# Analytic multi-loop results using finite fields and dataflow graphs with FiniteFlow

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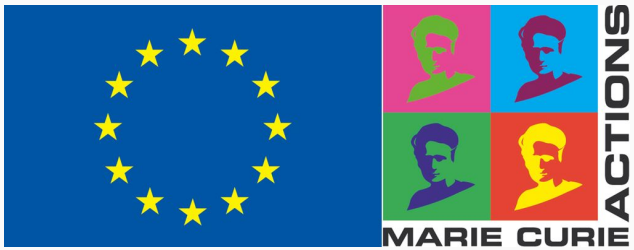



Based on:

T. P., JHEP 1907 (2019) 031, arXiv:1905.08019

# Introduction & motivation

Experiments at LHC

- high-accuracy (% level)
- large SM background
- high c.o.m. energy $\Rightarrow$ multi-particle states

We need scattering amplitudes

- describe hard partonic interaction
- high accuracy $\Rightarrow$ loops (% level $\sim$ 2 loops)
- multi-particle $\Rightarrow$ high multiplicity

Theoretical studies of amplitudes

- structures of QFT/gauge theories

# Two and higher loops

- Algebraic calculations for multi-loop amplitudes
  - preferred strategy @ $\ell \geq 2$ loops
    - faster/more stable evaluation
    - better suited for many multi-loop techniques
    - allows more tests, studies, etc. . . and better control
  - often characterized by high complexity

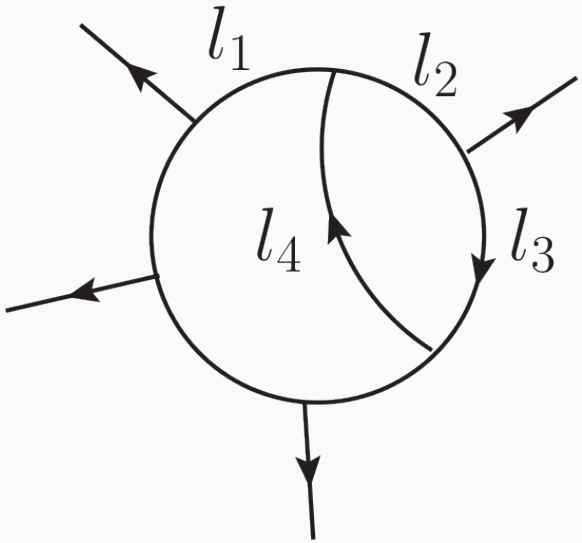
- Complexity can be a combination of
  - number of loops for high accuracy
  - number of legs for high multiplicity
  - numbers of scales (invariants, external/internal masses)

# Loop amplitudes

- An integrand contribution to $\ell$-loop amplitude

$$\mathcal{A} = \int_{-\infty}^{\infty} \left( \prod_{i=1}^{\ell} d^d k_i \right) \frac{\mathcal{N}}{D_1 \, D_2 \, D_3 \cdots}$$

  - rational function in the components of loop momenta $k_j$
  - polynomial numerator $\mathcal{N}$
  - quadratic denominators corresp. to loop propagators




$$D_j = l_j^2 - m_j^2$$

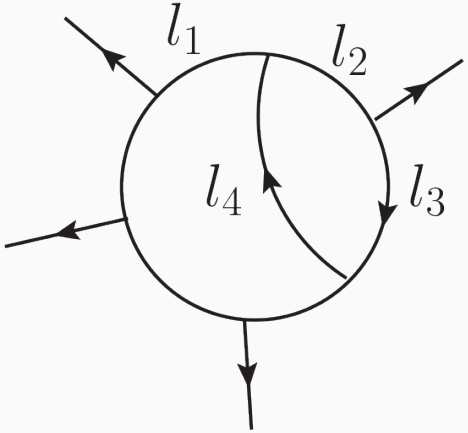
# Computing amplitudes: Step 1/3

- Write amplitudes as l.c. of Feynman integrals

$$\mathcal{A} = \sum_j a_j I_j$$

- Dependence on particle-content in rational coeff.s $a_j$
- The integrals should have a "nice" / "standard" form

$$I = \int_{-\infty}^{\infty} \left( \prod_{i=1}^{\ell} d^d k_i \right) \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} \dots}, \qquad \alpha_j \lesseqgtr 0$$




$$D_j = \begin{cases} l_j^2 - m_j^2 \\ l_j \cdot v_j - m_j^2 \end{cases}$$

Hard to do at
high multiplicity

# Computing amplitudes: Step 2/3

Chetyrkin, Tkachov (1981), Laporta (2000)

