High-Energy Amplitudes and Impact Factors at next-to-leading order

G. A. Chirilli

LPT-Orsay& CPHT-Polytechnique

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Outline

- Light-cone OPE versus OPE in color dipoles.
- High-energy scattering and Wilson lines.
- Factorization in rapidity: Feynman diagrams in a shock-wave background.
- NLO Photon Impact Factor
- Leading order and NLO BK equation.
- Conclusions.
- Outlook.
Incoherent Interactions

Bjorken Limit

\[ Q^2 \to \infty, \ s \to \infty \]

\[ x_B = \frac{Q^2}{s} \] fixed

\[ \text{resum} \ \alpha_s \ln \frac{Q^2}{\Lambda_{\text{QCD}}} \]
Incoherent-vs-Coherent

Incoherent Interactions

\[ Q^2 \to \infty, \ s \to \infty \]
\[ x_B = \frac{Q^2}{s} \text{ fixed} \]
\[ \text{resum} \ \alpha_s \ln \frac{Q^2}{\Lambda_{QCD}} \]

Bjorken Limit

Coherent Interactions

\[ Q^2 \text{ fixed, } s \to \infty \]
\[ x_B = \frac{Q^2}{s} \to 0 \]
\[ \text{resum} \ \alpha_s \ln \frac{1}{x_B} \]

Regge Limit

VS.
Light-cone expansion and DGLAP evolution in the NLO

$\mu^2$ - factorization scale (normalization point)

$k_{\perp}^2 > \mu^2$ - coefficient functions

$k_{\perp}^2 < \mu^2$ - matrix elements of light-ray operators (normalized at $\mu^2$)
Light-cone expansion and DGLAP evolution in the NLO

\( k_\perp^2 > \mu^2 \)

\( k_\perp^2 < \mu^2 \)

\( \mu^2 \) - factorization scale (normalization point)

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\( k_\perp^2 < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))

OPE in light-ray operators

\[ T\{j_\mu(x)j_\nu(y)\} = \frac{(x-y)\xi}{2\pi^2(x-y)^4} \left[ 1 + \frac{\alpha_s}{\pi} (\ln(x-y)^2 \mu^2 + C) \right] \bar{\psi}(x) \gamma_\mu \gamma_\xi \gamma_\nu [x, y] \psi(y) \]

\[ [x, y] \equiv Pe^{ig \int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y)} \] - gauge link
Light-cone expansion and DGLAP evolution in the NLO

\[ k_\perp^2 > \mu^2 \]

\[ k_\perp^2 < \mu^2 \]

\( \mu^2 \) - factorization scale (normalization point)

\( k_\perp^2 > \mu^2 \) - coefficient functions

\( k_\perp^2 < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))

Renorm-group equation for light-ray operators \( \Rightarrow \) DGLAP evolution of parton densities

\[
\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) = K_{\text{LO}} \bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\text{NLO}} \bar{\psi}(x)[x,y]\psi(y)
\]

\((x-y)^2 = 0\)
High-energy expansion in color dipoles in the NLO
High-energy expansion in color dipoles in the NLO

\[ Y > \eta \]

\[ Y < \eta \]

\[ \text{NLO Photon Impact Factor for DIS} \]

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$\eta$ - rapidity factorization scale

**Rapidity $Y > \eta$**

Coefficient function ("impact factor")

**Rapidity $Y < \eta$**

Matrix elements of (light-like) Wilson lines with rapidity divergence cut by $\eta$

$$ U_\eta^\mu = \text{Pexp} \left[ ig \int_{-\infty}^{\infty} dx^+ A_\eta^\mu (x_+, x_\perp) \right] $$

$$ A_\eta^\mu (x) = \int \frac{d^4k}{(2\pi)^4} \theta (e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu (k) $$
The high-energy operator expansion is

\[
T\{j_\mu(x)j_\nu(y)\} = \int d^2z_1 d^2z_2 \ I^{LO}_{\mu\nu}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\}
\]

\[
+ \int d^2z_1 d^2z_2 d^2z_3 \ I^{NLO}_{\mu\nu}(z_1, z_2, z_3, x, y) [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger \eta}\} \text{tr}\{\hat{U}_{z_2}^\eta \hat{U}_{z_2}^{\dagger \eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\}]
\]

