First Result for a Full Two-Loop Five-Gluon Amplitude

Simone Zoia

Max Planck Institute for Physics

Christmas Meeting 2019, Milano, 19th December 2019

European Research Council
Established by the European Commission

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

One year ago

Developments in scattering amplitudes for three-jet production at NNLO

Simone Zoia
zoia@mpp.mpg.de

Max Planck Institute for Physics, Munich

work in progress with Dmitry Chicherin, Thomas Gehrmann
Johannes Henn, Pascal Wasser, Yang Zhang

Milan Christmas Meeting, 21st December 2018

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

European Research Council

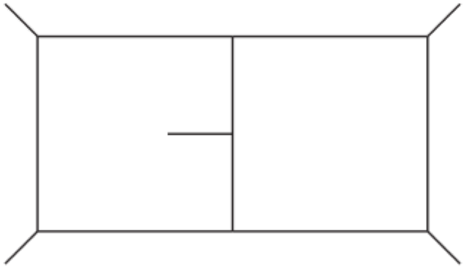
One week later…

- Symbol of the 2-loop 5-particle amplitude in N=4 super Yang-Mills

  [Abreu, Dixon, Herrmann, Page, Zeng '18] [Chicherin, Gehrmann, Henn, Wasser, Zhang, S.Z. '18]

- Missing non-planar integral family for massless 2-loop 5-particle amplitudes

  [Chicherin, Gehrmann, Henn, Wasser, Zhang, S.Z. '18]



This talk: 2-loop 5-gluon all-plus amplitude in Yang-Mills

[Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, S.Z. '19]
[Dunbar, Godwin, Perkins, Strong '19]

The pentagon team

Simon Badger
IPPP Durham

Dmitry Chicherin
MPI Munich

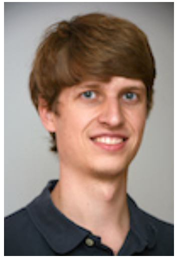
Gudrun Heinrich
MPI Munich

Thomas Gehrmann
U. Zurich

Johannes Henn
MPI Munich

Tiziano Peraro
U. Zurich

Yang Zhang
USTC Hefei



Pascal Wasser
U. Mainz

For many QCD processes, Next-to-Leading Order is insufficient

E.g. strong coupling from 3-jet/2-jet ratio: [CMS Collaboration ’13]

$$\alpha_s(M_Z) = 0.1148 \underset{\text{(exp.)}}{\pm 0.0014} \underset{\text{(PDF)}}{\pm 0.0018} \boxed{\underset{\text{(theory)}}{\pm 0.0050}}$$

Large theoretical uncertainty!

$\Longrightarrow$ Next-to-Next-to-Leading Order theory predictions needed!

Multi-jet processes are important for LHC phenomenology

State of the art: NNLO observables for 2→2 processes

Higher multiplicity is needed!

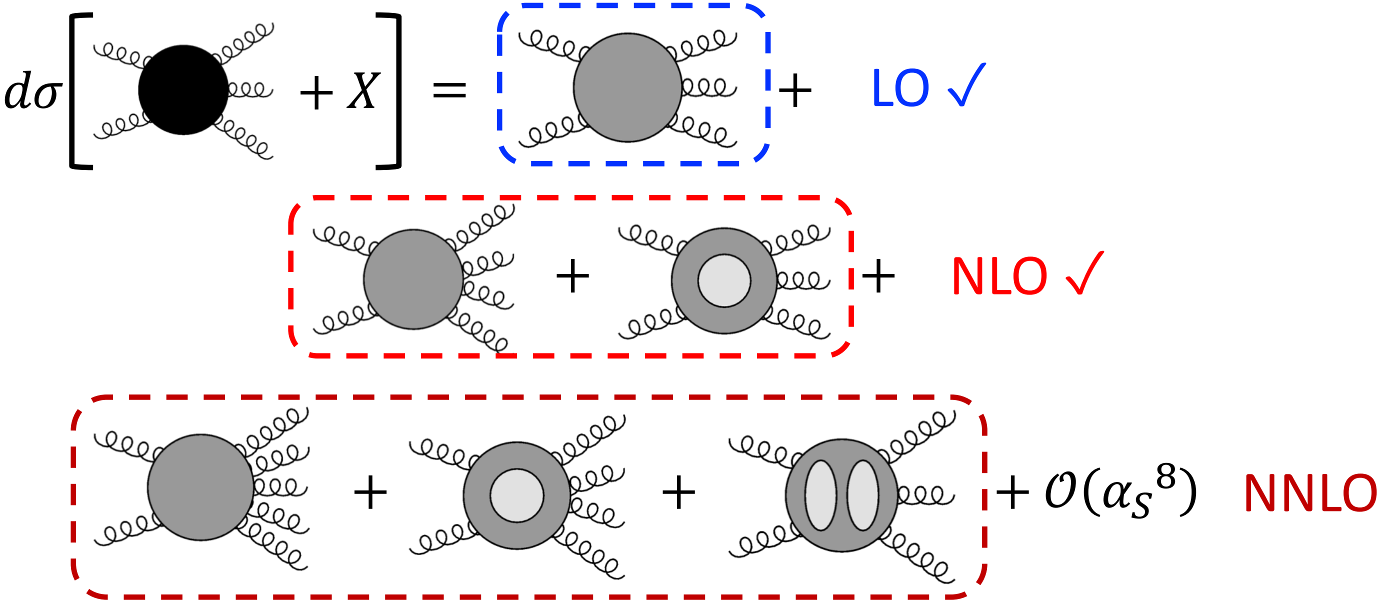
- Precision measurement of α_S,
- tests of Higgs couplings,
- new physics searches…