- Feynman integrals obey linear relations, e.g. IBPs

$$\int \left(\prod_j d^d k_j\right) \frac{\partial}{\partial k_j^\mu}\, v^\mu\, \frac{1}{D_1^{\alpha_1}\, D_2^{\alpha_2} \ldots} = 0, \qquad v^\mu = \begin{cases} p_i^\mu & \text{external} \\ k_i^\mu & \text{loop} \end{cases}$$

- Very large and sparse linear systems
- Reduce to linearly independent Master Integrals (MIs)
  $\{G_1, G_2, \ldots\} \subset \{I_j\}$

$$I_j = \sum_k c_{jk}\, G_k$$

# Computing amplitudes: Step 3/3

- The MIs can often be computed analytically
  - in terms of special functions (MPLs, elliptic, . . . )
  - most effective method is differential equations (DEs)
    Kotikov (1991), Gehrmann, Remiddi (2000)
  - can be simplified by the choice of MIs, e.g. UT integrals
    Henn (2013)

- Numerical methods may work depending on the process
  - the most successful is sector decomposition
    Binoth, Heinrich (2000)
  - can be improved via IBP reduction to a "better" basis of MIs

## Computing amplitudes (summary)

1. Integral representation $\mathcal{A} = \sum_j a_j I_j$
2. IBP reduction $I_j = \sum_k c_{jk} G_k$
3. Compute MIs $G_k$

## A major bottleneck

- Large intermediate expressions
- Intermediate stages much more complicated than final result

## Main idea of the talk

- Reconstruct analytic results from "numerical" evaluations
- Can be used for steps 1, 2 and help with step 3 (e.g. using DEs)

# Finite fields and functional reconstruction

## Functional reconstruction

- reconstruct analytic results from numerical evaluations
  - evaluation over finite fields $\mathcal{Z}_p$ (i.e. modulo prime integers $p$)
  - use machine-size integers, $p < 2^{64} \Rightarrow$ fast and exact
  - collect numerical evaluations and infer analytic result
- sidesteps large intermediate expressions & highly parallelizable
- first applications
  - IBPs and univ. reconstruction von Manteuffel, Schabinger (2014)
  - helicity amplitudes and multivariate reconstruction T.P. (2016)

# Some notable examples

- FinRed (private) [von Manteuffel]
  - several results for 4-loop form factors [von Manteuffel, Schabinger]
- FiniteFlow [T.P.]
  - Several two-loop five-point amplitudes
    [Badger, Brønnum-Hansen, Hartanto, T.P.;
    Badger, Chicherin, Gehrmann, Heinrich, Henn, T.P., Wasser, Zhang, Zoia]
  - Matter dependence of the four-loop cusp anomalous dimension
    [Henn, T.P., Stahlhofen, Wasser]
- Private code
  [Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov, Zeng]
  - analytic five-parton amplitudes
- Fire 6 [A.V. Smirnov, F.S. Chuharev]
  - Four-loop quark form factor with quartic fundamental colour factor [Lee, Smirnov, Smirnov, Steinhauser]

# The black-box interpolation problem

Given a rational function $f$ in the variables $\boldsymbol{z} = (z_1, \ldots, z_n)$ over $\mathcal{Q}$

- Reconstruct analytic form of $f$, given a numerical procedure

$$(\boldsymbol{z}, p) \longrightarrow \boxed{f} \longrightarrow f(\boldsymbol{z}) \bmod p.$$

- evaluate $f$ numerically for several $\boldsymbol{z}$ and $p$
- efficient multivariate reconstruction algorithms exist
  e.g. T.P. (2016,2019), Klappert, Lange (2019)
- upgrade analytic $f$ over $\mathcal{Q}$ using rational reconstruction algorithm [Wang (1981)] and Chinese remainder theorem

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**Question in this talk**

How to build the black box?

# Example: Scattering amplitudes over finite fields

T.P. (2016)

- External states (momenta and polarizations)
  - rational parametrization with momentum twistors variables

    Hodges (2009), Badger, Frellesvig, Zhang (2013), Badger (2016)
- Tree-level
  - diagrams or recursion relations (e.g. Berends-Giele)
- Loop integrands
  - Feynman diagrams and t'Hooft algebra
  - Unitarity cuts sewing tree-level currents
    - higher finite-dim. representation of internal states in dim. reg.
- Integrand reduction
  - linear fit to a "nice" integrand basis

# How to build the black box?

How to build a code for fast numerical evaluations of finite fields?