In the leading order the impact factor is Möbius invariant
In the NLO one should also expect conf. invariance since \(I^{NLO}_{\mu\nu}\) is given by tree diagrams
High-energy expansion in color dipoles in the NLO

\[ \eta > \eta \quad \text{and} \quad \eta < \eta \]

\[ \eta \text{- rapidity factorization scale} \]

Evolution equation for color dipoles

\[
\frac{d}{d\eta} \text{tr}\{U_x^n U_y^{\dagger n}\} = \frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \left[ \text{tr}\{U_x^n U_y^{\dagger n}\} \text{tr}\{U_x^n U_y^{\dagger n}\} \right] - N_c \text{tr}\{U_x^n U_y^{\dagger n}\} + \alpha_s K_{\text{NLO}} \text{tr}\{U_x^n U_y^{\dagger n}\} + O(\alpha_s^2)
\]

\[ K_{\text{NLO}} = ? \]

(Linear part of $K_{\text{NLO}} = K_{\text{NLO BFKL}}$)
Expansion of $F_2(x)$ in color dipoles in the next-to-leading order

\[ F_2(x_B) \simeq \int d^2z_1 d^2z_2 I^{LO}(z_1, z_2) \langle \text{tr}\{U_{z_1}^\eta U_{z_2}^\dagger \eta\} \rangle \]

\[ + \frac{\alpha_s}{\pi} \int d^2z_1 d^2z_2 d^2z_3 I^{NLO}(z_1, z_2, z_3) \langle \text{tr}\{U_{z_1}^\eta U_{z_3}^\dagger \eta\} \text{tr}\{U_{z_3} U_{z_2}^\dagger \eta\} \rangle \]

\[ \eta = \ln \frac{1}{x_B} \]

**plan**

- Calculate the NLO photon impact factor.
- Calculate the NLO evolution of color dipole.
Each path is weighted with the gauge factor $P e^{i g \int d x_{\mu} A_{\mu}}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x, y] = P e^{i g \int_0^1 d u (x-y)^\mu A_\mu (u x + (1-u) y)}$$

$$p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$
Each path is weighted with the gauge factor $P e^{i g \int d x_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
Each path is weighted with the gauge factor $P e^{i g \int dx^{\mu} A^{\mu}}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
\[ T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_\eta^{z_1} \hat{U}^{\dagger\eta}_{z_2}\} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) \left[ \text{tr}\{\hat{U}_\eta^{z_1} \hat{U}^{\dagger\eta}_{z_3}\} \text{tr}\{\hat{U}_\eta^{z_3} \hat{U}^{\dagger\eta}_{z_2}\} - N_c \text{tr}\{\hat{U}_\eta^{z_1} \hat{U}^{\dagger\eta}_{z_2}\} \right] \]
\[ T\{j_\mu(x)j_\nu(y)\} = \int d^2z_1 d^2z_2 I^{LO}_{\mu\nu}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 I^{NLO}_{\mu\nu}(z_1, z_2, z_3, x, y) [\text{tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\} \text{tr}\{\hat{U}_{z'}^\eta \hat{U}_{z'}^{\dagger \eta}\} - N_c \text{tr}\{\hat{U}_z^\eta \hat{U}_z^{\dagger \eta}\}] \]

LO Impact Factor diagram: \( I^{LO} \)

NLO Impact Factor diagrams: \( I^{NLO} \)
Conformal vectors:

\[ \kappa = \frac{\sqrt{s}}{2x_1}(p_1s - x_2p_2 + x_1) - \frac{\sqrt{s}}{2y_1}(p_1s - y_2p_2 + y_1) \]

\[ \zeta_1 = \sqrt{s}(\frac{p_1s}{s} + z_1^2p_2 + z_1) \quad \zeta_2 = \sqrt{s}(\frac{p_1s}{s} + z_2^2p_2 + z_2) \]

Here \( x^2 = -x_2^2 \) (similarly for \( y \))

\[ I^{LO} \propto \frac{2}{\pi^6} \int d^2z_1z_2 \frac{z_{12}^2}{x_1y_1(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)^3} \]

\[ \times \frac{\partial^2}{\partial x^\mu \partial y^\nu} \left[ -2(\kappa \cdot \zeta)(\kappa \cdot \zeta_2) + \kappa^2(\zeta_1 \cdot \zeta_2) \right] \]
The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln \sigma$ is not invariant.
The NLO impact factor is not Möbius invariant ⇒ the color dipole with the cutoff \( \eta = \ln \sigma \) is not invariant.

