d,\ldots)|^2 \to S_{abcd}|M(\ldots,a,d,\ldots)|^2 \quad \text{for} \quad b,c\to 0$$

At NNLO there are double and single unresoved configurations

Double unresolved

- triple collinear
- double single collinear
- soft-collinear
- double soft

Single unresolved

- soft
- collinear

Subtraction Methods

All based on the factorization properties of phase space and matrix elements in soft and collinear limits

$$|M(\ldots,a,b,c,\ldots)|^2 \to P_{abc\to X}|M(\ldots,X,\ldots)|^2 + ang \quad \text{for} \quad a\|b\|c$$

$$|M(\ldots,a,b,c,d,\ldots)|^2 \to S_{abcd}|M(\ldots,a,d,\ldots)|^2 \quad \text{for} \quad b,c\to 0$$

At NNLO there are double and single unresoved configurations

Double unresolved

- triple collinear
- double single collinear
- soft-collinear
- double soft

Single unresolved

- soft
- collinear

Idea: construct subtraction terms that

- Approximate the m+2 matrix elements in all singular limits
- Are sufficiently simple to be integrated analytically

Antenna Subtraction @ NNLO

Structure of NNLO m-jet cross section:

$$\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO} = &\int_{\mathrm{d}\Phi_{m+2}} \left(\mathrm{d}\hat{\sigma}^{R}_{NNLO} - \mathrm{d}\hat{\sigma}^{S}_{NNLO}\right) + \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{S}_{NNLO} \\
&+ \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\hat{\sigma}^{V,1}_{NNLO} - \mathrm{d}\hat{\sigma}^{VS,1}_{NNLO}\right) + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{VS,1}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{MF,1}_{NNLO} \\
&+ \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{V,2}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{MF,2}_{NNLO}.
\end{aligned}$$

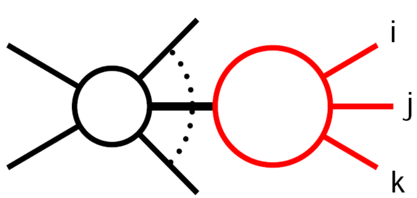
$d\sigma^{S}_{NNLO}$: real radiation subtraction term for $d\sigma^{R}_{NNLO}$

$d\sigma^{VS,1}_{NNLO}$: One-loop virtual subtraction term for $d\sigma^{V,1}_{NNLO}$

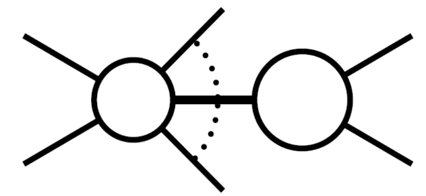
$d\sigma^{V,2}_{NNLO}$: two-loop virtual corrections

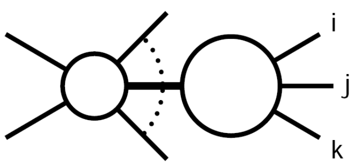
Each of the differences above is finite and can be integrated numerically

Antenna Subtraction: building block @ NLO

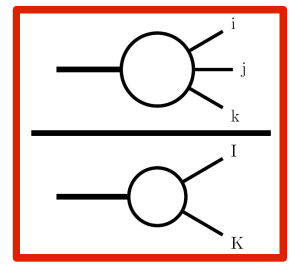


→ X^0_{ijk} ×

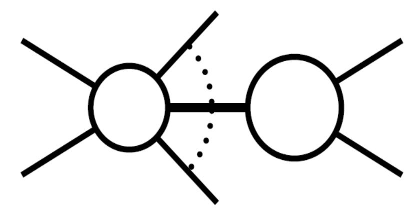




→



×



$$X^0_{ijk} = S_{ijk,IK}\frac{|M^0_{ijk}|^2}{|M^0_{IK}|^2}$$

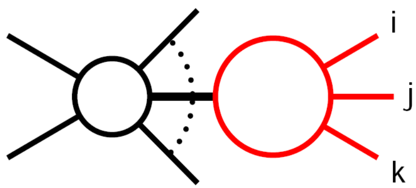
&

$$\mathrm{d}\Phi_{X_{ijk}} = \frac{\mathrm{d}\Phi_3}{P_2}$$

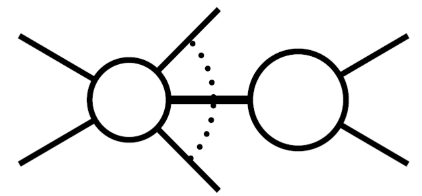
- Antenna functions contain all singular configurations of parton j emitted between two hard color-connected partons i & k
- An appropriate mapping of momenta $\{p_i, p_j, p_k\} \rightarrow \{p_I, p_K\}$ leads to the factorization of the phase space