process	known	desired
$pp \to 2\,\mathrm{jets}$	$\mathrm{N^2LO_{QCD}}$ $\mathrm{NLO_{QCD}+NLO_{EW}}$	
$pp \to 3\,\mathrm{jets}$	$\mathrm{NLO_{QCD}}$	$\mathrm{N^2LO_{QCD}}$

Table I.2: Precision wish list: jet final states.

from "Les Houches 2017. Standard Model Working Group Report"

Five-particle cross sections at NNLO



Double real   Real-virtual   Double virtual

Amplitudes known
Non-trivial phase-space

HIC SUNT LEONES

Dramatic recent progress

- All QCD amplitudes known analytically in the *planar* limit

  [Gehrmann, Henn, Lo Presti '15][Dunbar, Perkins '16][Badger, Brønnum-Hansen, Hartanto, Peraro '18]
  [Abreu, Dormans, Febres Cordero, Ita, Page '18][Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov '19]

- NNLO QCD corrections to 3-photon production in the *planar* limit

  [Chawdhry, Czakon, Mitov, Poncelet '19]

- *Symbols* of N=4 super Yang-Mills and N=8 supergravity amplitudes

  [Abreu, Dixon, Herrmann, Page, Zeng '18 '19] [Chicherin, Gehrmann, Henn, Wasser, Zhang, S.Z. '18 '19]

- Full-color five-gluon all-plus amplitude in Yang-Mills theory

  [Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, S.Z. '19]
  [Dunbar, Godwin, Perkins, Strong '19]

  → this talk

A remarkably compact expression for the hard function

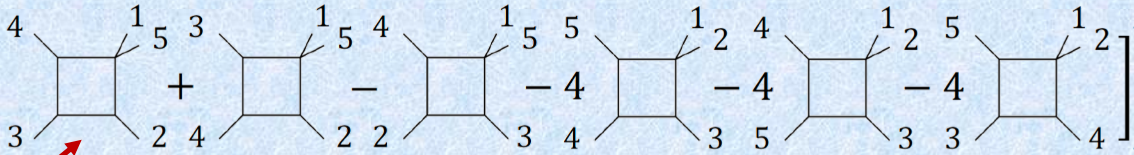
Permutations of external legs

Color $SU(N_c)$

$\kappa = \dfrac{d_s - 2}{6}$ with $d_s = g^{\mu}{}_{\mu}$ gluon spin dimension

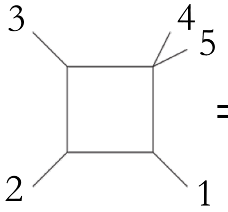
$$\mathcal{H}^{(2)}_{\substack{\text{double}\\ \text{trace}}} = \sum_{S_5/\Sigma} \mathrm{Tr}(12)[\mathrm{Tr}(345) - \mathrm{Tr}(543)] \sum_{\Sigma} \left\{ 6\,\kappa^2 \left[\frac{\langle 24\rangle[14][23]}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} + 9\,\frac{\langle 24\rangle[12][23]}{\langle 12\rangle\langle 34\rangle\langle 45\rangle^2} \right] \right.$$

$$\left. + \kappa\,\frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \left[\right. \right.$$



$$\left. \left. \right] \right\}$$

Finite part of the one-mass box



$$= \mathrm{Li}_2\left(1 - \frac{s_{12}}{s_{45}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{23}}{s_{45}}\right) + \log^2\left(\frac{s_{12}}{s_{23}}\right) + \frac{\pi^2}{6}$$

