# High-Energy Amplitudes and Impact Factors at next-to-leading order

G. A. Chirilli

LPT-Orsay& CPHT-Polytechnique

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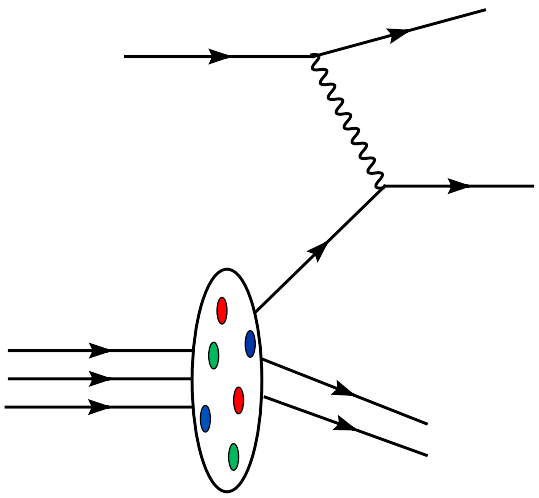
# Outline

- Light-cone OPE versus OPE in color dipoles.
- High-energy scattering and Wilson lines.
- Factorization in rapidity: Feynman diagrams in a shock-wave background.
- NLO Photon Impact Factor
- Leading order and NLO BK equation.
- Conclusions.
- Outlook.

# Incoherent-vs-Coherent

Incoherent Interactions



Bjorken Limit

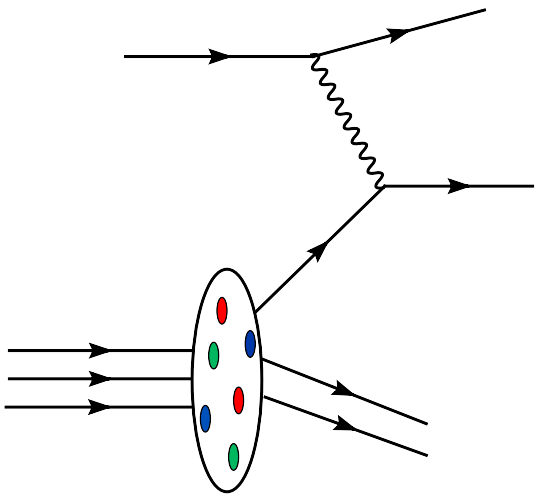
$$Q^2 \to \infty, \; s \to \infty$$

$$x_{\mathrm{B}} = \frac{Q^2}{s} \quad \text{fixed}$$

$$\text{resum} \quad \alpha_s \ln \frac{Q^2}{\Lambda_{\mathrm{QCD}}}$$

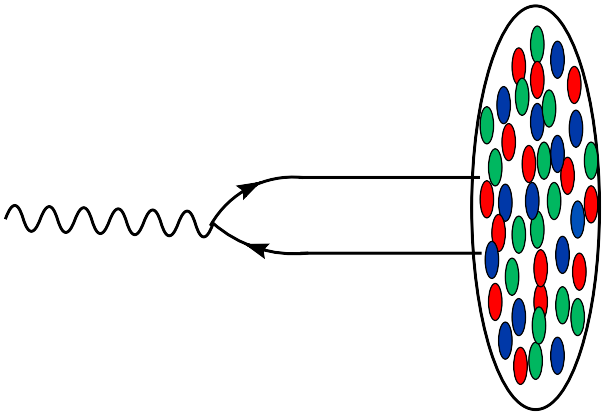
# Incoherent-vs-Coherent

Incoherent Interactions



Coherent Interactions

VS.



Bjorken Limit

$$Q^2 \to \infty, \; s \to \infty$$

$$x_B = \frac{Q^2}{s} \quad \text{fixed}$$

resum $\alpha_s \ln \frac{Q^2}{\Lambda_{QCD}}$

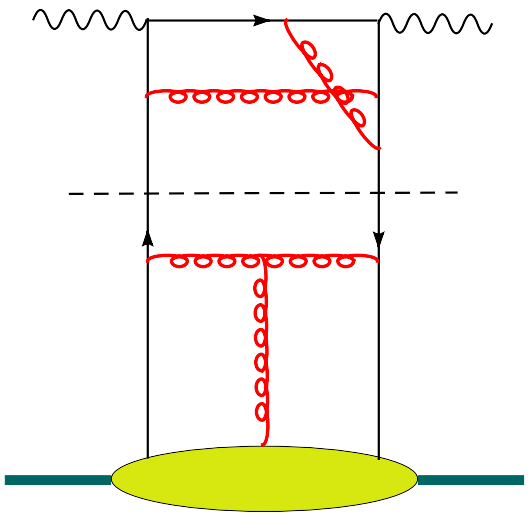
Regge Limit

$$Q^2 \;\; \text{fixed}, \; s \to \infty$$

$$x_B = \frac{Q^2}{s} \to 0$$

resum $\alpha_s \ln \frac{1}{x_B}$

# Light-cone expansion and DGLAP evolution in the NLO



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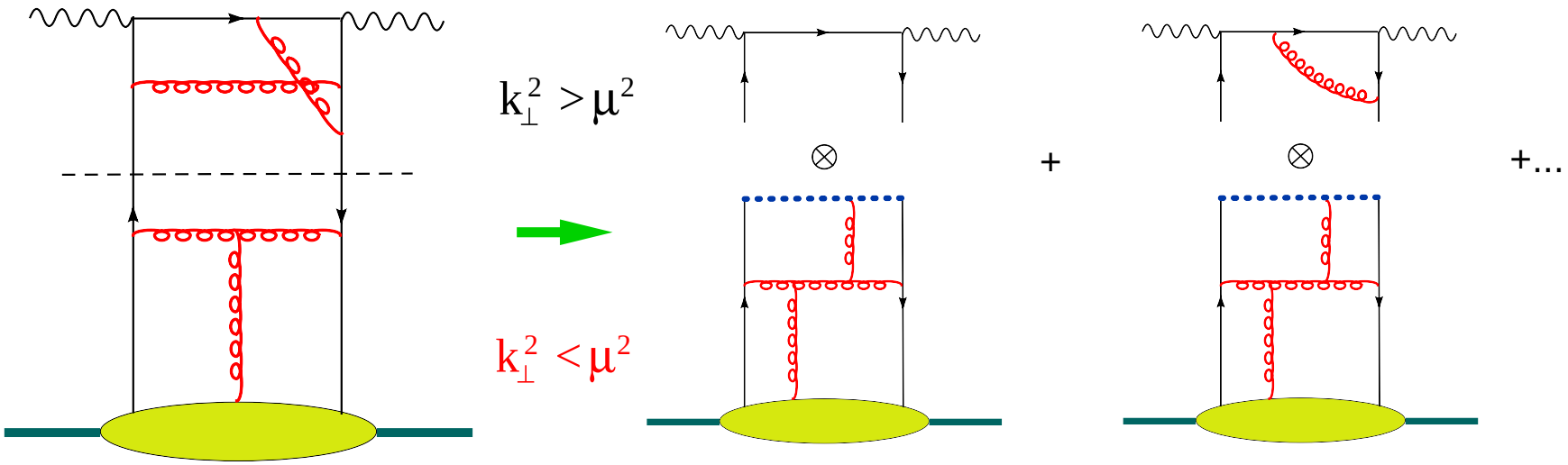
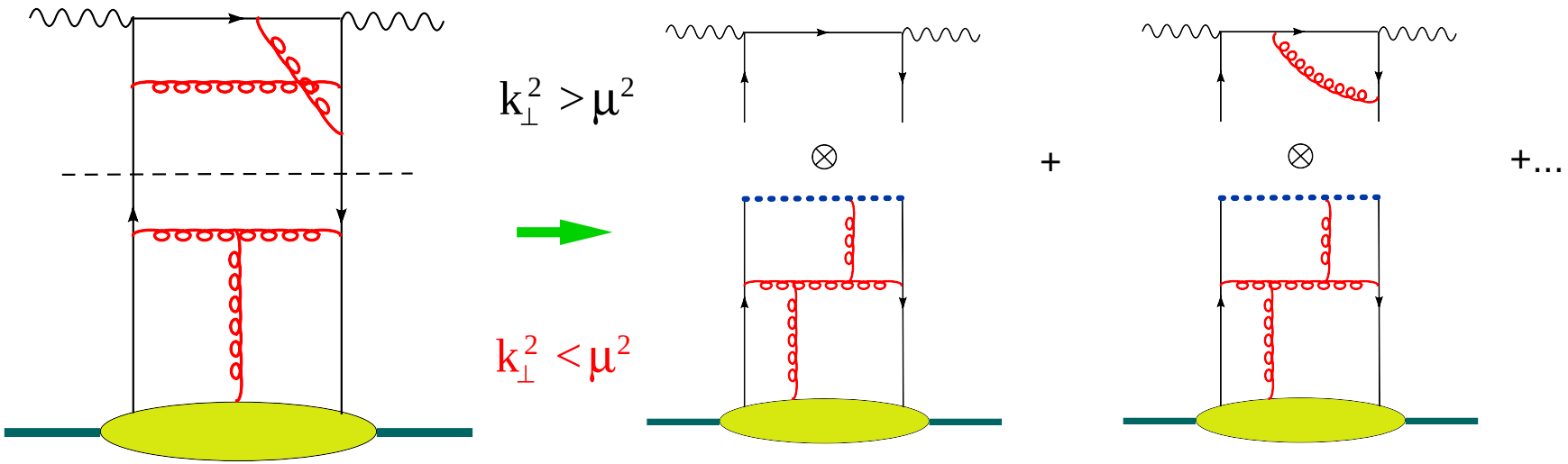