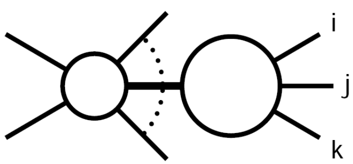
$$\sum_{m+1} d\phi_{m+1}\left|M_{m+1}\right|^2 J_m^{(m+1)} \rightarrow \sum_{m+1} d\phi_m\left|M_m\right|^2 J_m^{(m)} \sum_j d\phi_{X^0_{ijk}} X^0_{ijk}$$

Antenna Subtraction: building block @ NLO

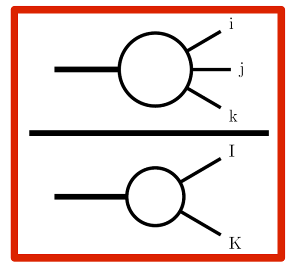


$$\rightarrow \quad X^0_{ijk} \quad \times$$

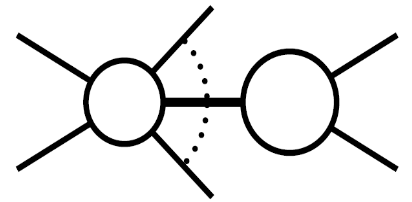




$$\rightarrow$$



$$\times$$



$$X^0_{ijk} = S_{ijk,IK} \frac{|M^0_{ijk}|^2}{|M^0_{IK}|^2}$$

&

$$d\Phi_{X_{ijk}} = \frac{d\Phi_3}{P_2}$$

$$\sum_{m+1} \int d\phi_{m+1} |M_{m+1}|^2 J_m^{(m+1)} \rightarrow \sum_{m+1} \int d\phi_m |M_m|^2 J_m^{(m)} \int \sum_j d\phi_{X^0_{ijk}} X^0_{ijk}$$

Integrated analytically
explicit ϵ-poles cancel the loop ones

Momenta mapping must be such that:

$$p_i + p_j + p_k = p_I + p_K$$

$$p_I^2 = 0, \; p_K^2 = 0$$

Observables do not depend on individual monmenta p_i, p_j, p_k

Antenna Functions @ NNLO

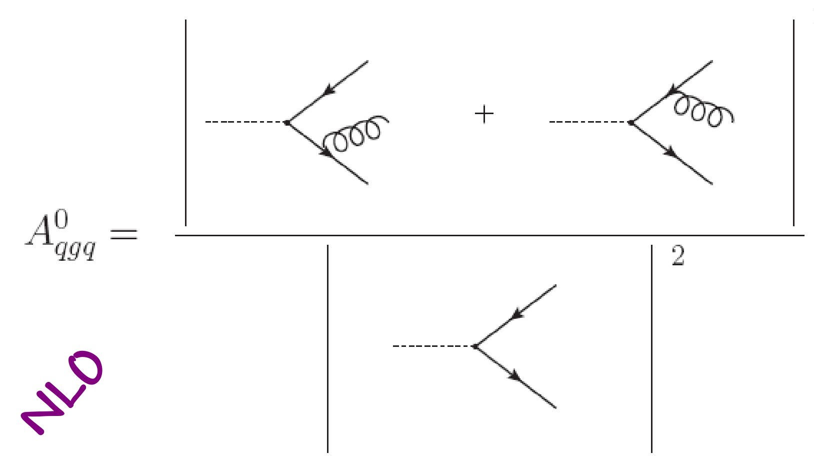
Antenna functions: derived from physical matrix elements normalized to two-parton matrix elements

$q\bar{q}$ from $\gamma \to q\,\bar{q}$

$q\,g$ from $\tilde{\chi} \to \tilde{g}\,g$

$g\,g$ from $H \to g\,g$

eg.



They refer to colour-ordered pair of hard partons

$q\bar{q}$, $q\,g$, $g\,g$ with radiations in between

NNLO: two unresolved partons (real or virtual)