We can consider a few options:

1. Low-level coding (e.g. in C/C++/Fortran)?
   - ✓ very good runtime efficiency
   - ✗ harder to program
   - ✗ limits usability
2. Low-level coding + high-level interfaces?
   - common algorithms in C++ (e.g. linear solvers, fits, etc. . . )
   - high-level wrapper (e.g. for Mathematica/Python)
   - ✓ good efficiency and usability
   - ✗ not flexible
   - ✗ these algorithms are often intermediate steps

# How to build the black box?

Observations:

- A typical multi-loop algorithm involves several steps
  - solving linear systems
  - substitutions / changes of variables
  - etc. . .
- Large simplifications often occur at the very last stages
  - it's best to do everything numerically
  - only the final expression reconstructed analytically
- Many algorithms share common "building blocks"

# FiniteFlow: using data flow graphs

FiniteFlow [T.P. (2019)] has three main components

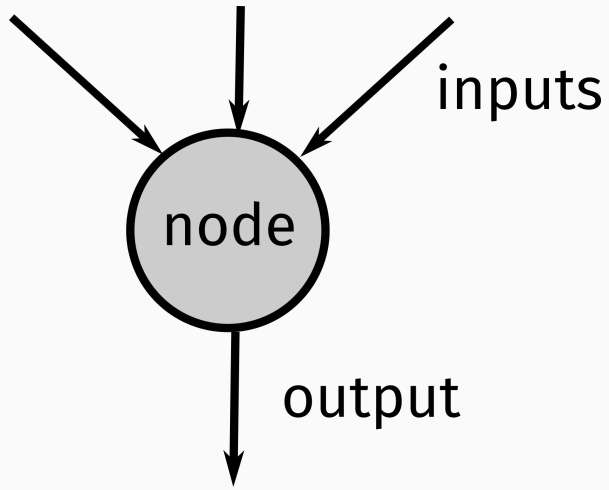
1. "basic" algorithms in C++ over finite fields
   - dense/sparse linear solvers, linear fits, evaluating rat. functions, list manipulations, etc. . .
2. higher-level framework to combine them into complex ones
   - output of a basic algorithm is input of others
   - graphical representation of your calculation (dataflow graphs)
3. multivariate reconstruction algorithms

## FiniteFlow

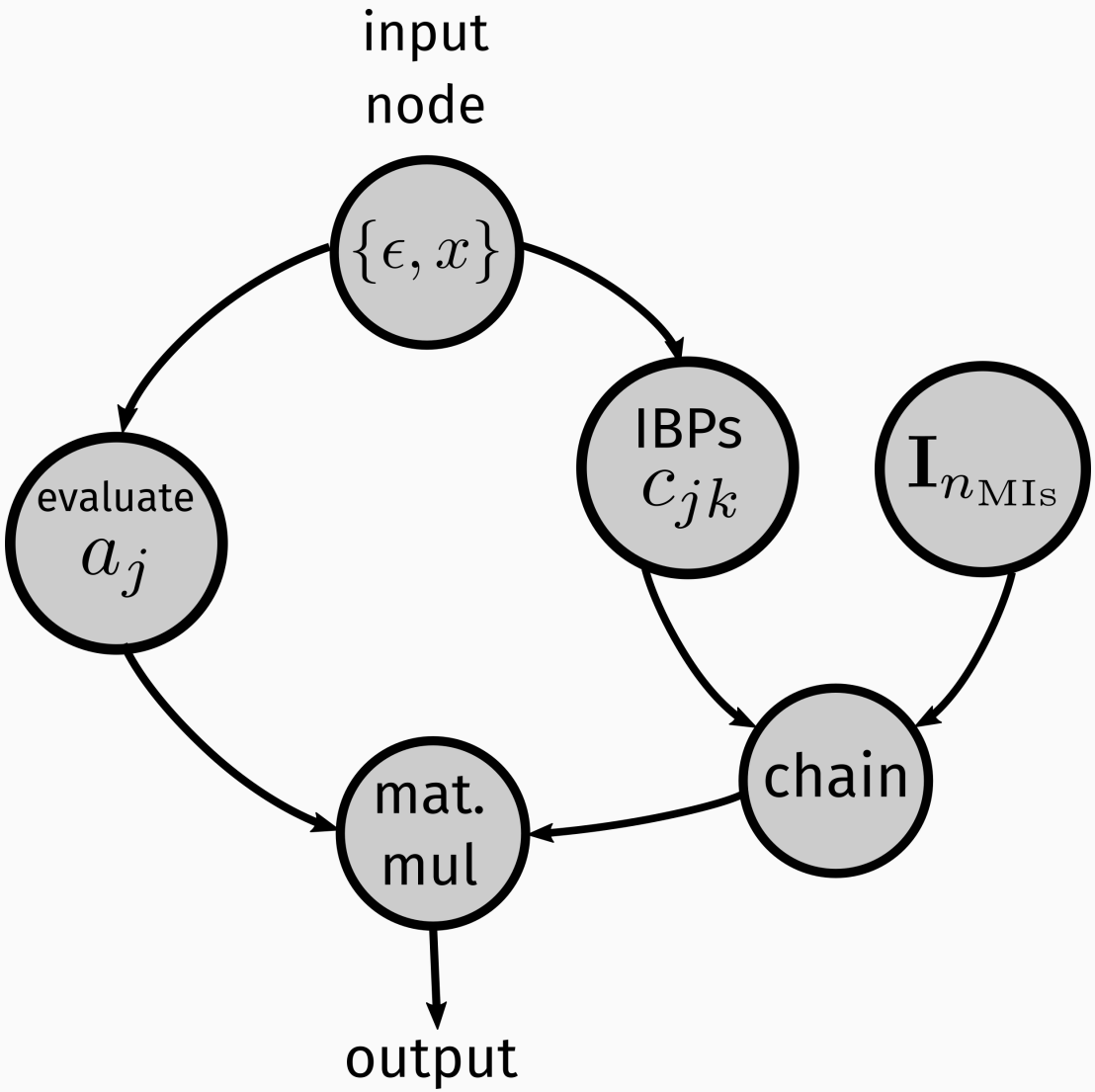
- build complex algorithms without any low-level programming
  (e.g. from Mathematica interface)
- many methods for amplitudes can be cast in this framework