However, if we define a composite operator (a - analog of \( \mu^{-2} \) for usual OPE)

\[
[\text{Tr}\{\hat{U}_{z_1}^n \hat{U}_{z_2}^\dagger \eta\}]^{\text{conf}} = \text{Tr}\{\hat{U}_{z_1}^n \hat{U}_{z_2}^{\dagger \eta}\} \\
+ \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^n \hat{U}_{z_3}^{\dagger \eta}\} \text{tr}\{\hat{U}_{z_3}^n \hat{U}_{z_2}^{\dagger \eta}\} - \text{Tr}\{\hat{U}_{z_1}^n \hat{U}_{z_2}^{\dagger \eta}\}\right] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2)
\]

the impact factor becomes conformal in the NLO.
Operator expansion in conformal dipoles

\[
T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2 z_1 d^2 z_2 \ I^{LO}_{\mu\nu}(z_1, z_2, x, y) \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger\eta_{z_2}\}^{\text{conf}}
\]

\[
+ \int d^2 z_1 d^2 z_2 d^2 z_3 \ I^{NLO}_{\mu\nu}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger\eta_{z_3}\} \text{tr}\{\hat{U}^\eta_{z_3} \hat{U}^\dagger\eta_{z_2}\} - \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^\dagger\eta_{z_2}\} \right]
\]
Operator expansion in conformal dipoles

\[ T\{ \hat{j}_\mu(x)\hat{j}_\nu(y) \} = \int d^2z_1 d^2z_2 \ I^{\text{LO}}_{\mu\nu}(z_1, z_2, x, y) \text{tr}[\{ \hat{U}_z^\eta \hat{U}^\dagger_\eta \}]^{\text{conf}} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 \ I^{\text{NLO}}_{\mu\nu}(z_1, z_2, z_3, x, y) \bigg[ \frac{1}{N_c} \text{tr}\{ \hat{U}_z^\eta \hat{U}^\dagger_\eta \} \text{tr}\{ \hat{U}_{z_3}^\eta \hat{U}^\dagger_{z_3} \} - \text{tr}\{ \hat{U}_{z_1}^\eta \hat{U}^\dagger_{z_2} \} \bigg] \]

\[ I^{\text{NLO}}_{\mu\nu} = - I^{\text{LO}}_{\mu\nu} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a_s^2}{z_{13}^2 z_{23}^2} + \text{conf.} \]

The new NLO impact factor is conformally invariant.
Operator expansion in conformal dipoles

\[ T\{j_\mu(x)j_\nu(y)\} = \int d^2z_1d^2z_2 \, I^{\text{LO}}_{\mu\nu}(z_1, z_2, x, y) \text{tr}\{\{\hat{U}_1^\eta \hat{U}_2^\eta\}\}^{\text{conf}} \]

\[ + \int d^2z_1d^2z_2d^2z_3 \, I^{\text{NLO}}_{\mu\nu}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_1^\eta \hat{U}_3^\eta\}\text{tr}\{\hat{U}_2^\eta \hat{U}_3^\eta\} - \text{tr}\{\hat{U}_1^\eta \hat{U}_2^\eta\}\right] \]

\[ I^{\text{NLO}}_{\mu\nu} = - I^{\text{LO}}_{\mu\nu} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a_s^2}{z_{13}^2 z_{23}^2} z_3^2 + \text{conf.} \]

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N} = 4$ SYM theory (where the $\beta$-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.
Operator expansion in conformal dipoles

\[ T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1d^2z_2 \, I^{\text{LO}}_{\mu\nu}(z_1, z_2, x, y)\text{tr}\{\hat{U}_\eta \, \hat{U}^\dagger_\eta\}\rfloor_{\text{conf}} \]

\[ + \int d^2z_1d^2z_2d^2z_3 \, I^{\text{NLO}}_{\mu\nu}(z_1, z_2, z_3, x, y)\left[ \frac{1}{N_c} \text{tr}\{\hat{U}_\eta \, \hat{U}^\dagger_\eta\}\text{tr}\{\hat{U}_\eta \, \hat{U}^\dagger_\eta\} - \text{tr}\{\hat{U}_\eta \, \hat{U}^\dagger_\eta\}\right] \]

\[ I^{\text{NLO}}_{\mu\nu} = - I^{\text{LO}}_{\mu\nu} \, \frac{\alpha_s N_c}{4\pi} \int dz_3 \, \frac{z_{12}^2 z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} \alpha_s^2}{z_{13}^2 z_{23}^2} \, Z_3^2 + \text{conf.} \]