$\mu^2$ - factorization scale (normalization point)

$k_\perp^2 > \mu^2$ - coefficient functions

$k_\perp^2 < \mu^2$ - matrix elements of light-ray operators (normalized at $\mu^2$)

# Light-cone expansion and DGLAP evolution in the NLO




$\mu^2$ - factorization scale (normalization point)

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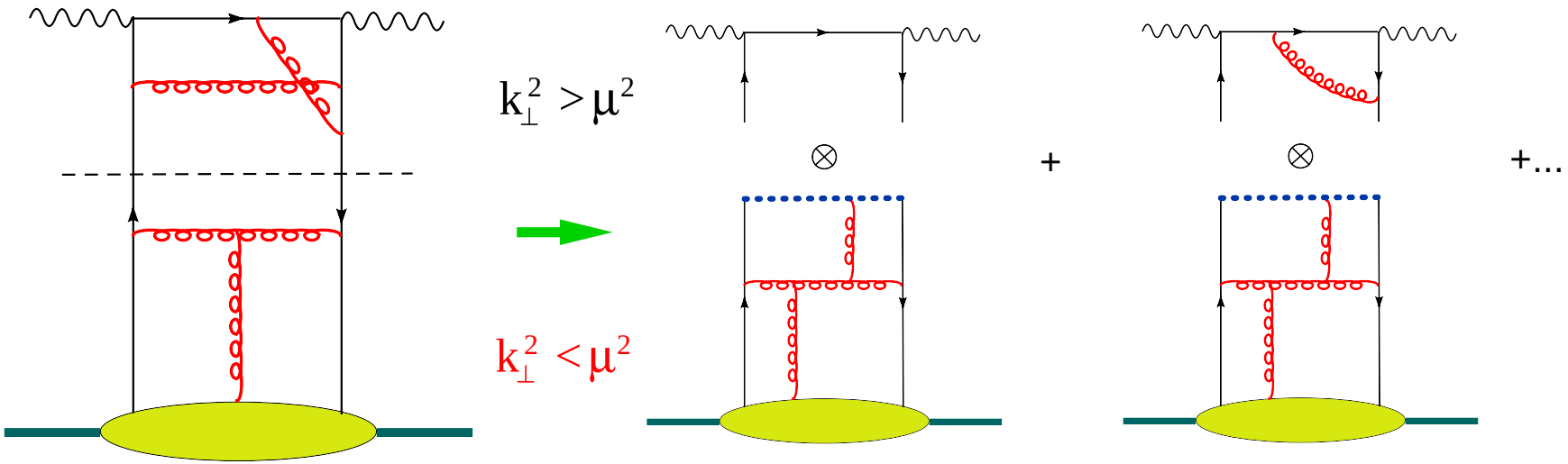
$k_\perp^2 < \mu^2$ - matrix elements of light-ray operators (normalized at $\mu^2$)

OPE in light-ray operators $\qquad (x-y)^2 \to 0$

$$T\{j_\mu(x) j_\nu(y)\} = \frac{(x-y)_\xi}{2\pi^2 (x-y)^4}\left[1 + \frac{\alpha_s}{\pi}(\ln(x-y)^2\mu^2 + C)\right]\bar{\psi}(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y)$$

$$[x,y] \equiv Pe^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)}$$ - gauge link

# Light-cone expansion and DGLAP evolution in the NLO




$\mu^2$ - factorization scale (normalization point)

$k_\perp^2 > \mu^2$ - coefficient functions

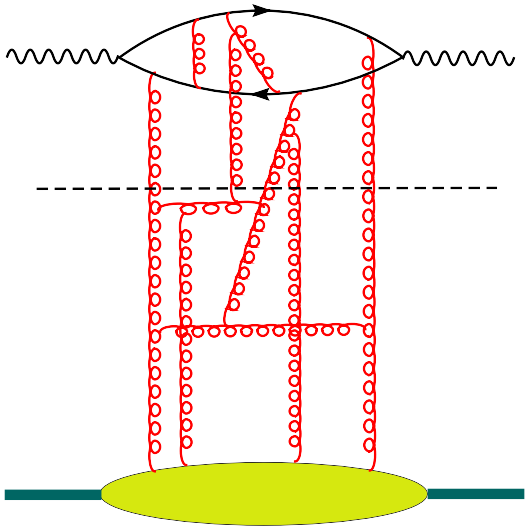
$k_\perp^2 < \mu^2$ - matrix elements of light-ray operators (normalized at $\mu^2$)

Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities

$(x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) = K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar{\psi}(x)[x,y]\psi(y)$$