We are ready to tackle all two-loop five-parton amplitudes in QCD

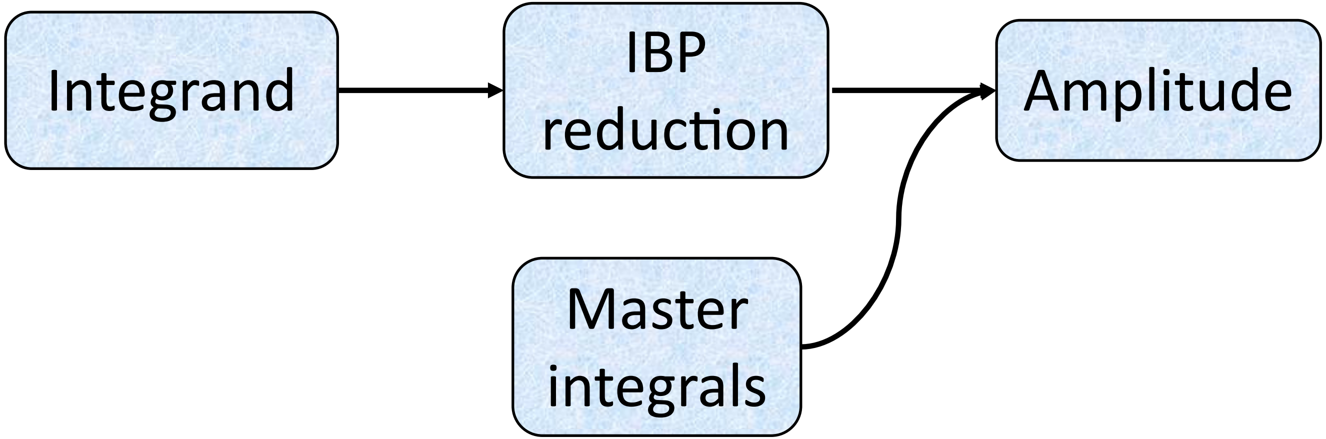
The all-plus helicity configuration is simple…

but

- We computed **all** the required Feynman integrals
- Our toolkit can be straightforwardly applied to all the other five-parton amplitudes

STAY TUNED!

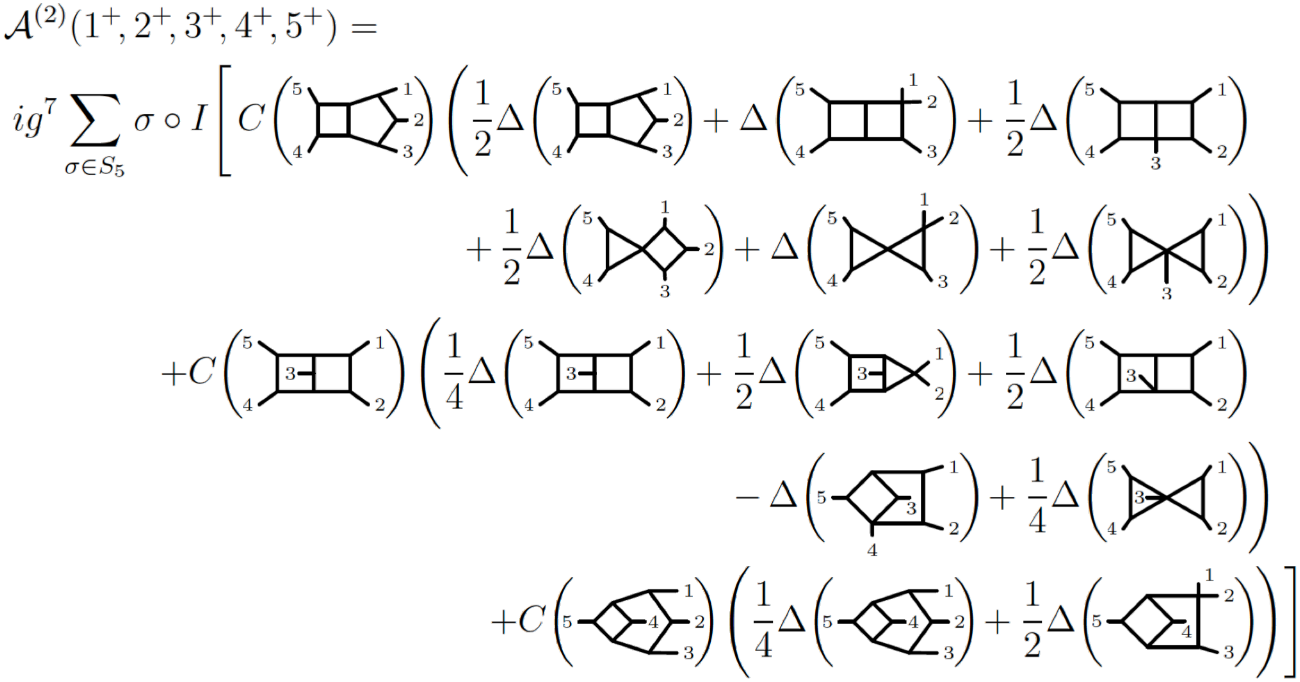
The workflow* of scattering amplitudes



* simplified

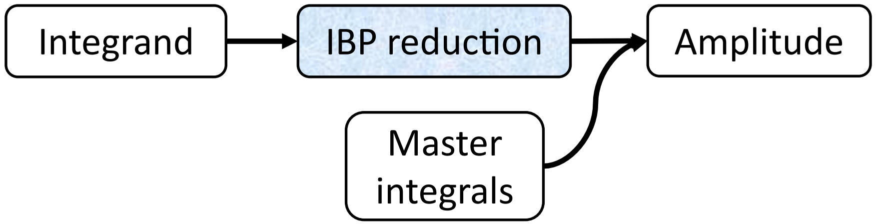
The two-loop five-gluon all-plus integrand