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N} = 4$ SYM theory (where the $\beta$-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

**Analogy:**

When the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator must be corrected by finite counter-terms order by order in perturbation theory.
\[ \Delta \equiv (x - y), \quad x^* = x^+ \sqrt{s/2}, \quad y^* = x^- \sqrt{s/2}, \quad R \equiv -\frac{\Delta^2 z_2}{x^* y^* Z_1 Z_2} \]

\[ Z_1 = -\frac{(x-z_1)^2}{x^*} + \frac{(y-z_1)^2}{y^*}, \quad Z_2 = -\frac{(x-z_2)^2}{x^*} + \frac{(y-z_2)^2}{y^*} \]

\[
I_{\mu \nu}^{NLO}(x, y) = -\frac{\alpha_s N_c^2}{8 \pi^2 x^*_y^2} \int d^2z_1 d^2z_2 U^{\text{conf}}(z_1, z_2) \left\{ \frac{1}{Z_1^2 Z_2^2} \partial_\mu \partial_\nu \ln \frac{\Delta^2}{x^* y^*} \right. \\
+ 2 \left( \frac{\partial_\mu Z_1}{Z_1^3 Z_2^3} \right) \left( \frac{\partial_\nu Z_2}{Z_1^3 Z_2^3} \right) \left[ \ln \frac{1}{R} + \frac{1}{2R} - 2 \right] + \frac{2}{Z_1^4 Z_2^2} \left( \partial_\mu \ln \frac{\Delta^2}{x^* y^*} \right) \right. \\
- \frac{1}{2} \left[ \ln \frac{1}{R} + \frac{3}{2R^2} - 2 \right] \left( \frac{x^*_y^*}{\Delta^2} \right)^3 + \frac{1}{R} \left[ \frac{4 \ln R}{R(1 - R)} - \frac{1}{R} + 2 \ln R - 4 \right] + 2 \left( \frac{\partial_\mu Z_1}{Z_1^3 Z_2^3} \right) \left( \frac{\partial_\nu Z_2}{Z_1^3 Z_2^3} \right) \left[ \frac{\ln R}{R(1 - R)} - \frac{1}{R} \right] \\
- \left( \partial_\mu \partial_\nu \ln \frac{\Delta^2}{x^* y^*} \right) \left[ \frac{\ln R}{R(1 - R)} - 2 \right] + \left( z_1 \leftrightarrow z_2 \right) \\
- 2 \frac{z_1^2 z_2}{Z_1^3 Z_2^3} \left[ 4 \ln R - \frac{2 \pi^2}{3} + 2 (\ln R - 1) - 6 + \frac{1}{R} \ln R + \frac{3}{2R} - \frac{5}{2} \right] \left( \frac{x^*_y^*}{\Delta^2} \right) \left[ \frac{\ln R}{R(1 - R)} - \frac{1}{R} \right] \}
\]
Different tensor structures appearing at NLO

$$\frac{z_{12}^2}{Z_1 Z_2} \partial_x^\mu \partial_y^\nu \frac{\Delta^2}{x_* y_*}$$

$$\frac{(\partial_x^\mu Z_1)}{Z_1} \left( \partial_y^\nu \frac{\Delta^2}{x_* y_*} \right) + \frac{(\partial_y^\nu Z_1)}{Z_1} \left( \partial_x^\mu \frac{\Delta^2}{x_* y_*} \right) + (z_1 \leftrightarrow z_2)$$

$$\frac{(\partial_x^\mu Z_1) (\partial_y^\nu Z_1)}{Z_1^2} + \frac{(\partial_x^\mu Z_2) (\partial_y^\nu Z_2)}{Z_2^2}$$

$$\frac{1}{Z_1 Z_2} \left[ (\partial_x^\mu Z_1) (\partial_y^\nu Z_2) + (\partial_x^\mu Z_2) (\partial_y^\nu Z_1) \right]$$
Photon Impact Factor at NLO