# FiniteFlow: using data flow graphs

- FiniteFlow uses (simplified) data flow graphs
  - Nodes represent numerical algorithms
  - Arrows represent lists of numerical values
- In my implementation, a node has
  - 0 or more lists (arrows) of input values
  - 1 list (arrow) of output values



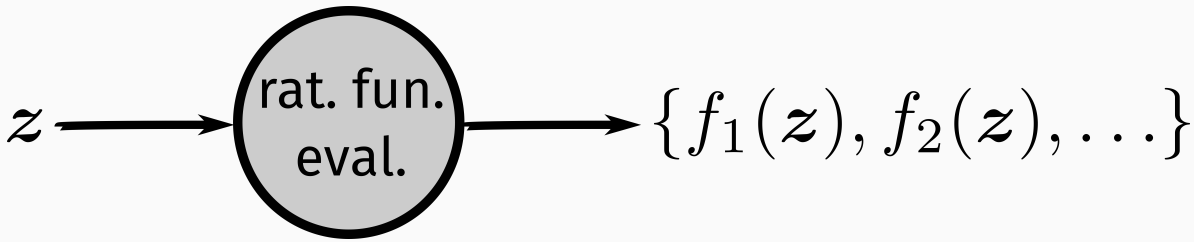
# Example of a graph



# Example: Evaluation of rational functions

- input: a list of values $\boldsymbol{z} = (z_1, \ldots, z_n)$
- output: a list of rational functions $\{f_1, f_2, \ldots\}$ at $\boldsymbol{z}$

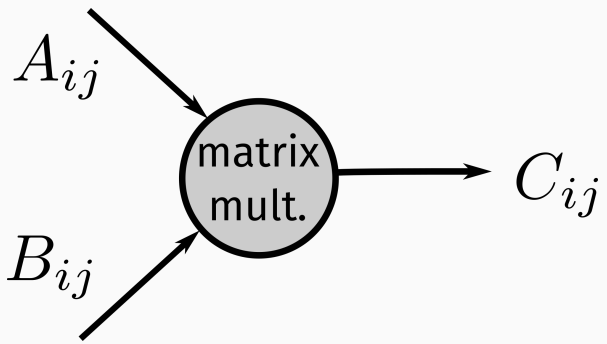
$$f_i(\boldsymbol{z}) = \frac{p_i(\boldsymbol{z})}{q_i(\boldsymbol{z})} = \frac{\sum_\alpha n_{i,\alpha}\, \boldsymbol{z}^\alpha}{\sum_\beta d_{i,\beta}\, \boldsymbol{z}^\beta},$$