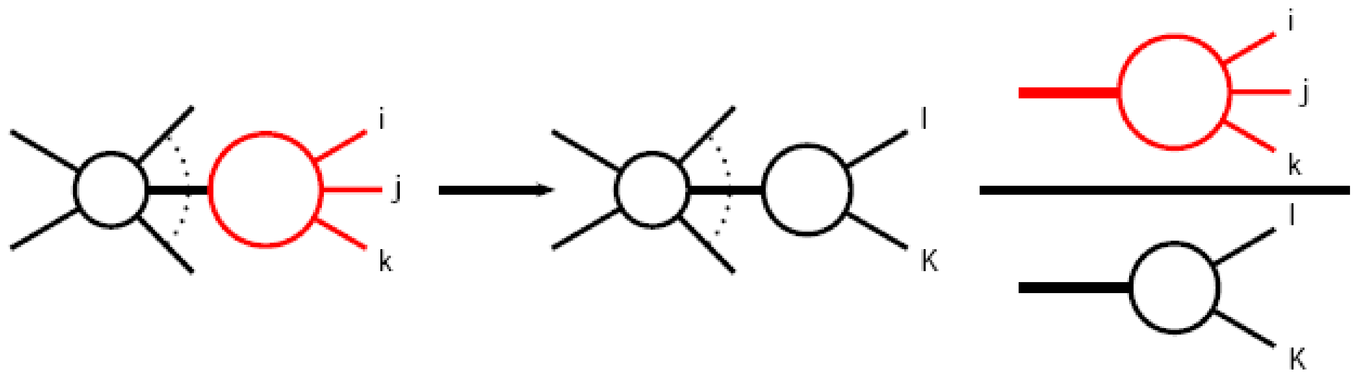
→ tree level four-parton antenna

→ one-loop three parton antenna

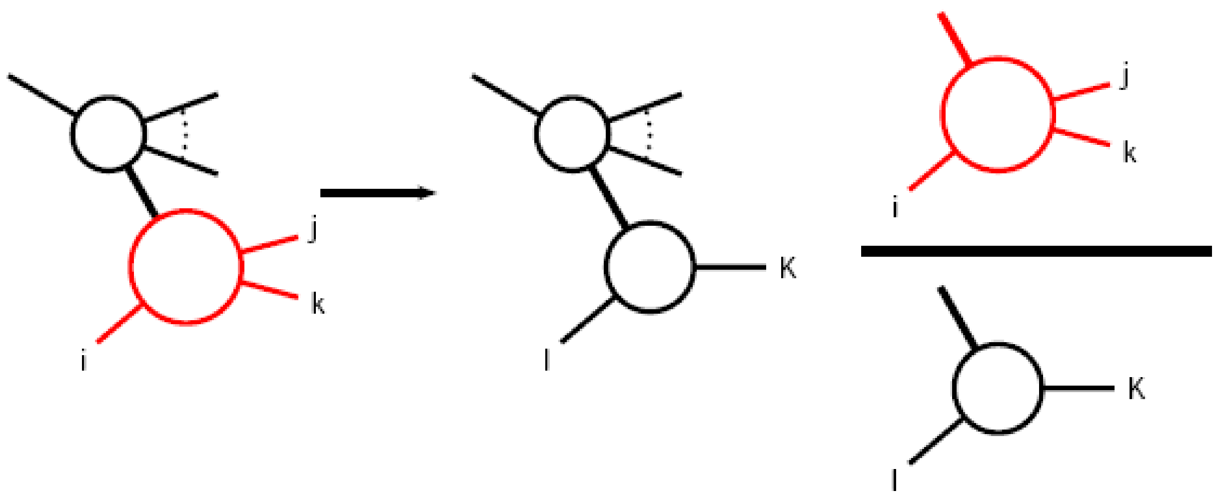
→ product of two three-parton antenna

Possible Configurations for Two Unresolved Partons

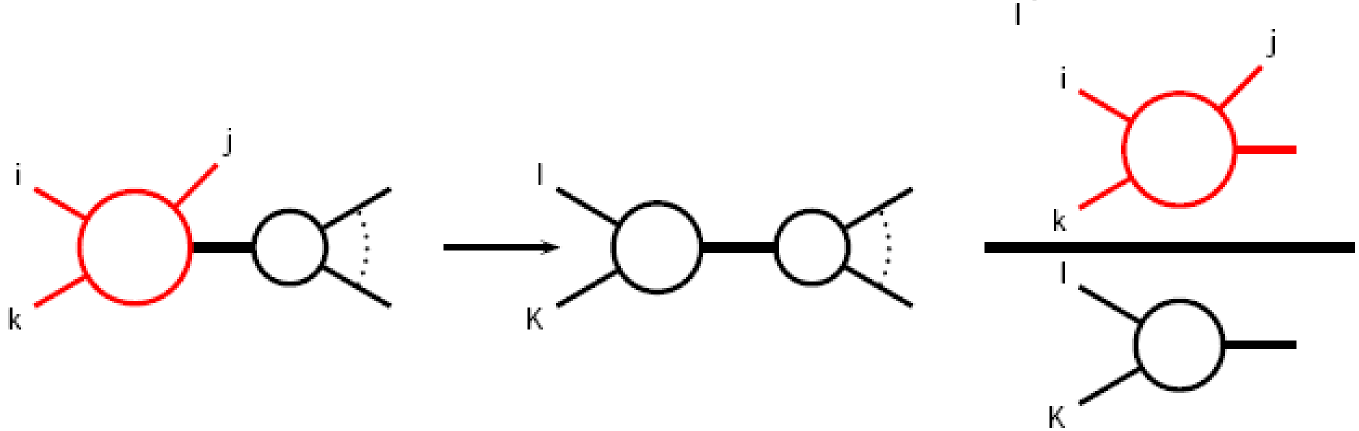
Pictures correspond to NLO for simplicity