[Badger, Mogull, Ochirov, O'Connel '15]



Numerators with up to degree five/six in the loop momentum

Integration-by-parts relations



Integration-by-parts identities [Chetyrkin, Tkachov '81]

Any Feynman integral I can be "IBP-reduced" to a finite number of master integrals

$$I(s, \varepsilon) = \sum_i c_i(s, \varepsilon)\, g_i(s, \varepsilon)$$

$$D = 4 - 2\,\varepsilon$$

$$\mathrm{s} = (s_{12}, s_{23}, s_{34}, s_{45}, s_{51})$$

Dramatic progress in IBP reduction from finite field techniques

Numerical evaluation over finite fields + rational reconstruction

[Schabinger, von Manteuffel ’15][Peraro ’16, ’19]
[Maierhöfer, Usovitch ’18] [Smirnov, Chukharev ’19]

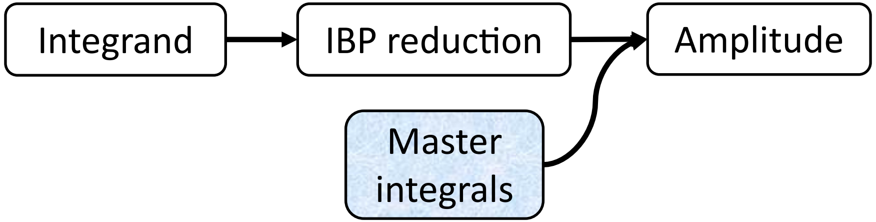
using **FiniteFlow** [Peraro ‘19]

- Faster Integration-by-Parts (IBP) reduction
- Better scaling for multi-scale problems

Further optimisation: reconstruct only the final physical answer

→ Tiziano's talk

The workflow of scattering amplitudes



Integration-by-Parts identities

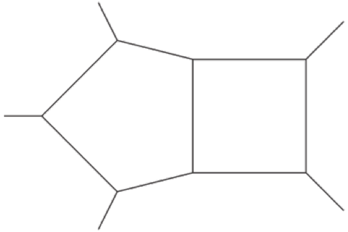
$$I(s,\varepsilon) = \sum_i c_i(s,\varepsilon)\, g_i(s,\varepsilon)$$

Master integrals

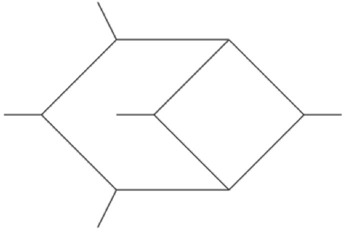
$$D = 4 - 2\,\varepsilon$$

$$\mathrm{s} = (s_{12}, s_{23}, s_{34}, s_{45}, s_{51})$$

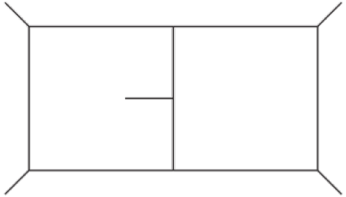
The integral families for massless five-particle scattering at two loops



[Gehrmann, Henn, Lo Presti ’15, ’18]
[Papadopoulos, Tommasini, Wever 15’]



[Böhm, Georgoudis, Larsen, Schönemann, Zhang ’18]
[Abreu, Page, Zeng 18’]
[Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser ’18]



[Abreu, Dixon, Herrmann, Page, Zeng ’18]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, S.Z. ’18]
[Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, S.Z. ‘19]

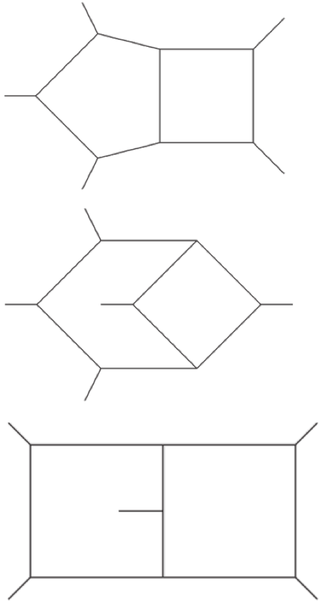
All master integrals for massless 2-loop 5-particle scattering amplitudes are known