# Example: Matrix multiplication

- Two lists as input
  1. entries of a matrix $A$
  2. entries of a matrix $B$
- use row-major order to store them as a list
- ouput: entries of matrix $C$ such that

$$C_{ij} = \sum_k A_{ik} \, B_{kj}$$



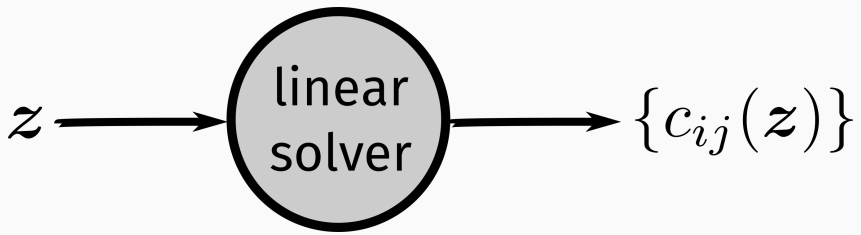
# Example: Linear solver

- A $n \times m$ linear system with parametric rational entries

$$\sum_{j=1}^{m} A_{ij}\, x_j = b_i, \quad (i = 1, \ldots, n), \qquad A_{ij} = A_{ij}(\boldsymbol{z}), \quad b_i = b_i(\boldsymbol{z})$$

- input: list of values for paramers $\boldsymbol{z} = (z_1, \ldots, z_n)$
- output: solution $c_{ij} = c_{ij}(\boldsymbol{z})$ such that

$$x_i = \sum_{j \in \text{indep}} c_{ij}\, x_j + c_{i0} \qquad (i \notin \text{indep})$$



# IBP reduction

- IBPs are large and sparse linear systems
- they reduce Feynman integrals $I_j$ to a lin. indep. set of MIs $G_j$

$$I_i = \sum_j c_{ij}\, G_j$$

- amplitudes and other multi-loop objects can be reduced mod IBPs

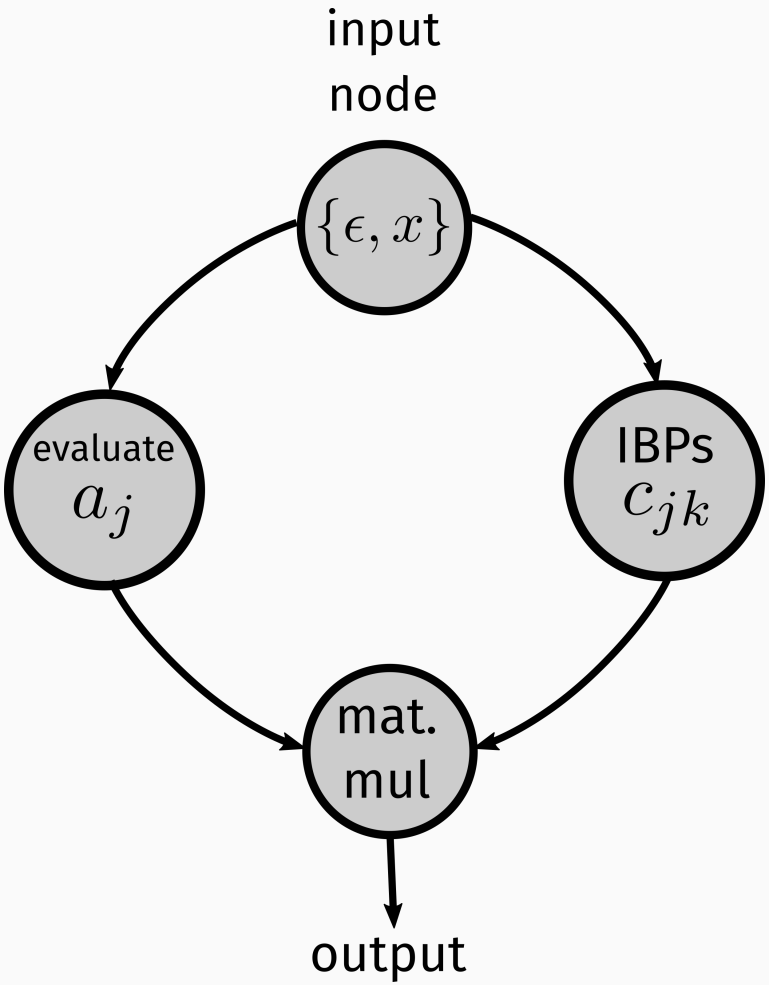
$$A = \sum_j a_j\, I_j = \sum_{jk} a_j\, c_{jk}\, G_k = \sum_j A_j\, G_j, \quad \text{with } A_j = \sum_k a_k\, c_{kj}$$

- final results for $A_k$ often much simpler than $c_{ij}$

$\Rightarrow$ solve IBPs numerically and compute $A_j$ via a matrix multiplication