Final-final antenna



Initial-final antenna



Initial-initial antenna

Integrated Subtracted Terms for Two Unresolved Partons

- Final-final antenna: A. Gehrmann, T. Gehrmann, N. Glover

  4 master integrals: cut three-loop self energies, 1 scale

  applied to ee → 3jets A. Gehrmann, T. Gehrmann, N. Glover, G. Heinrich; S. Weinzierl

- Initial-final antenna: A. Daleo, A.. Gehrmann, T. Gehrmann, G. Luisoni

  about 10 master integrals: cut two-loop boxes, 1 scale

  Sufficient for DIS (2+1)-jet production

- Initial-initial antenna: R. B, A. Gehrmann-De Ridder, M. Ritzmann

  about 30 master integrals: cut two-loop boxes, 2 scales

  Required for any process with two hadronic initial states, eg. V+j

Integrated Subtracted Terms for Two Unresolved Partons

- Final-final antenna: A. Gehrmann, T. Gehrmann, N. Glover

  4 master integrals: cut three-loop self energies, 1 scale

  applied to ee → 3jets A. Gehrmann, T. Gehrmann, N. Glover, G. Heinrich; S. Weinzierl

- Initial-final antenna: A. Daleo, A.. Gehrmann, T. Gehrmann, G. Luisoni

  about 10 master integrals: cut two-loop boxes, 1 scale

  Sufficient for DIS (2+1)-jet production

- Initial-initial antenna: R. B, A. Gehrmann-De Ridder, M. Ritzmann

  about 30 master integrals: cut two-loop boxes, 2 scales

  Required for any process with two hadronic initial states, eg. V+j

This talk

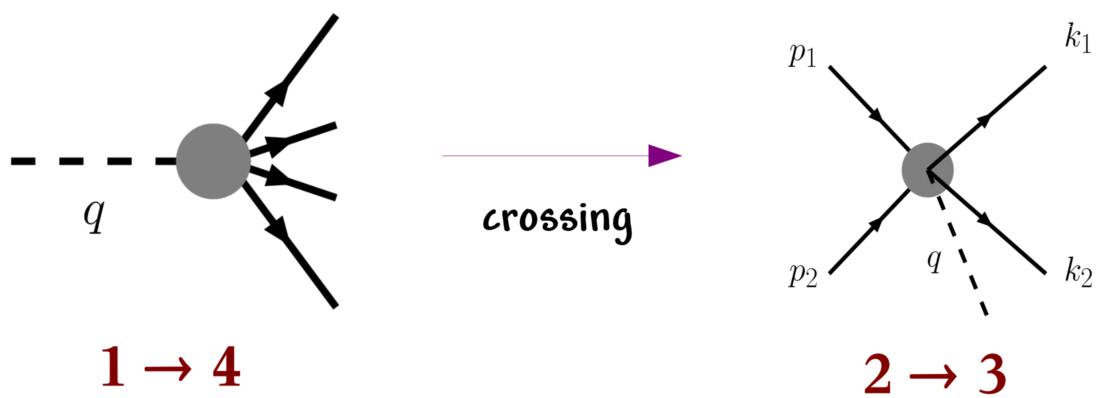
Initial-initial Antenna Functions: Double Real Radiation $2 \to 3$

- Obtain antenna functions for double real radiation by crossing $1 \to 4$ NNLO antenna

  each final-final antenna produces 6 initial-initial antennae

  depending on symmetries of the antenna, some of the 6 antennae can be identical

Kinematics:



$$p_1 + p_2 \to q + k_1 + k_2$$

with $q^2 > 0$

Phase space factorization (Daleo, Gehrmann, Maitre):