Computed using the method of the differential equations in the canonical form [Henn ‘13]

$$\boxed{d\vec{g}(s,\varepsilon) = \varepsilon\, d\tilde{A}(s) \;\cdot \vec{g}(s,\varepsilon)}$$

- Refined procedure to find canonical bases
- Boundary constants determined analytically from physical constraints

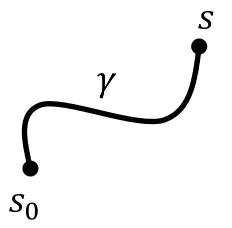
All integrals known analytically in the physical scattering region for all permutations of the external legs



Pentagon functions

Proposed in [Chicherin, Henn, Mitev ’17], confirmed in [Abreu, Dixon, Herrmann, Page, Zeng ’18] [Chicherin, Gehrmann, Henn, Wasser, Zhang, S.Z. ‘18]

Iterated integrals along contour γ



$$[W_{i_1}, \ldots, W_{i_n}]_{s_0} = \int_\gamma d\log W_{i_1}(s) \ \ldots \ d\log W_{i_n}(s)$$

boundary point

$\{W_i(s)\} = 31-$letter alphabet $\qquad n =$ transcendental weight

The letters encode all possible physical and spurious singularities of amplitudes

Letter	s notation	momentum notation	cylic
W_1	s_{12}	$2p_1 \cdot p_2$	$+\ (4)$
W_6	$s_{34} + s_{45}$	$2p_4 \cdot (p_3 + p_5)$	$+\ (4)$
W_{11}	$s_{12} - s_{45}$	$2p_3 \cdot (p_4 + p_5)$	$+\ (4)$
W_{16}	$s_{45} - s_{12} - s_{23}$	$2p_1 \cdot p_3$	$+\ (4)$
W_{21}	$s_{34} + s_{45} - s_{12} - s_{23}$	$2p_3 \cdot (p_1 + p_4)$	$+\ (4)$
W_{26}	$\frac{s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}-s_{12}s_{15}-s_{45}s_{15}-\sqrt{\Delta}}{s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}-s_{12}s_{15}-s_{45}s_{15}+\sqrt{\Delta}}$	$\frac{\mathrm{tr}[(1-\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}{\mathrm{tr}[(1+\gamma_5)\not{p}_4\not{p}_5\not{p}_1\not{p}_2]}$	$+\ (4)$
W_{31}	$\sqrt{\Delta}$	$\mathrm{tr}[\gamma_5\not{p}_1\not{p}_2\not{p}_3\not{p}_4]$	

Adapted from [Gehrmann, Henn, Lo Presti '18]

$\Delta = \det\left(2\, p_i \cdot p_j\right)_{i,j=1,\ldots,4}$

Letters encode all possible physical and spurious singularities of amplitudes

Soft & collinear limits

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W_{31}	$\sqrt{\Delta}$	$\mathrm{tr}[\gamma_5\not{p}_1\not{p}_2\not{p}_3\not{p}_4]$	

Complex phases

Adapted from [Gehrmann, Henn, Lo Presti '18]

$$\Delta = \det\left(2\, p_i \cdot p_j\right)_{i,j=1,\ldots,4}$$

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W_{31}	$\sqrt{\Delta}$	$\mathrm{tr}[\gamma_5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4]$	

Complex phases

Gram determinant

$$\Delta = \det(2\, p_i \cdot p_j)_{i,j=1,\ldots,4}$$

Adapted from [Gehrmann, Henn, Lo Presti '18]

Pentagon functions in terms of familiar functions (to mathematicians and particle theorists)

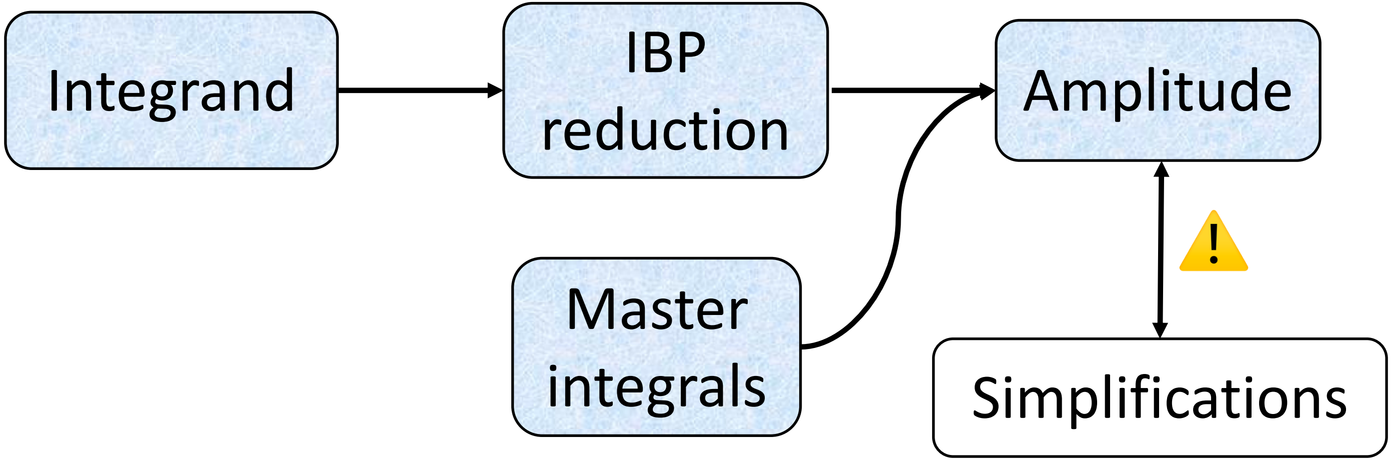
At NNLO, up to four iterations (weight) needed:

$$\int d\log W_a \int d\log W_b \int d\log W_c \int d\log W_d = [W_a, W_b, W_c, W_d]_{s_0}$$

- Up to weight 2: logarithms and dilogarithms
- In general: Goncharov polylogarithms

Well studied
Numerical routines $\Longrightarrow$ numerical and analytical control

Amplitude assembly – Ideal world



Amplitude = $\varepsilon \otimes$ rational functions $\otimes$ pentagon functions

Infrared singularities factorize

$$\mathcal{A} \;=\; \mathcal{Z} \cdot \mathcal{A}^f$$

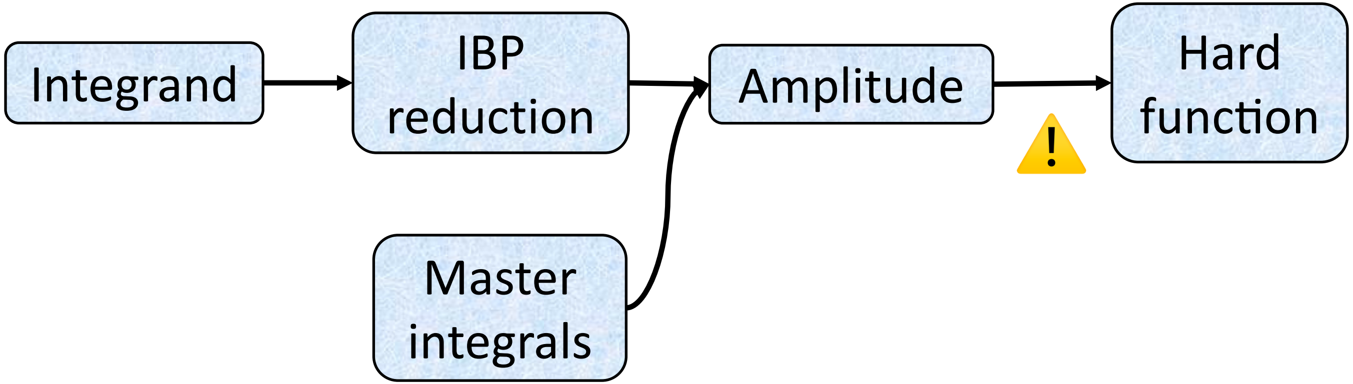
Finite

Captures all IR singularities (poles in ε)
Matrix in color space

We can define an IR safe hard function $\mathcal{H} = \lim_{\varepsilon \longrightarrow 0} \mathcal{A}^f$

- Truly new piece of information
- Much simpler than the amplitude

Directly target the hard function



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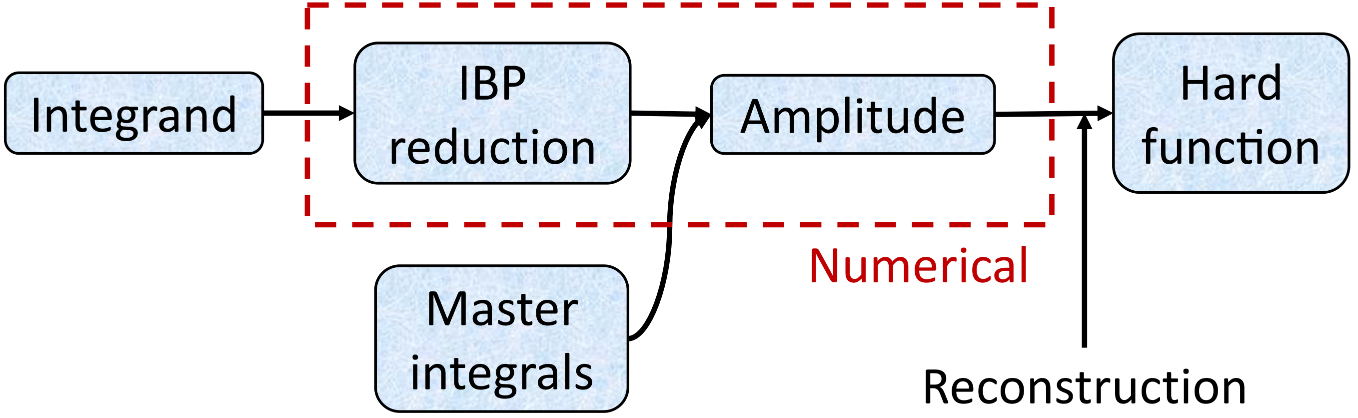