- Similar approach for differential equations for MIs

# IBP reduction

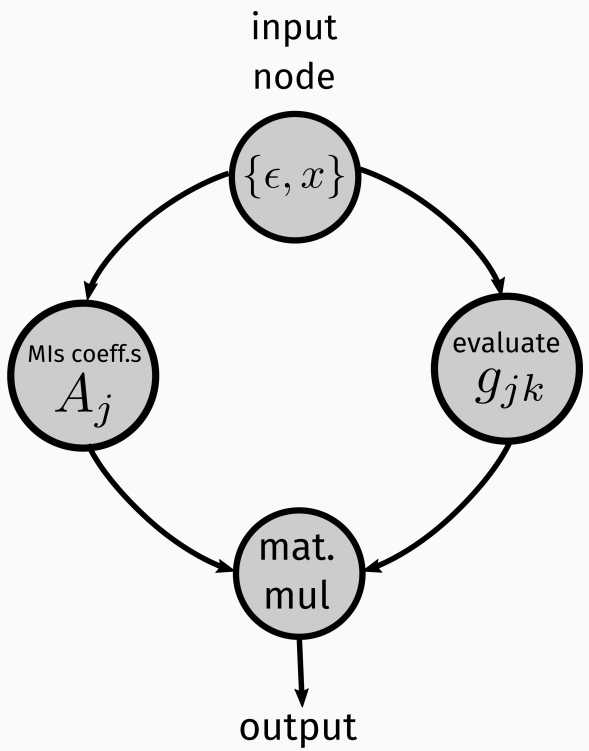


# Coefficients of the $\epsilon$-expansion

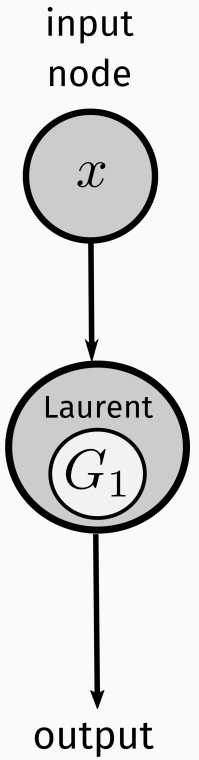
If MIs are known analytically in terms of special functions $f_k$

$$G_j = \sum_k g_{jk}(\epsilon, x)\, f_k + \mathcal{O}(\epsilon),$$

$G_1 =$



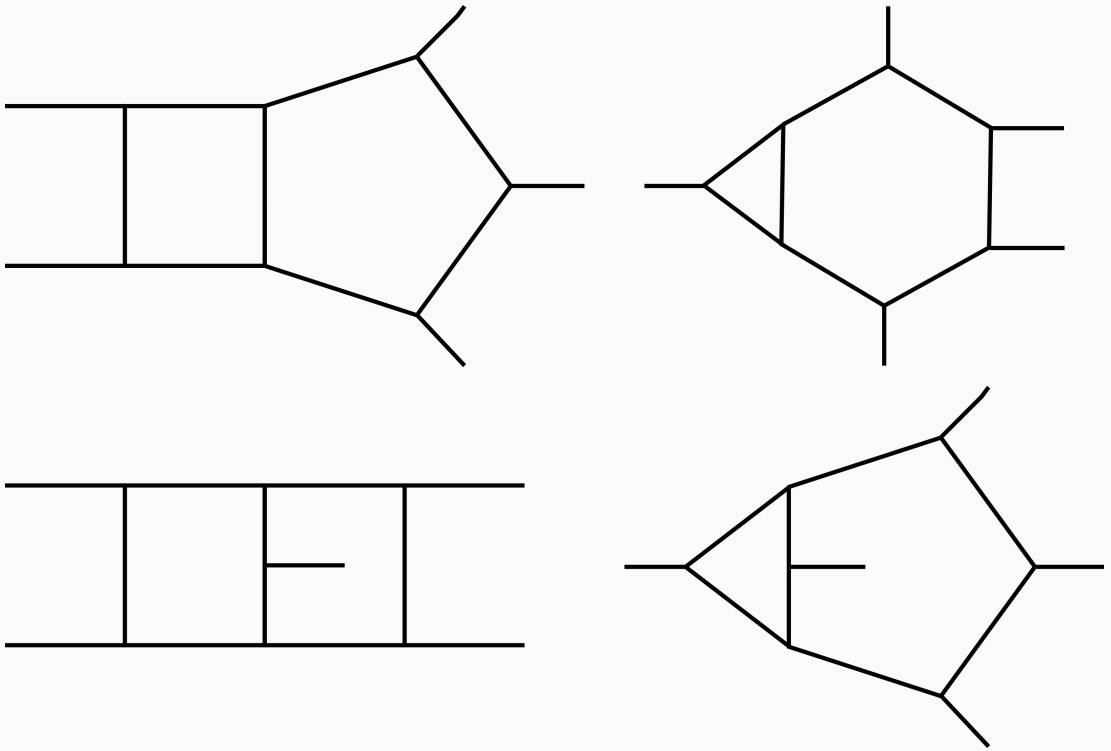

$G_2 =$



# Cutting-edge applications of FiniteFlow

- Five-point two-loop amplitudes
  - Several planar results for five partons and $W + 4$ partons