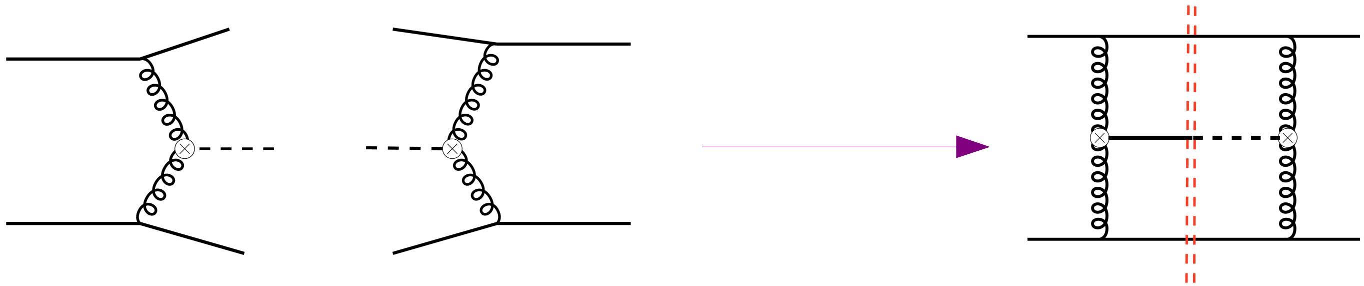
$$d\,\phi_{m+2}(k_1, \ldots, k_{m+2}; p_1, p_2) = d\,\phi_m(\tilde{k}_1, \ldots, \tilde{k}_i, \tilde{k}_l, \ldots, \tilde{k}_{m+2}; x_1\, p_1, x_2\, p_2)$$
$$[dk_j][dk_k]\, d\,x_1\, d\,x_2\, J\, \delta(q^2 - x_1\, x_2\, s_{12})\, \delta(2(x_2\, p_2 - x_1\, p_1)\cdot q)$$

Factorization achieved with the Lorentz boost: $q \to \tilde{q} = x_1\, p_1 + x_2\, p_2\, ;\ k \to \tilde{k}$

With: $$J = s_{12}(x_1(s_{12} - s_{1j} - s_{1k}) + x_2(s_{12} - s_{2j} - s_{2k}))$$

Initial-initial Antenna Functions: Double Real Radiation $2 \to 3$

Integration: inclusive three-particle phase space integrals with q^2, x_1, x_2 fixed.



Map phase space integrals into cut loop integrals using unitarity (Anastasiou, Melnikov)

Cutkosky rules: $$\delta\left(q_i^2 - m_i^2\right) \Rightarrow \frac{1}{q_i^2 - m_i^2 - i\epsilon} - \frac{1}{q_i^2 - m_i^2 + i\epsilon}$$

Apply to $$\delta\left(q^2 - x_1 x_2 s_{12}\right),\ \delta\left(2\left(x_2 p_2 - x_1 p_1\right)\cdot q\right)$$

- Mass-shell conditions for auxiliary propagators ⇒ constraints on the phase space
- Use integration by parts identities and Laporta algorithm to reduce all phase space integrals into a small set of master integrals: ~ 30

Initial-initial Antenna Functions: Double Real Radiation $2 \to 3$

$$x_1 = \left(\frac{s_{12} - s_{j2} - s_{k2}}{s_{12}} \, \frac{s_{12} - s_{1j} - s_{1k} - s_{j2} - s_{k2} + s_{jk}}{s_{12} - s_{1j} - s_{1k}} \right)^{\frac{1}{2}}$$

$$x_2 = \left(\frac{s_{12} - s_{1j} - s_{1k}}{s_{12}} \, \frac{s_{12} - s_{1j} - s_{1k} - s_{j2} - s_{k2} + s_{jk}}{s_{12} - s_{j2} - s_{k2}} \right)^{\frac{1}{2}}$$

In double unresolved case, x1 and x2 satisfy the limits:

i) j and k soft: $x_1 \to 1$, $x_2 \to 1$,

ii) j soft and $k_k = z_1 p_1 \parallel p_1$: $x_1 \to 1 - z_1$, $x_2 \to 1$,

iii) $k_j = z_1 p_1 \parallel p_1$ and $k_k = z_2 p_2 \parallel p_2$: $x_1 \to 1 - z_1$, $x_2 \to 1 - z_2$,

iv) $k_j + k_k = z_1 p_1 \parallel p_1$: $x_1 \to 1 - z_1$, $x_2 \to 1$,

$+ \quad p_1 \leftrightarrow p_2, \quad k_j \leftrightarrow k_k$

Masters have to be calculated in three regions of phase space:

- $x_1 \neq 1,\ x_2 \neq 1$
- $x_1 \neq 1,\ x_2 = 1 \parallel x_1 = 1,\ x_2 \neq 1$
- $x_1 = 1,\ x_2 = 1$

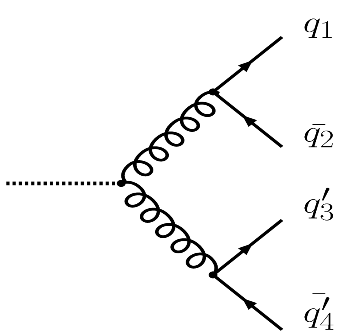
The B_4^0 , H_4^0 , $\tilde{E}_4^0$ Antennae