Taming the rational functions

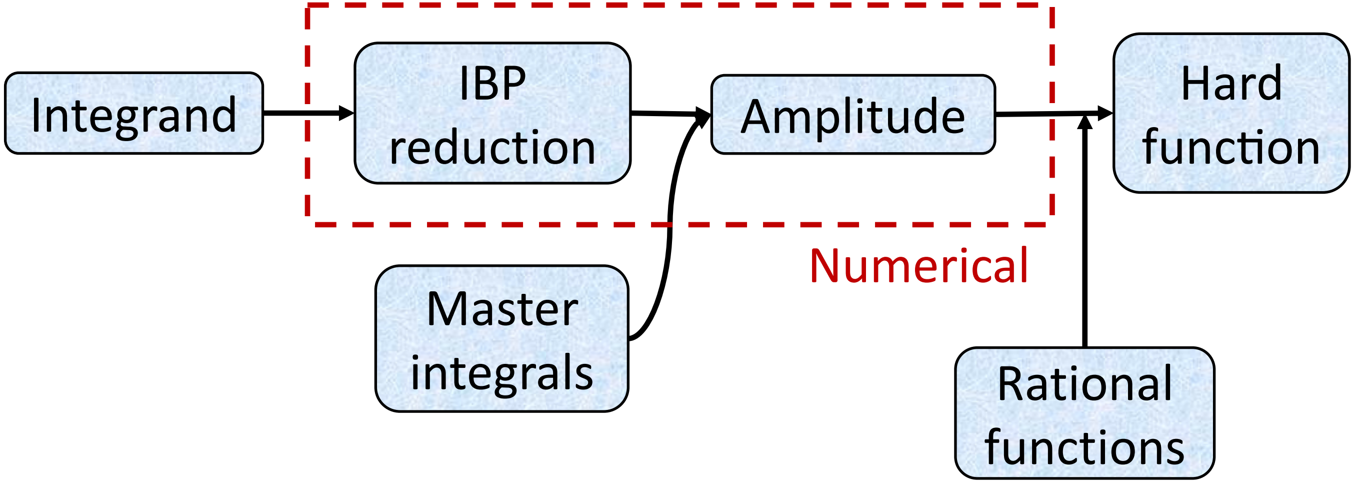
- Rational reconstruction over finite fields

- Leading singularity analysis of the integrand

- Analysis of lower loop/planar amplitudes ✓

  Planar two-loop five-gluon all-plus amplitude [Gehrmann, Henn, Lo Presti ‘15]

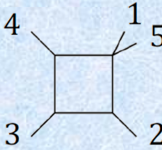
Directly target the hard function



A remarkably compact expression for the non-planar two-loop hard function

$$\mathcal{H}^{(2)}_{\substack{\text{double}\\\text{trace}}} = \sum_{S_5/\Sigma} \mathrm{Tr}(12)[\mathrm{Tr}(345) - \mathrm{Tr}(543)] \sum_{\Sigma} \Bigg\{ 6\,\kappa^2 \left[\frac{\langle 24\rangle[14][23]}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} + 9\,\frac{\langle 24\rangle[12][23]}{\langle 12\rangle\langle 34\rangle\langle 45\rangle^2} \right]$$

$$+ \kappa\,\frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} \Bigg[$$



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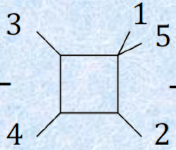

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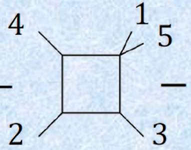
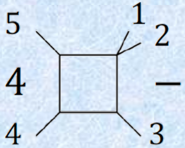


$-\,4$



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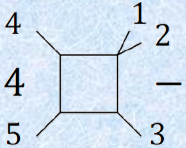
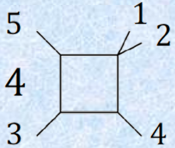