    [Badger, Brønnum-Hansen, Hartanto, T.P. (2017-2019)]
  - all-plus five gluon non-planar [Badger, Chicherin, Gehrmann, Heinrich, Henn, T.P., Wasser, Zhang, Zoia (2019)]



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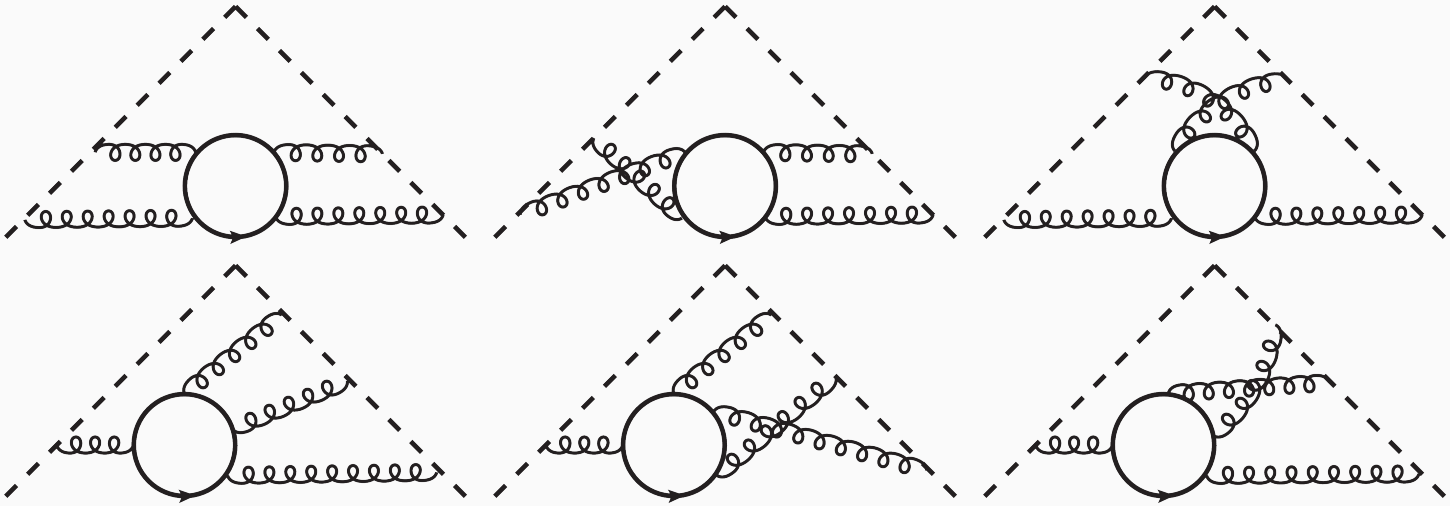
- Matter dependence of the 4-loop cusp anomalous dimension

  [Henn, T.P., Stahlhofen, Wasser (2019)]



# Public codes

- FiniteFlow

  `https://github.com/peraro/finiteflow`

  - C++ code
  - Mathematica interface (strongly recommended)