# High-energy expansion in color dipoles in the NLO



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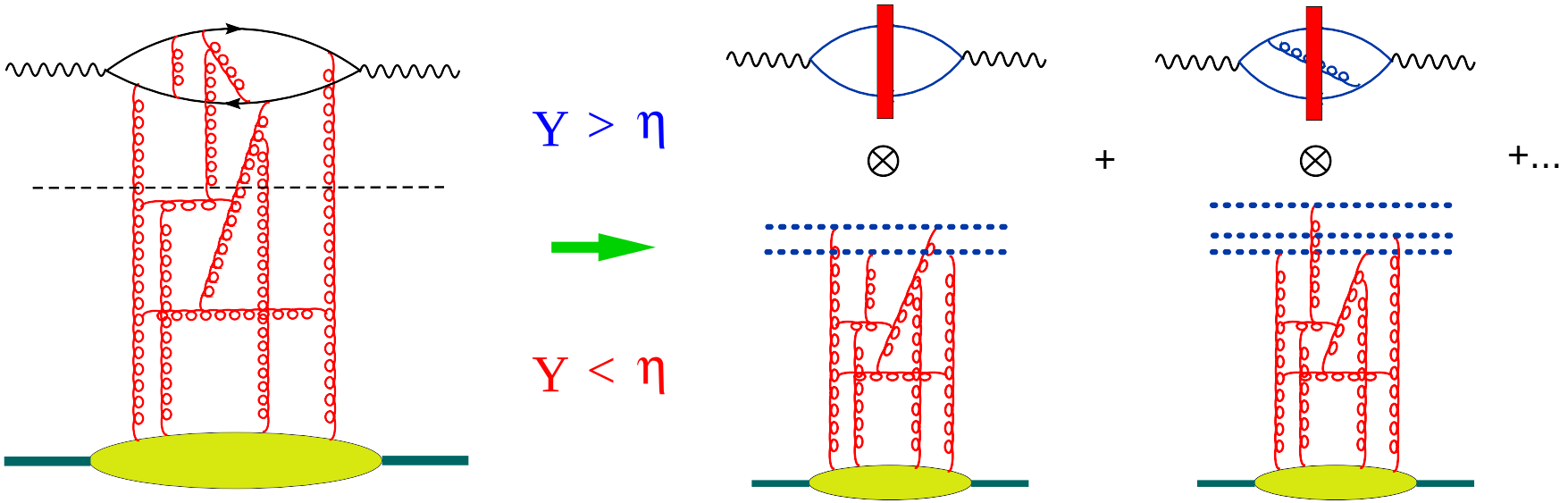

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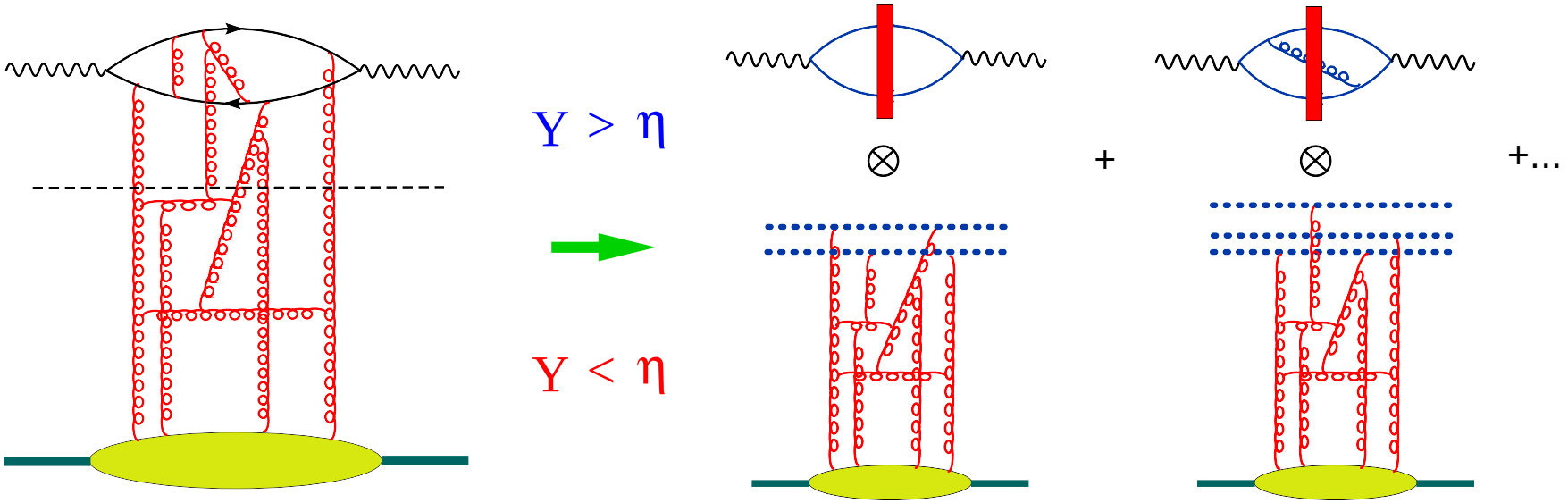



$\eta$ - rapidity factorization scale

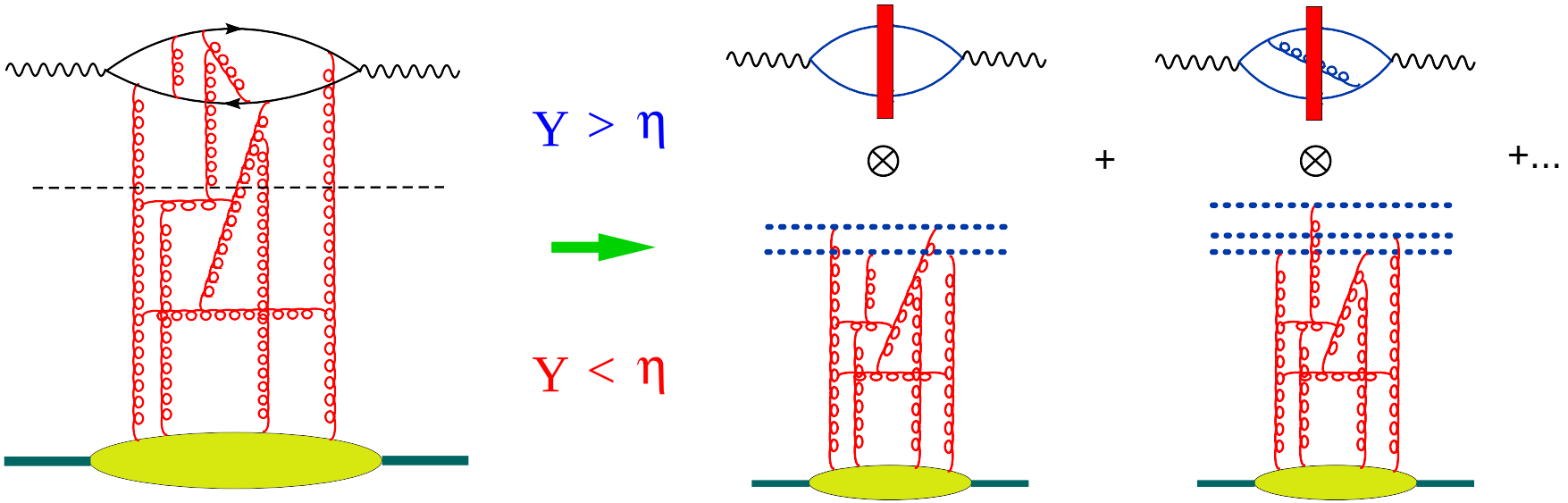
Rapidity Y > $\eta$ - coefficient function ("impact factor")

Rapidity Y < $\eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by $\eta$

$$U_x^\eta = \mathrm{Pexp}\left[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