$B_4^0(q, q', \bar{q}', \bar{q})$ collapses to the hard partons $q\,\bar{q}$

$\tilde{E}_4^0(q, q', \bar{q}', g)$ collapses to the hard partons $q\,g$

$H_4^0(q, \bar{q}, q', \bar{q}')$ collapses to the hard partons $g\,g$

Singularities taken care of by these antennae (final-final as example)



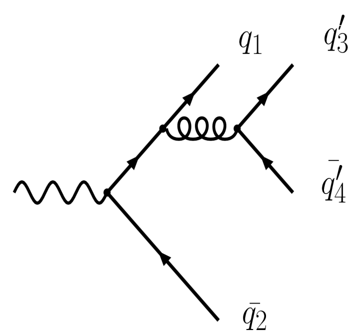
H_4^0

single unresolved:

1 || 2 or 3 || 4

double unresolved:

1 || 2 & 3 || 4



$+ \; q_1 \leftrightarrow \bar{q}_2$

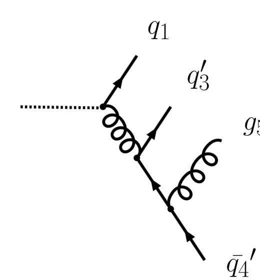
B_4^0

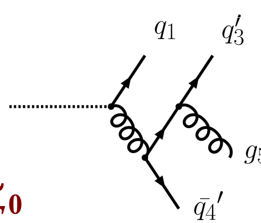
single unresolved:

3 || 4

double unresolved:

1 || 3 || 4
or
2 || 3 || 4
or
$3 \longrightarrow 0$, $4 \longrightarrow 0$





$\tilde{E}_4^0$

single unresolved:

5 || 3 or 5 || 4
or
$5 \longrightarrow 0$

double unresolved:

3 || 4 || 5

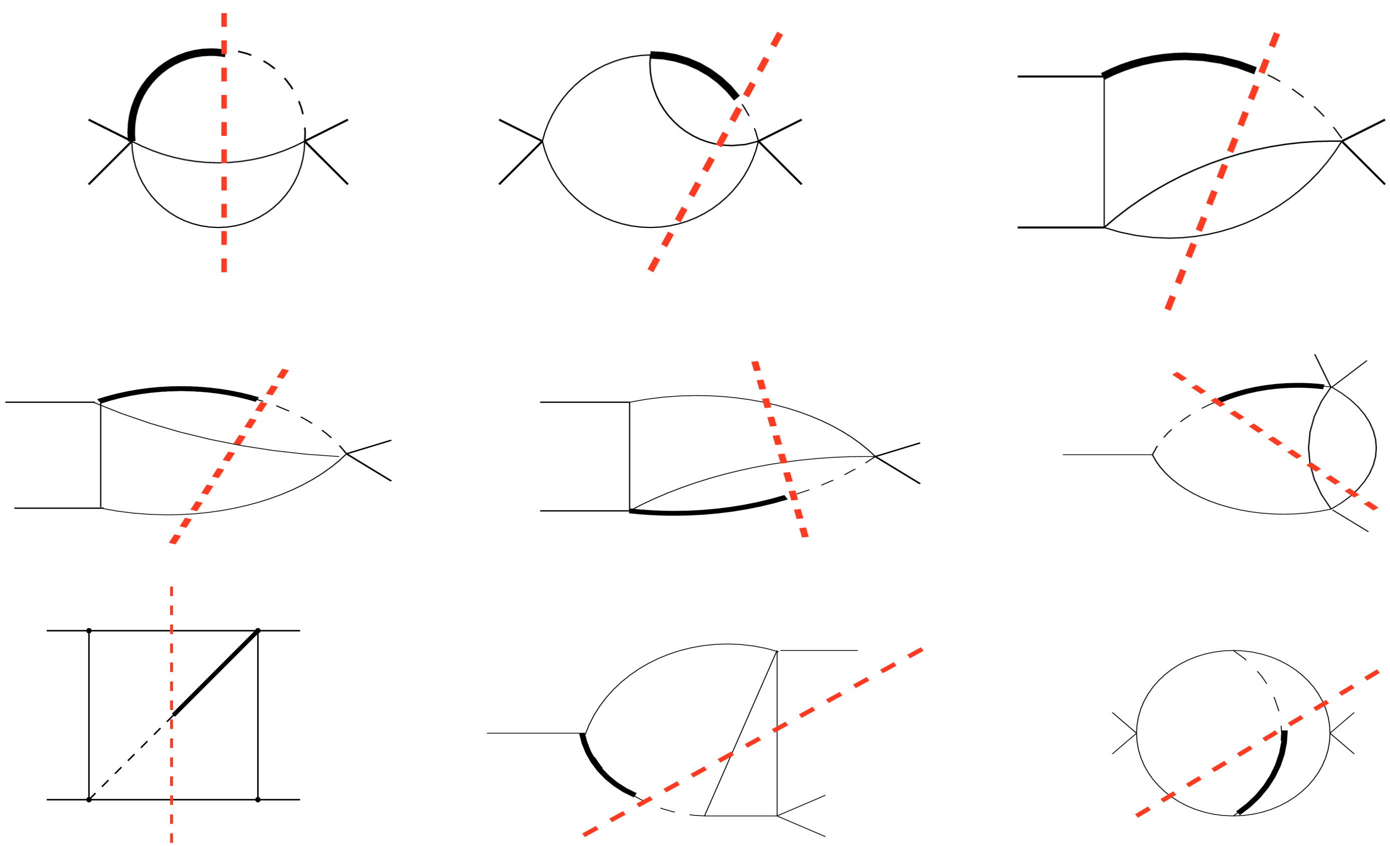
The B_4^0 , H_4^0 , $\tilde{E}_4^0$ Antennae