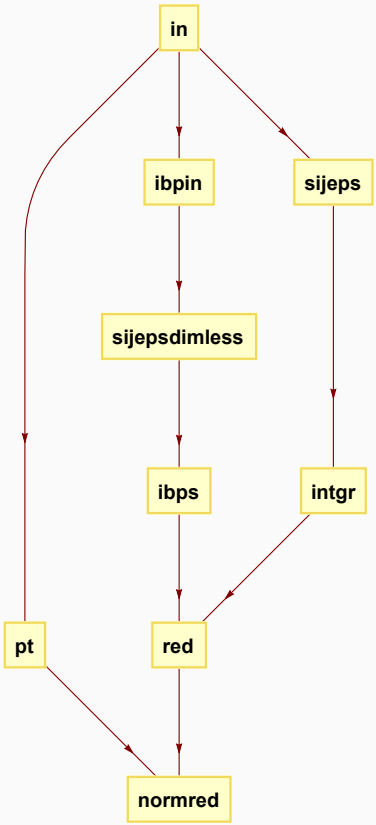
- FiniteFlow MathTools

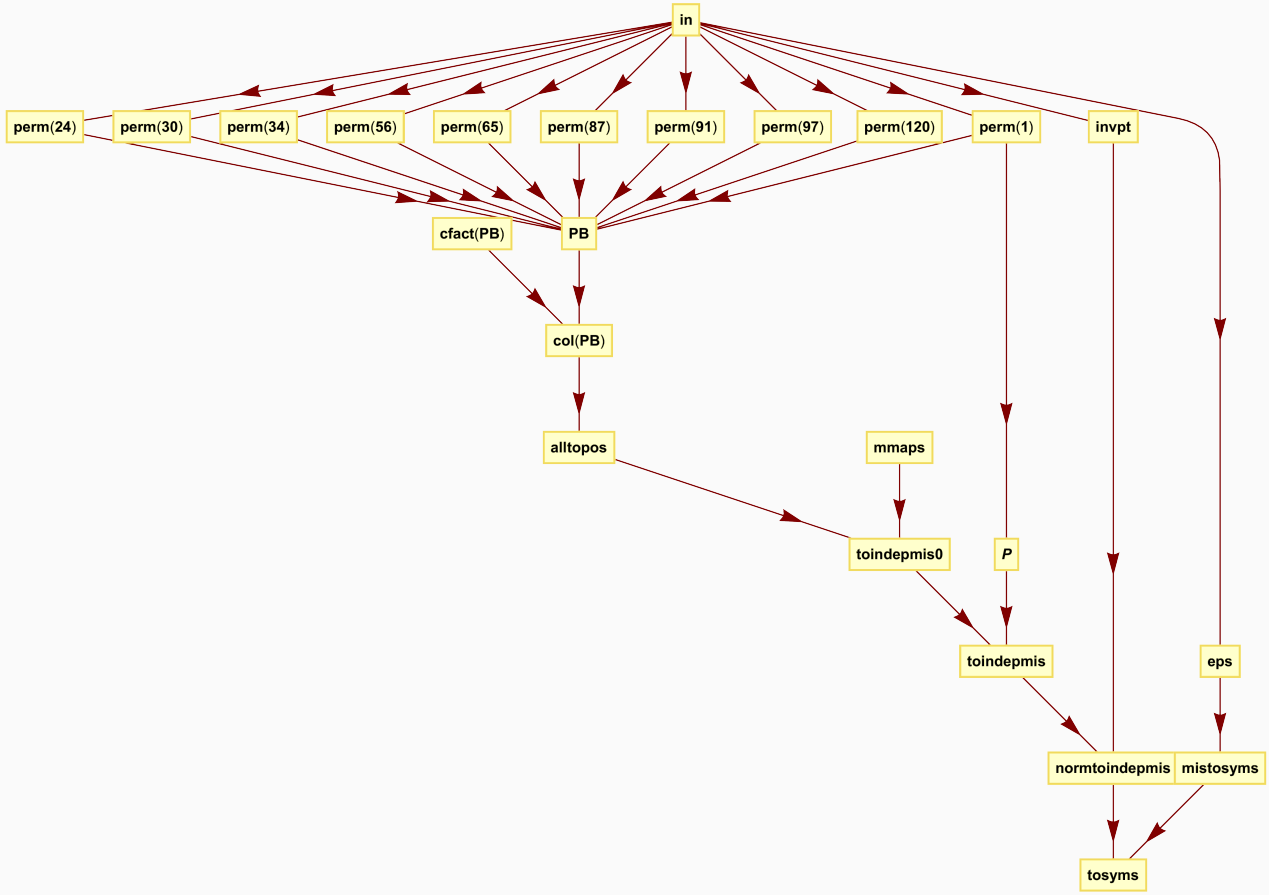
  `https://github.com/peraro/finiteflow-mathtools`

  - packages FFUtils, LiteMomentum, LiteIBP, Symbols
  - examples (amplitudes, IBPs, diff. equations and many more)

# Example of graphs in FiniteFlow

Piecing together the all-plus five gluon amplitude (only planar contributions are shown)



# Summary & Outlook

## Summary

- Finite fields and functional reconstruction
  - enhance the possibilities of our theoretical predictions
  - new results unattainable with traditional computer algebra
  - public code FiniteFlow
- Progress on 2-loop 5-point and other complex processes

## Outlook

- More applications
  - massive processes, phase-space integrals, ...
- High level of automation for higher-loop predictions