# High-energy expansion in color dipoles in the NLO




The high-energy operator expansion is

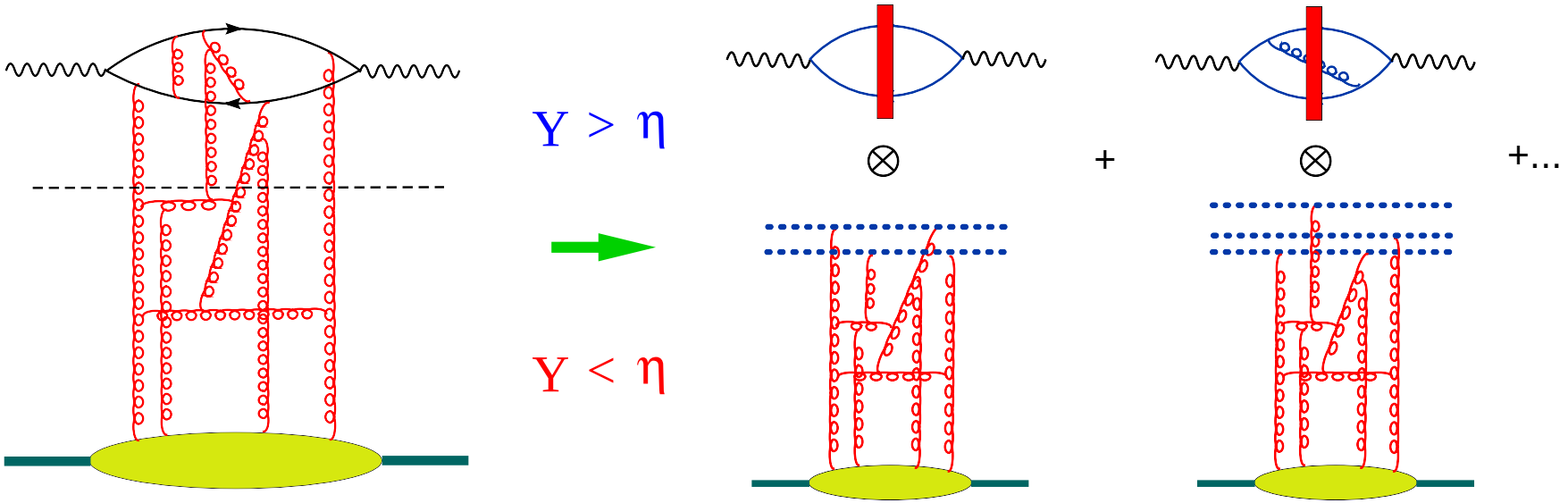
$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

In the leading order the impact factor is Möbius invariant

In the NLO one should also expect conf. invariance since $I^{\rm NLO}_{\mu\nu}$ is given by tree diagrams

# High-energy expansion in color dipoles in the NLO



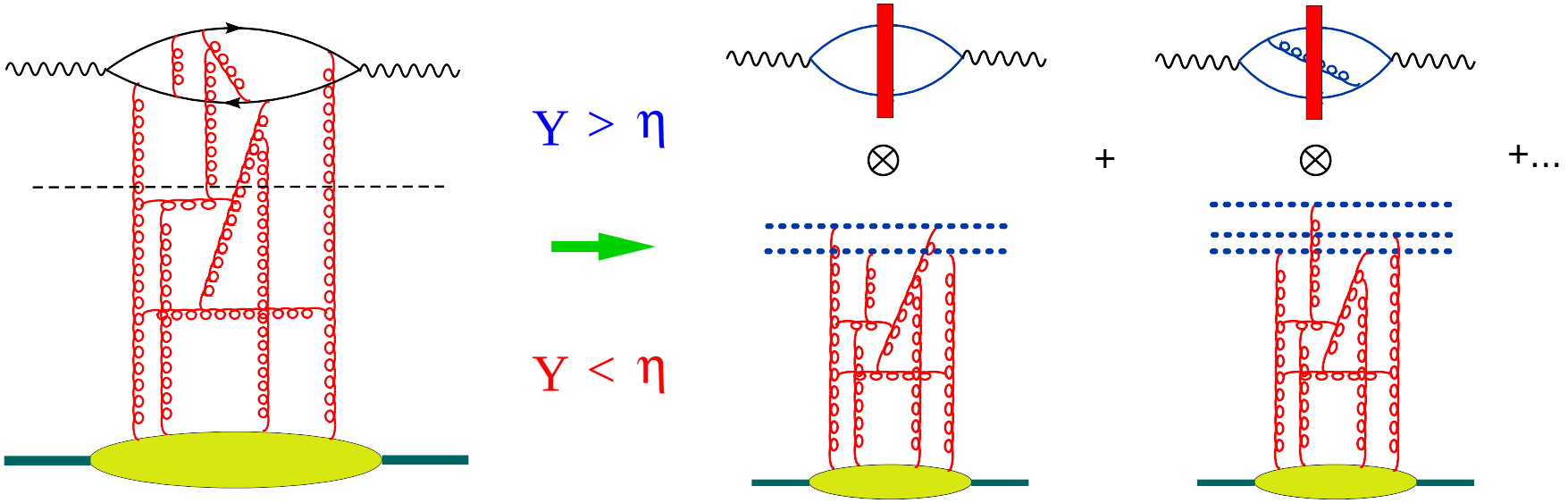

$\eta$ - rapidity factorization scale

Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} - N_c\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2)$$

$K_{\mathrm{NLO}}$=? (Linear part of $K_{\mathrm{NLO}}$ = $K_{\mathrm{NLO\ BFKL}}$)

# Expansion of $F_2(x)$ in color dipoles in the next-to-leading order




$$F_2(x_B) \simeq \int d^2z_1 d^2z_2\, I^{LO}(z_1,z_2)\langle \mathrm{tr}\{U^{\eta}_{z_1} U^{\dagger\,\eta}_{z_2}\}\rangle \qquad \eta = \ln\frac{1}{x_B}$$

$$+\frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\, I^{NLO}(z_1,z_2,z_3)\langle \mathrm{tr}\{U^{\eta}_{z_1} U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U_{z_3} U^{\dagger\eta}_{z_2}\}\rangle$$

## plan

- Calculate the NLO photon impact factor.
- Calculate the NLO evolution of color dipole.

# Propagation in the shock wave: Wilson line (Spectator frame)


Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.


$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x,y] = Pe^{ig\int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)} \qquad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

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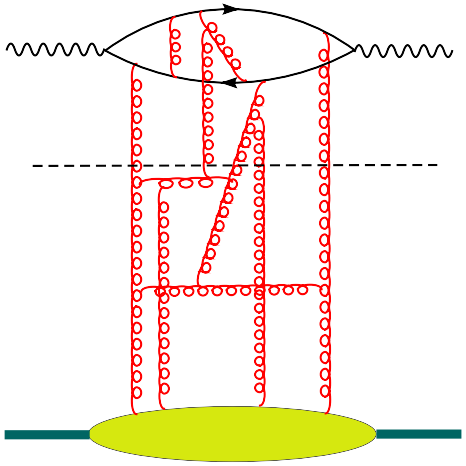

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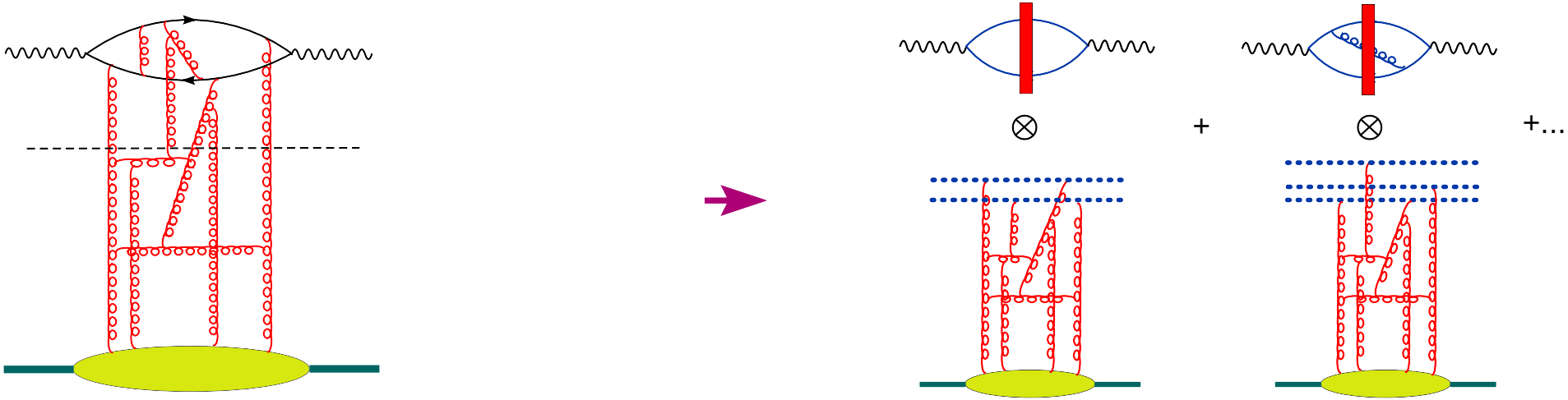
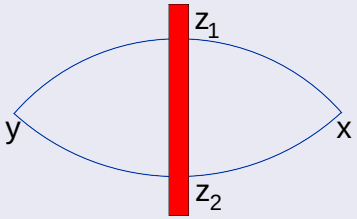


# LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

## LO Impact Factor diagram: $I^{\rm LO}$



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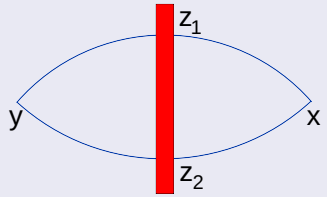
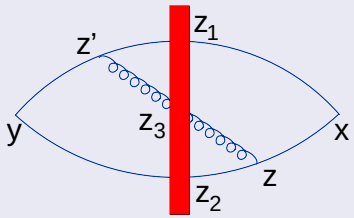
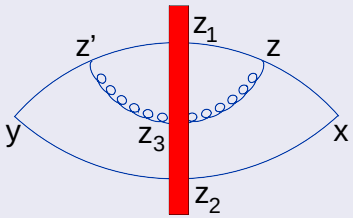
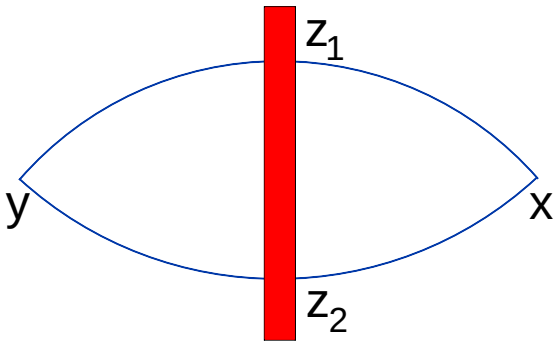



## NLO Impact Factor diagrams: $I^{\rm NLO}$



# LO Impact Factor




Conformal vectors:

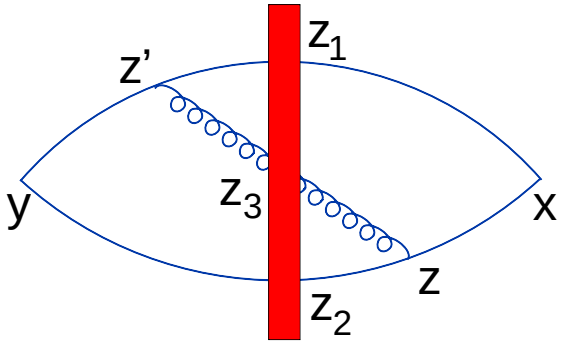
$$\kappa = \frac{\sqrt{s}}{2x_*}\left(\frac{p_1}{s} - x^2 p_2 + x_\perp\right) - \frac{\sqrt{s}}{2y_*}\left(\frac{p_1}{s} - y^2 p_2 + y_\perp\right)$$

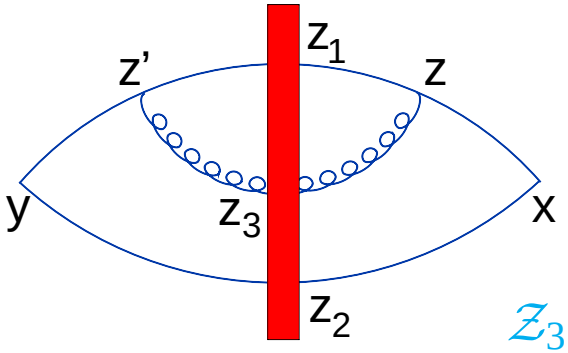
$$\zeta_1 = \sqrt{s}\left(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp}\right), \qquad \zeta_2 = \sqrt{s}\left(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp}\right)$$

Here $x^2 = -x_\perp^2$, (similarly for $y$)

$$I^{LO} \propto \frac{2}{\pi^6}\int d^2z_{1\perp} d^2z_{2\perp} \mathrm{tr}\{U_{z_{1\perp}} U^\dagger_{z_{2\perp}}\} \frac{z_{12\perp}^2}{x_*^2 y_*^2 (\kappa\cdot\zeta_1)^3 (\kappa\cdot\zeta_2)^3}$$
$$\times \frac{\partial^2}{\partial x^\mu \partial y^\nu}\left[-2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2) + \kappa^2(\zeta_1\cdot\zeta_2)\right]$$

# NLO Impact Factor



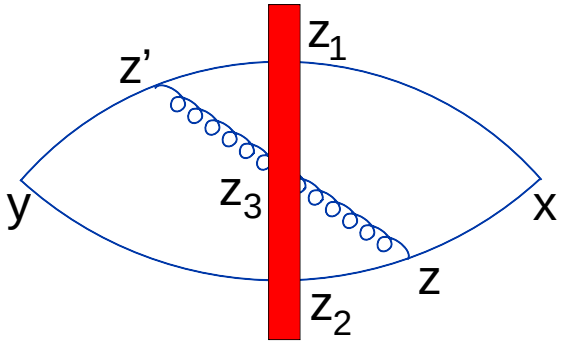



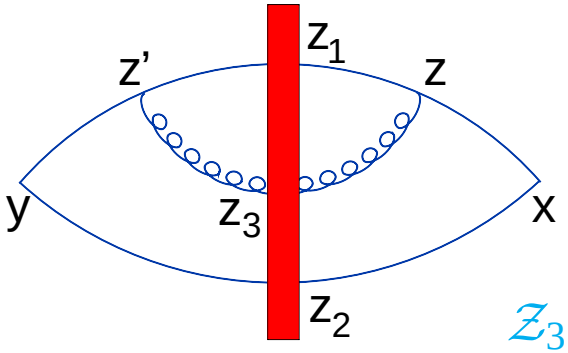

$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\rm NLO}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I_{\mu\nu}^{\rm LO} \times \frac{\alpha_s}{2\pi}\frac{z_{13}^2}{z_{12}^2 z_{23}^2}\ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

# NLO Impact Factor

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The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

However, if we define a composite operator ($a$ - analog of $\mu^{-2}$ for usual OPE)

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} \;=\; \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}$$
$$+\;\frac{\alpha_s}{4\pi}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\frac{1}{N_c}\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{a z_{12}^2}{z_{13}^2 z_{23}^2} \;+\; O(\alpha_s^2)$$

the impact factor becomes conformal in the NLO.

# Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\; I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)\mathrm{tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\; I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

# Operator expansion in conformal dipoles

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$$I^{\rm NLO}_{\mu\nu} = -\,I^{\rm LO}_{\mu\nu}\,\frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} as^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + \mathrm{conf.}$$

The new NLO impact factor is conformally invariant.

# Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$
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$$I^{\rm NLO}_{\mu\nu} = -I^{\rm LO}_{\mu\nu}\frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} as^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N}=4$ SYM theory (where the $\beta$-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

# Operator expansion in conformal dipoles

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$$+\int d^2z_1 d^2z_2 d^2z_3\, I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - \mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = -I^{\rm LO}_{\mu\nu}\frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} as^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + \mathrm{conf.}$$

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N}=4$ SYM theory (where the $\beta$-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

**Analogy:**

When the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator must be corrected by finite counter-terms order by order in perturbation theory.

$$\Delta \equiv (x-y), \qquad x_* = x^+\sqrt{s/2}, \qquad y_* = x^-\sqrt{s/2}, \qquad R \equiv -\frac{\Delta^2 z_{12\perp}^2}{x_* y_* \mathcal{Z}_1 \mathcal{Z}_2}$$

$$\mathcal{Z}_1 = -\frac{(x-z_1)^2}{x_*} + \frac{(y-z_1)^2}{y_*}, \qquad \mathcal{Z}_2 = -\frac{(x-z_2)^2}{x_*} + \frac{(y-z_2)^2}{y_*}$$

$$I^{NLO}_{\mu\nu}(x,y) = -\frac{\alpha_s N_c^2}{8\pi^7 x_*^2 y_*^2}\int d^2z_1 d^2z_2\, \mathcal{U}^{\rm conf}(z_1,z_2)\Bigg\{\Bigg[\frac{1}{\mathcal{Z}_1^2\mathcal{Z}_2^2}\partial^x_\mu\partial^y_\nu \ln\frac{\Delta^2}{x_* y_*}$$

$$+ 2\frac{(\partial^x_\mu \mathcal{Z}_1)(\mathcal{Z}_2\partial^y_\nu)}{\mathcal{Z}_1^3\mathcal{Z}_1^3}\Big[\ln\frac{1}{R} + \frac{1}{2R} - 2\Big] + \frac{2(\partial^x_\mu \mathcal{Z}_1)(\partial^y_\nu \mathcal{Z}_1)}{\mathcal{Z}_1^4\mathcal{Z}_2^2}\Big[\ln\frac{1}{R} - \frac{1}{2R}\Big]$$

$$- \frac{1}{2}\Big[\frac{\partial^x_\mu \mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\partial^y_\nu \ln\frac{\Delta^2}{x_* y_*} + \frac{\partial^y_\nu \mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\partial^x_\mu \ln\frac{\Delta^2}{x_* y_*}\Big]\Big(1-\frac{1}{R}\Big) - \frac{1}{2\mathcal{Z}_2^2}\Big[\Big(\partial^x_\mu\frac{1}{\mathcal{Z}_1^2}\Big)\partial^y_\nu R + \Big(\partial^y_\nu\frac{1}{\mathcal{Z}_1^2}\Big)\partial^x_\mu R\Big]\frac{\ln R}{1-R}$$

$$- \Big(\partial^x_\mu\partial^y_\nu\frac{\Delta^2}{x_* y_*}\Big)\frac{R^3}{z_{12}^4}\Big[\frac{1}{R} + \frac{3}{2R^2} - 2\Big]\Big(\frac{x_* y_*}{\Delta^2}\Big)^3 + \frac{1}{R}\Big[\frac{\partial^x_\mu \mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\Big(\partial^y_\nu \ln\frac{\Delta^2}{x_* y_*}\Big) + \frac{\partial^y_\nu \mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\Big(\partial^x_\mu \ln\frac{\Delta^2}{x_* y_*}\Big)\Big]$$

$$+4\frac{(\partial^x_\mu \mathcal{Z}_1)(\partial^y_\nu \mathcal{Z}_2)}{\mathcal{Z}_1^3\mathcal{Z}_2^3}\Big[4{\rm Li}_2(1-R) - \frac{2\pi^2}{3} + 2(\ln R - 1)\Big(\ln R - \frac{1}{R}\Big)\Big]$$

$$+ 2\frac{(\partial^x_\mu \mathcal{Z}_1)(\partial^y_\nu \mathcal{Z}_2)}{\mathcal{Z}_1^3\mathcal{Z}_2^3}\Big[\frac{\ln R}{R(1-R)} - \frac{1}{R} + 2\ln R - 4\Big] + 2\frac{(\partial^x_\mu \mathcal{Z}_1)(\partial^y_\nu \mathcal{Z}_1)}{\mathcal{Z}_1^4\mathcal{Z}_2^2}\Big[\frac{\ln R}{R(1-R)} - \frac{1}{R}\Big]$$

$$-\Big(\frac{\partial^x_\mu \mathcal{Z}_1}{\mathcal{Z}_1^3\mathcal{Z}_2^2}\partial^y_\nu \ln\frac{\Delta^2}{x_* y_*} + \frac{\partial^y_\nu \mathcal{Z}_2}{\mathcal{Z}_2^3\mathcal{Z}_1^2}\partial^x_\mu \ln\frac{\Delta^2}{x_* y_*}\Big)\Big[\frac{\ln R}{R(1-R)} - 2\Big] + (z_1 \leftrightarrow z_2)\Bigg]$$

$$-2\frac{z_{12\perp}^2}{\mathcal{Z}_1^3\mathcal{Z}_2^3}\Big[4{\rm Li}_2(1-R) - \frac{2\pi^2}{3} + 2\Big(\ln\frac{1}{R} + \frac{1}{R} + \frac{1}{2R^2} - 3\Big)\ln\frac{1}{R} - \Big(6 + \frac{1}{R}\Big)\ln R + \frac{3}{2R} - \frac{5}{2}\Big]\partial^x_\mu\partial^y_\nu\frac{\Delta^2}{x_* y_*}\Bigg\}$$

## Different tensor structures appearing at NLO

$$\frac{z_{12\perp}^2}{\mathcal{Z}_1\mathcal{Z}_2}\partial_\mu^x\partial_\nu^y\frac{\Delta^2}{x_*y_*}$$

$$\frac{\left(\partial_\mu^x\mathcal{Z}_1\right)}{\mathcal{Z}_1}\left(\partial_\nu^y\frac{\Delta^2}{x_*y_*}\right)+\frac{\left(\partial_\nu^y\mathcal{Z}_1\right)}{\mathcal{Z}_1}\left(\partial_\mu^x\frac{\Delta^2}{x_*y_*}\right)+\left(z_1\leftrightarrow z_2\right)$$

$$\frac{\left(\partial_\mu^x\mathcal{Z}_1\right)\left(\partial_\nu^y\mathcal{Z}_1\right)}{\mathcal{Z}_1^2}+\frac{\left(\partial_\mu^x\mathcal{Z}_2\right)\left(\partial_\nu^y\mathcal{Z}_2\right)}{\mathcal{Z}_2^2}$$

$$\frac{1}{\mathcal{Z}_1\mathcal{Z}_2}\left[\left(\partial_\mu^x\mathcal{Z}_1\right)\left(\partial_\nu^y\mathcal{Z}_2\right)+\left(\partial_\mu^x\mathcal{Z}_2\right)\left(\partial_\nu^y\mathcal{Z}_1\right)\right]$$

# Photon Impact Factor at NLO

## Conformal vectors

$$\kappa^\mu = \frac{\sqrt{s}}{2x_*}\left(\frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu\right) - \frac{\sqrt{s}}{2y_*^\mu}\left(\frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu\right)$$

$$\zeta_1^\mu = \left(\frac{p_1^\mu}{s} + z_{1\perp}^2 p_2^\mu + z_{1\perp}^\mu\right), \qquad \zeta_2^\mu = \left(\frac{p_1^\mu}{s} + z_{2\perp}^2 p_2^\mu + z_{2\perp}^\mu\right)$$

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DIS photon impact factor is a linear combinationof the following tensor basis