Conformal vectors

\[
\kappa^\mu = \frac{\sqrt{s}}{2x^*} \left( \frac{p_1^\mu}{s} - x^2 p_2^\mu + x_{\perp}^\mu \right) - \frac{\sqrt{s}}{2y^*} \left( \frac{p_1^\mu}{s} - y^2 p_2^\mu + y_{\perp}^\mu \right)
\]

\[
\zeta_1^\mu = \left( \frac{p_1^\mu}{s} + z_1^2 p_2^\mu + z_{1\perp}^\mu \right), \quad \zeta_2^\mu = \left( \frac{p_1^\mu}{s} + z_2^2 p_2^\mu + z_{2\perp}^\mu \right)
\]
Conformal vectors

\[ \kappa^\mu = \frac{\sqrt{s}}{2x_*} \left( \frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu \right) - \frac{\sqrt{s}}{2y_*} \left( \frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu \right) \]

\[ \zeta_1^\mu = \left( \frac{p_1^\mu}{s} + z_1^2 p_2^\mu + z_1^\mu_\perp \right), \quad \zeta_2^\mu = \left( \frac{p_1^\mu}{s} + z_2^2 p_2^\mu + z_2^\mu_\perp \right) \]

DIS photon impact factor is a linear combination of the following tensor basis

\[ \mathcal{I}_{1}^{\mu\nu} = g^{\mu\nu} \]

\[ \mathcal{I}_{2}^{\mu\nu} = \frac{\kappa^{\mu} \kappa^{\nu}}{\kappa^2} \]

\[ \mathcal{I}_{3}^{\mu\nu} = \frac{\kappa^{\mu} \zeta_1^{\nu} + \kappa^{\nu} \zeta_1^{\mu}}{\kappa \cdot \zeta_1} + \frac{\kappa^{\mu} \zeta_2^{\nu} + \kappa^{\nu} \zeta_2^{\mu}}{\kappa \cdot \zeta_2} \]

\[ \mathcal{I}_{4}^{\mu\nu} = \frac{\kappa^{2} \zeta_1^{\mu} \zeta_1^{\nu}}{(\kappa \cdot \zeta_1)^2} + \frac{\kappa^{2} \zeta_2^{\mu} \zeta_2^{\nu}}{(\kappa \cdot \zeta_2)^2} \]

\[ \mathcal{I}_{5}^{\mu\nu} = \frac{\zeta_1^{\mu} \zeta_2^{\nu} + \zeta_2^{\mu} \zeta_1^{\nu}}{\zeta_1 \cdot \zeta_2} \]

Cornalba, Costa, Penedones (2010)
Regularization of the rapidity divergence

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[ \int_{0}^{\infty} \frac{d\alpha}{\alpha} = \int_{-\infty}^{\infty} d\eta = \infty \]

Regularization by: slope

\[ U_{\eta}(x_{\perp}) = \text{Pexp}\left\{ ig \int_{-\infty}^{\infty} du \, n_{\mu} A^{\mu}(un + x_{\perp}) \right\} \]

\[ n^{\mu} = p_{1}^{\mu} + e^{-2\eta} p_{2}^{\mu} \]

Regularization by: Rigid cut-off (used in NLO)

\[ U_{x}^{\eta} = \text{Pexp}\left[ ig \int_{-\infty}^{\infty} du \, p_{1}^{\mu} A_{\mu}^{\eta}(up_{1} + x_{\perp}) \right] \]

\[ A_{\mu}^{\eta}(x) = \int \frac{d^{4}k}{(2\pi)^{4}} \theta(e^{\eta} - |\alpha_{k}|) e^{-ik \cdot x} A_{\mu}(k) \]

\[ k^{\mu} = \alpha_{k} p_{1}^{\mu} + \beta_{k} p_{2}^{\mu} + k_{\perp}^{\mu} \]
To get the evolution equation, consider the dipole with the rapidities up to $\eta_1$ and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to $\eta_2$).

In the frame $\parallel$ to $\eta_1$ the gluons with $\eta < \eta_1$ are seen as pancake.
Leading order: BK equation

\[ \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} + \ldots \Rightarrow \]

\[ \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}^\dagger_y\} \rangle_{\text{shockwave}} \]

\[ x_\bullet = \sqrt{\frac{s}{2}} x^- \quad x_* = \sqrt{\frac{s}{2}} x^+ \]
Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{2}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{\eta_{2}} \]
Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{ t^{a} U_{z} t^{b} U_{z}^{\dagger} \} \Rightarrow (U_{x} U_{y}^{\dagger})^{n_{1}} \rightarrow (U_{x} U_{y}^{\dagger})^{n_{2}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x} U_{z}^{\dagger} U_{z} U_{y}^{\dagger})^{n_{2}} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}} \text{Tr}\{ \hat{U}(x_{\perp}) \hat{U}^{\dagger}(y_{\perp}) \} \]