$B_4^0(q, q', \bar{q}', \bar{q})$ collapses to the hard partons $q\,\bar{q}$

$\tilde{E}_4^0(q, q', \bar{q}', g)$ collapses to the hard partons $q\,g$

$H_4^0(q, \bar{q}, q', \bar{q}')$ collapses to the hard partons $g\,g$

13 masters are involved in the calculation of $B_4^0, H_4^0, \tilde{E}_4^0$, only scalar ones are shown



The B_4^0 , H_4^0 , $\tilde{E}_4^0$ Antennae

- Compute the master integrals analytically using differential equations and a basis of generalized harmonic polylogarithms (GHPLs) of dimension two

- Boundaries obtained by calculating the master integrals in the soft limit $x1 \to 1$, $x2 \to 1$

$x_1 \neq 1, x_2 \neq 1$ → Completed

GHPLs weight 2, epsilon expansion of masters at most up to ϵ^2

$x_1 \neq 1, x_2 = 1 \parallel x_1 = 1, x_2 \neq 1$

HPLs weight 3, epsilon expansion of masters at most up to ϵ^3

$x_1 = 1, x_2 = 1$

epsilon expansion of masters at most up to ϵ^4

} Easier than first two regions, and is underway

Checked that all initial-initial antennae reproduce the splitting functions and eikonal factors in collinear and soft limits. (Campbell, Glover; Catani, Grazzini)

Initial-initial Antenna Functions: Real Radiation at One-loop 2 →2

- Obtain antenna functions by crossing one-loop 1 → 3 NNLO antennae
- Kinematics: $p_1 + p_2 \to k_1 + q$ with $q^2 > 0$
- Phase space factorization (Daleo, Maitre, Gehrmann):

$$
\begin{aligned}
\mathrm{d}\Phi_{m+1}(k_1,\ldots,k_{m+1};p_1,p_2) &= \mathrm{d}\Phi_m(\tilde{k}_1,\ldots,\tilde{k}_i,\tilde{k}_k,\ldots,\tilde{k}_{m+1};x_1p_1,x_2p_2) \\
&\quad \delta(x_1-\hat{x}_1)\,\delta(x_2-\hat{x}_2)\,[\mathrm{d}k_j]\,\mathrm{d}x_1\,\mathrm{d}x_2
\end{aligned}
$$

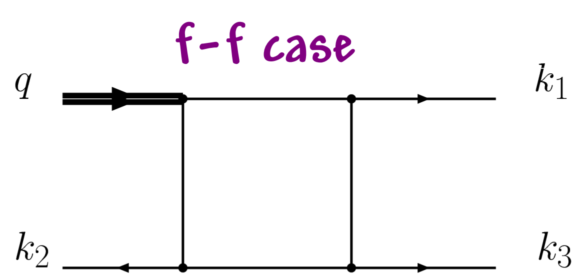
$$
\hat{x}_1 = \left(\frac{s_{12}-s_{j2}}{s_{12}}\,\frac{s_{12}-s_{1j}-s_{j2}}{s_{12}-s_{1j}}\right)^{\frac{1}{2}}
$$

$$
\hat{x}_2 = \left(\frac{s_{12}-s_{1j}}{s_{12}}\,\frac{s_{12}-s_{1j}-s_{j2}}{s_{12}-s_{j2}}\right)^{\frac{1}{2}}
$$

Easier than the double-unresolved radiation case since the phase space is over constrained

Initial-initial Antenna Functions: Real Radiation at One-loop 2 → 2

- Obtain antenna functions by crossing one-loop 1 → 3 NNLO antennae
- Kinematics: $p_1 + p_2 \to k_1 + q$ with $q^2 > 0$
- One-loop 2 → 2: box integrals already known for the final-final case where all the invariants $0 < s_{ij} < 1$