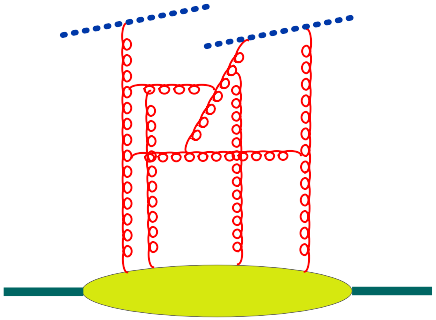
$$\mathcal{I}_1^{\mu\nu} = g^{\mu\nu} \qquad \mathcal{I}_2^{\mu\nu} = \frac{\kappa^\mu \kappa^\nu}{\kappa^2}$$

$$\mathcal{I}_3^{\mu\nu} = \frac{\kappa^\mu \zeta_1^\nu + \kappa^\nu \zeta_1^\mu}{\kappa \cdot \zeta_1} + \frac{\kappa^\mu \zeta_2^\nu + \kappa^\nu \zeta_2^\mu}{\kappa \cdot \zeta_2}$$

$$\mathcal{I}_4^{\mu\nu} = \frac{\kappa^2 \zeta_1^\mu \zeta_1^\nu}{(\kappa \cdot \zeta_1)^2} + \frac{\kappa^2 \zeta_2^\mu \zeta_2^\nu}{(\kappa \cdot \zeta_2)^2} \qquad \mathcal{I}_5^{\mu\nu} = \frac{\zeta_1^\mu \zeta_2^\nu + \zeta_2^\mu \zeta_1^\nu}{\zeta_1 \cdot \zeta_2}$$

Cornalba, Costa, Penedones (2010)

# Regularization of the rapidity divergence



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty$$

**Regularization by: slope**

$$U^\eta(x_\perp) = \mathrm{Pexp}\left\{ig \int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\right\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

**Regularization by: Rigid cut-off (used in NLO)**

$$U_x^\eta = \mathrm{Pexp}\left[ig \int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\right]$$

$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik\cdot x} A_\mu(k)$$

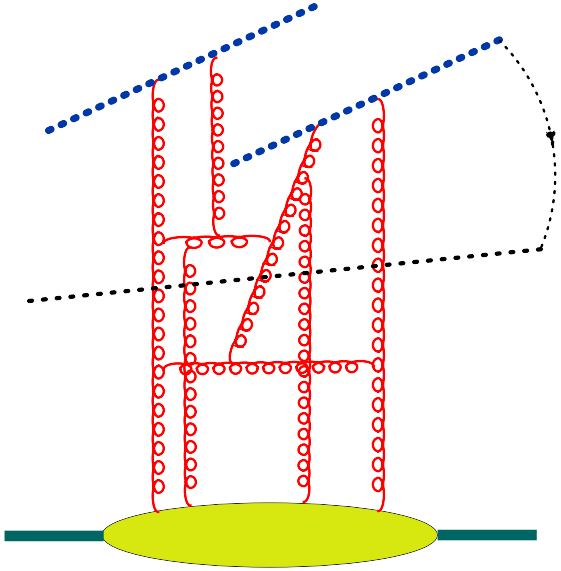
$$k^\mu = \alpha_k\, p_1^\mu + \beta_k\, p_2^\mu + k_\perp^\mu$$

# Evolution Equation

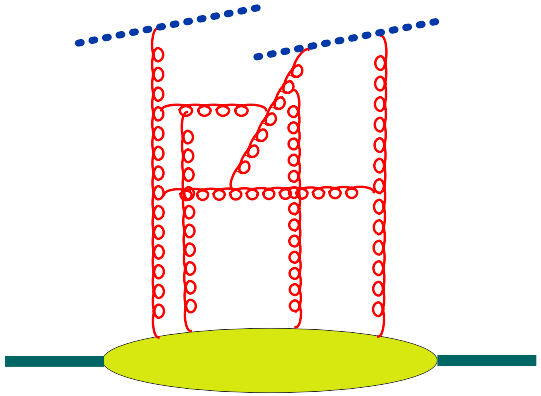
$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle$$

To get the evolution equation, consider the dipole with the rapidies up to $\eta_1$ and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to $\eta_2$).

In the frame $||$ to $\eta_1$ the gluons with $\eta < \eta_1$ are seen as pancake.



$$\alpha_s(\eta_1 - \eta_2)K_{\mathrm{evol}} \quad \otimes$$

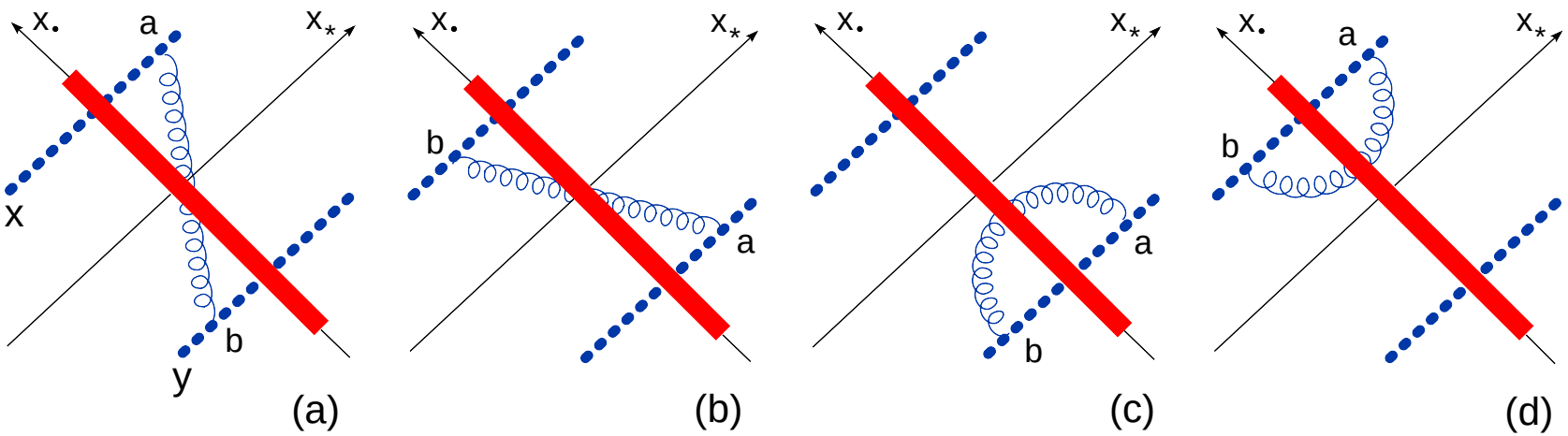


Particles with different rapidity perceive each other as Wilson lines.

# Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$




$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

# Non linear evolution equation: BK equation

$$U_z^{ab} = \mathrm{Tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

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$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

**BK equation:** **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1,\ \alpha_s^2 A^{1/3} \sim 1$)

(s for semi-classical)