\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_{s}N_{c}}{2\pi^{2}} \int \frac{d^{2}z (x - y)^{2}}{(x - z)^{2}(y - z)^{2}} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]

Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{2}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{\eta_{2}} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}} \text{Tr}\{\hat{U}(x_{\perp})\hat{U}(y_{\perp})\} \]


\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_{s}N_{c}}{2\pi^{2}} \int \frac{d^{2}z}{(x - z)^{2}(y - z)^{2}} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}
\]


LLA for DIS in pQCD \Rightarrow BFKL

(LLA: \( \alpha_{s} \ll 1, \alpha_{s}\eta \sim 1 \))
Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^\dagger\} \Rightarrow (U_{x}U_{y}^\dagger)^{\eta_{1}} \rightarrow (U_{x}U_{y}^\dagger)^{\eta_{2}} + \alpha_{s}(\eta_{1} - \eta_{2})U_{x}U_{z}^\dagger U_{z}U_{y}^\dagger \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}}\text{Tr}\{\hat{U}(x_{\perp})\hat{U}^\dagger(y_{\perp})\} \]


\[ \frac{d}{d\eta}\hat{U}(x, y) = \frac{\alpha_{s}N_{c}}{2\pi^{2}} \int \frac{d^{2}z}{(x - z)^{2}(y - z)^{2}} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]


LLA for DIS in pQCD ⇒ BFKL

(\text{LLA: } \alpha_{s} \ll 1, \alpha_{s}\eta \sim 1)

LLA for DIS in sQCD ⇒ BK eqn

(\text{LLA: } \alpha_{s} \ll 1, \alpha_{s}\eta \sim 1, \alpha_{s}^{2}A^{1/3} \sim 1)

(s for semi-classical)
Non linear evolution equation at NLO

\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \\
\int \frac{d^2z}{2\pi^2} \left( \alpha_s \frac{(x-y)^2}{(x-z)(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z) \right) \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right] + \\
\alpha_s^2 \int d^2z d^2z' \left( K_4(x,y,z,z') \{U_x, U_z^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z') \{U_x, U_{z'}^\dagger, U_z', U_z, U_{z'}^\dagger, U_y^\dagger\} \right)
\]

\( K_{NLO} \) is the next-to-leading order correction to the dipole kernel and \( K_4 \) and \( K_6 \) are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.
Sample of diagrams of the NLO gluon contribution
NLO evolution of composite “conformal” dipoles in QCD

\[
\frac{d}{d\eta} [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}]^{\text{conf}} = \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left( [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_2}^\dagger\} - N_c \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}]^{\text{conf}} \times \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( b \ln z_{12}^2 \mu^2 + b \frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] + \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z_4}{z_3^4 z_4^2} \left\{ -2 + \frac{z_{14}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{14}^2 z_{23}^2 - z_{24}^2 z_{13}^2)} \ln \frac{z_{14}^2 z_{23}^2}{z_{24}^2 z_{13}^2} \right\} \times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger \hat{U}_{z_4}^\dagger \hat{U}_{z_2}^\dagger \hat{U}_{z_3}^\dagger \hat{U}_{z_4}^\dagger\} - (z_4 \to z_3)] \times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \text{tr}\{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_4}^\dagger \hat{U}_{z_3}^\dagger \hat{U}_{z_2}^\dagger \hat{U}_{z_3}^\dagger \hat{U}_{z_4}^\dagger\} - (z_4 \to z_3)] \right\} \}
\]

\[ b = \frac{11}{3} N_c - \frac{2}{3} n_f \]

I. Balitsky and G.A.C

\[ K_{\text{NLO BK}} = \text{Running coupling part} + \text{Conformal "non-analytic" (in j) part} + \text{Conformal analytic (N = 4) part} \]

Linearized \( K_{\text{NLO BK}} \) reproduces the known result for the forward NLO BFKL kernel.
NLO Amplitude in $\mathcal{N}=4$ SYM theory and in QCD

Factorization in rapidity

- NLO amplitude in QCD: (in preparation)
NLO Amplitude in $\mathcal{N}=4$ SYM theory and in QCD

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Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- NLO photon impact factor in coordinate space has been calculated and presented: the result is conformal.
- The NLO BK kernel in QCD and $\mathcal{N} = 4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.
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Outlook

- Fourier transform of the NLO Photon Impact Factor.
- NLO amplitude of $\gamma^* \gamma^*$ scattering (QCD).