$q^2 = s_{12} + s_{13} + s_{23}$ and $y_{ij} = s_{ij}/q^2$

$$
\begin{aligned}
&\mathrm{Box}(s_{13}, s_{23}, s_{12}) \\
&= \left(\frac{(4\pi)^\epsilon}{16\pi^2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right)(-q^2)^{-2-\epsilon}\frac{2i}{\epsilon^2}\frac{1}{y_{13}y_{23}} \\
&\times\left[\left(\frac{y_{13}y_{23}}{1-y_{13}}\right)^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{1-y_{13}-y_{23}}{1-y_{13}}\right)\right. \\
&+\left(\frac{y_{13}y_{23}}{1-y_{23}}\right)^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{1-y_{13}-y_{23}}{1-y_{23}}\right) \\
&\left.-\left(\frac{y_{13}y_{23}}{(1-y_{13})(1-y_{23})}\right)^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;\frac{1-y_{13}-y_{23}}{(1-y_{13})(1-y_{23})}\right)\right],
\end{aligned}
$$

For initial-initial case: cross two legs to the initial state and q to final state

- two of the invariants sij become negative
- analytic continuation of 2F1 is needed as well as extraction of end point singularities

Conclusions and Outlook

First analytical results for the integrated initial-initial antennae:

the $B_4^0(q, q', \bar{q}', \bar{q})$, $H_4^0(q, \bar{q}, q', \bar{q}')$, $\tilde{E}_4^0(q, q', \bar{q}', g)$

collapse to the hard partons: $q\bar{q}$, gg, qg respectively

Next to do:

- Complete the set of the integrated initial-initial antennae
- Cross check of initial-initial antennae with NNLO Drell-Yan coefficient functions

Potential applications:

V+jet, pp $\rightarrow$ 2 jets, W-pair production

Backup slides

Antenna Functions @ NNLO

Antenna functions: derived from physical matrix elements normalized to two-parton matrix elements

$$q\,\bar{q} \quad \text{from} \quad \gamma \to q\,\bar{q} + X$$
$$q\,g \quad \text{from} \quad \tilde{\chi} \to \tilde{g}\,g + X$$
$$g\,g \quad \text{from} \quad H \to g\,g + X$$

	tree level	one loop
quark-antiquark		
$qg\bar{q}$	$A_3^0(q,g,\bar{q})$	$A_3^1(q,g,\bar{q})$, $\tilde{A}_3^1(q,g,\bar{q})$, $\hat{A}_3^1(q,g,\bar{q})$
$qgg\bar{q}$	$A_4^0(q,g,g,\bar{q})$, $\tilde{A}_4^0(q,g,g,\bar{q})$	
$qq'\bar{q}'\bar{q}$	$B_4^0(q,q',\bar{q}',\bar{q})$	
$qq\bar{q}\bar{q}$	$C_4^0(q,q,\bar{q},\bar{q})$	
quark-gluon		
qgg	$D_3^0(q,g,g)$	$D_3^1(q,g,g)$, $\hat{D}_3^1(q,g,g)$
$qggg$	$D_4^0(q,g,g,g)$	
$qq'\bar{q}'$	$E_3^0(q,q',\bar{q}')$	$E_3^1(q,q',\bar{q}')$, $\tilde{E}_3^1(q,q',\bar{q}')$, $\hat{E}_3^1(q,q',\bar{q}')$
$qq'\bar{q}'g$	$E_4^0(q,q',\bar{q}',g)$, $\tilde{E}_4^0(q,q',\bar{q}',g)$	
gluon-gluon		
ggg	$F_3^0(g,g,g)$	$F_3^1(g,g,g)$, $\hat{F}_3^1(g,g,g)$
$gggg$	$F_4^0(g,g,g,g)$	
$gq\bar{q}$	$G_3^0(g,q,\bar{q})$	$G_3^1(g,q,\bar{q})$, $\tilde{G}_3^1(g,q,\bar{q})$, $\hat{G}_3^1(g,q,\bar{q})$
$gq\bar{q}g$	$G_4^0(g,q,\bar{q},g)$, $\tilde{G}_4^0(g,q,\bar{q},g)$	
$q\bar{q}q'\bar{q}'$	$H_4^0(q,\bar{q},q',\bar{q}')$	

A. Gehrmann-De Ridder, T. Gehrmann, N. Glover

$$X_4^0(i,j,k,l) \quad 1 \to 4$$
$$X_3^1(i,j,k) \quad 1 \to 3 \quad \text{1-loop}$$

NNLO unintegrated antennae