# Non linear evolution equation at NLO

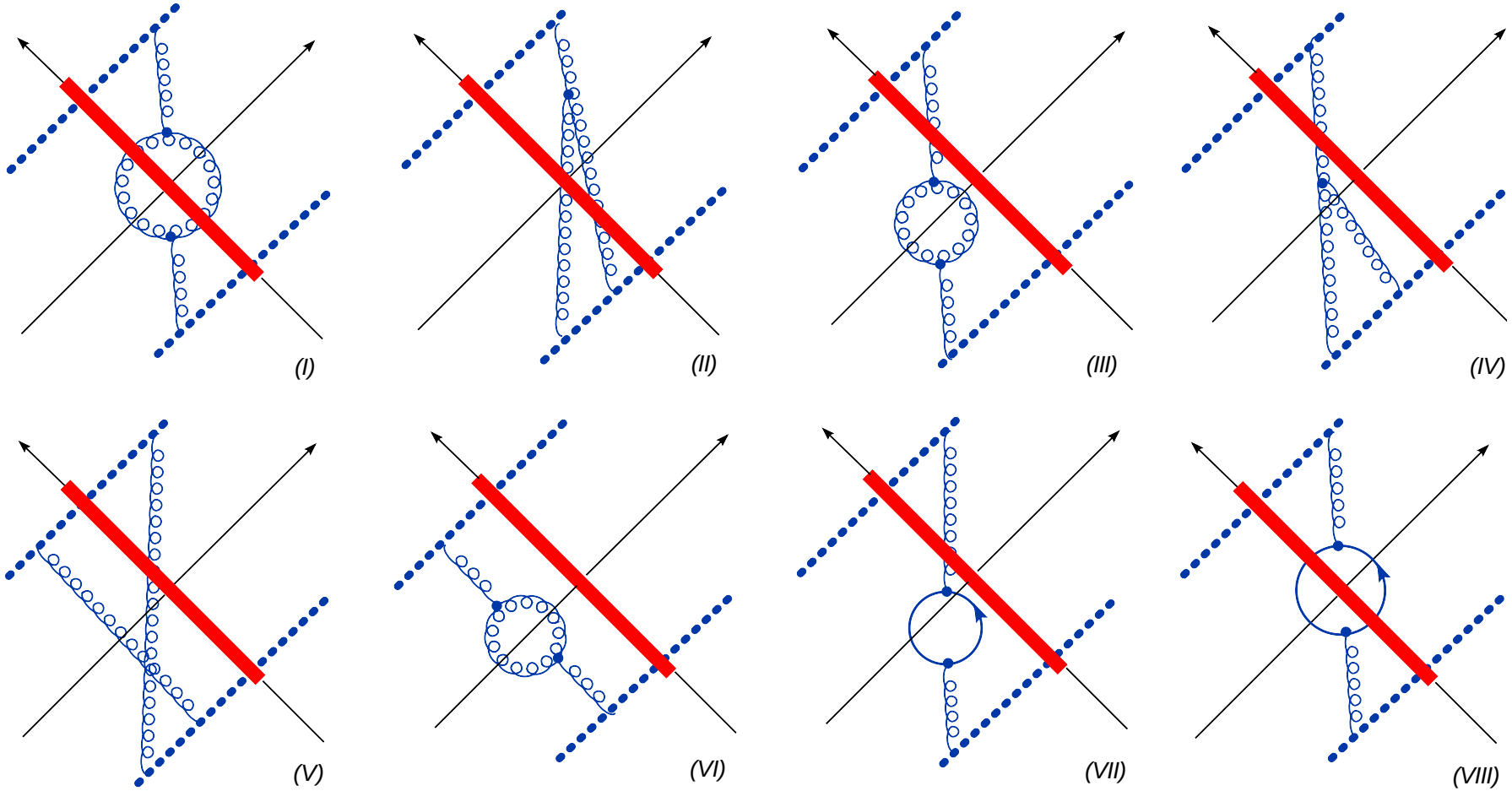
$$\frac{d}{d\eta} Tr\{U_x U_y^\dagger\} =$$

$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_x U_z^\dagger\}Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] +$$

$$\alpha_s^2 \int d^2z d^2z' \left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

$K_{NLO}$ is the next-to-leading order correction to the dipole kernel and K4 and K6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

# Sample of diagrams of the NLO gluon contribution



# NLO evolution of composite "conformal" dipoles in QCD

$$\frac{d}{d\eta}[\mathrm{tr}\{\hat{U}_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{{z_{14}}^2z_{23}^2+{z_{24}}^2z_{13}^2-4z_{12}^2z_{34}^2}{2({z_{14}}^2z_{23}^2-{z_{24}}^2z_{13}^2)}\ln\frac{{z_{14}}^2z_{23}^2}{{z_{24}}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2{z_{24}}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2{z_{24}}^2}{{z_{14}}^2z_{23}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\mathrm{tr}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_4}\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\hat{U}_{z_4}\hat{U}^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

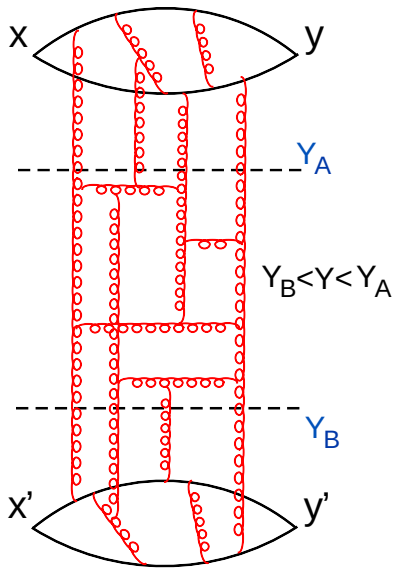
$$b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

I. Balitsky and G.A.C

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel.

## Factorization in rapidity




- NLO amplitude in $\mathcal{N}$=4 SYM: I. Balitsky and G.A.C. (2009)
- NLO amplitude in QCD: (in preparation)

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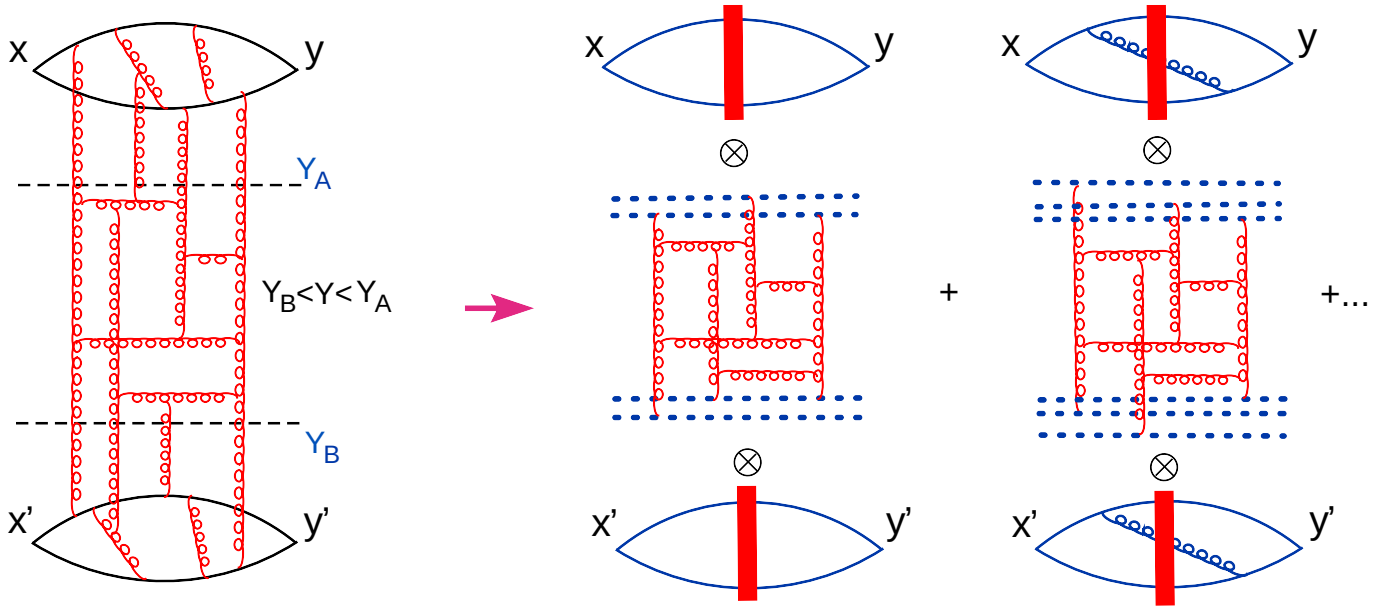


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# Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- NLO photon impact factor in coordinate space has been calculated and presented: the result is conformal.
- The NLO BK kernel in QCD and $\mathcal{N} = 4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.

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## Outlook

- Fourier transform of the NLO Photon Impact Factor.
- NLO amplitude of $\gamma^* \gamma^*$ scattering (QCD).