$-\,4$



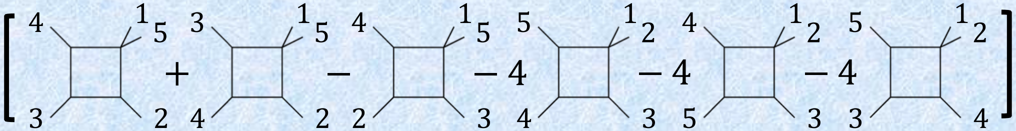
$$\Bigg]\Bigg\}$$

- ➢ All weight 1, 3 and 4 iterated integrals cancel out
- ➢ Valid in all physical regions ($s_{ij} \to s_{ij} + i0$)

Correct factorization in the collinear limits ✓

Hints of conformal symmetry in the leading transcendental weight part

$$+\kappa\,\frac{[15]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle}$$



Manifestly conformally invariant rational factors

$$k_{\alpha\dot\alpha}\,\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} = 0 \qquad k_{\alpha\dot\alpha} = \sum_{i=1}^{5}\frac{\partial^2}{\partial\lambda_i^{\alpha}\,\partial\tilde\lambda_i^{\dot\alpha}}$$

[Witten '03]

Related to conformal invariance of one-loop amplitude

$$A_1^{(1,0)} \propto \kappa\left(\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}+\frac{[23]^2}{\langle 45\rangle\langle 51\rangle\langle 14\rangle}+\frac{[52]^2}{\langle 41\rangle\langle 13\rangle\langle 34\rangle}\right)$$

[Henn, Power, S.Z. '19]

But this is another story!

Summary

- Very first analytic, complete 2-loop 5-particle amplitude
    - ✓ Non-planar
    - ✓ Function level

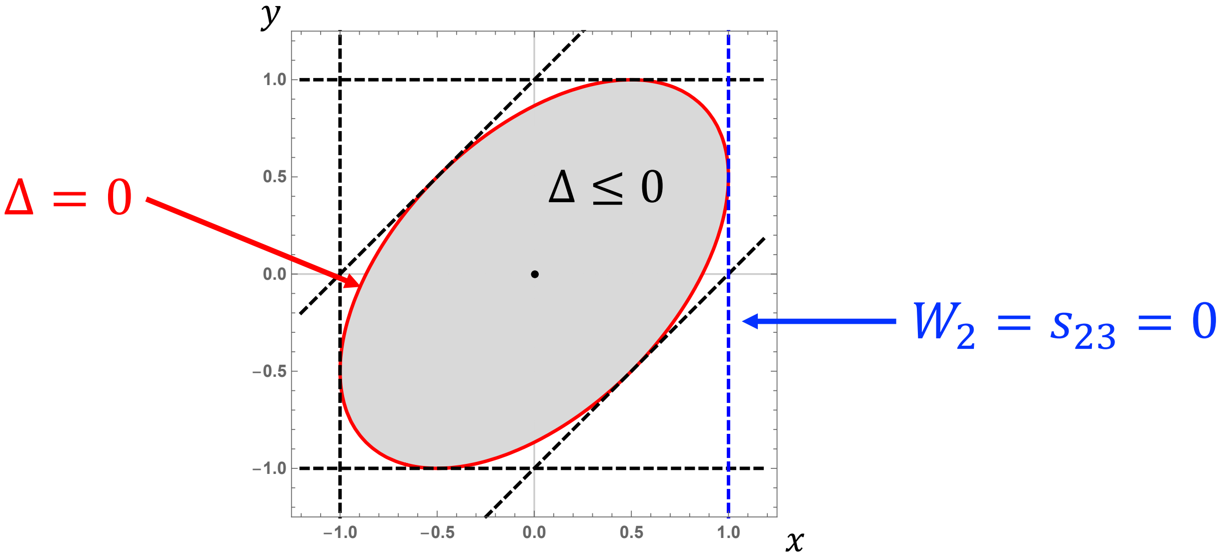
  Intriguing conformal symmetry properties

- All two-loop master integrals for generic five-particle QCD amplitudes known analytically in the physical region
    - ✓ Full analytical and numerical control over the integrals

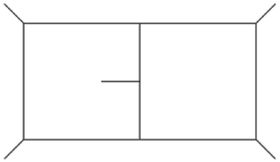
Physical $\boldsymbol{s_{12}}$ channel

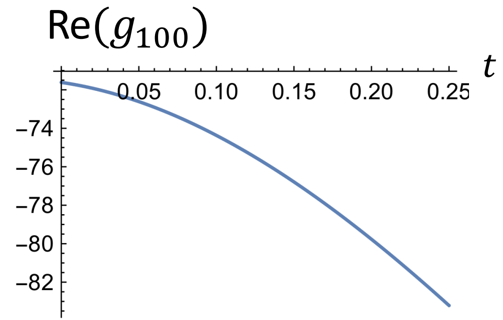
Positive s-channel energies, negative t-channel energies, real momenta $\Delta \leq 0$

2D slice: $(s_{12}, s_{23}, s_{34}, s_{45}, s_{51}) = (3, -1+x, 1, 1, -1+y)$



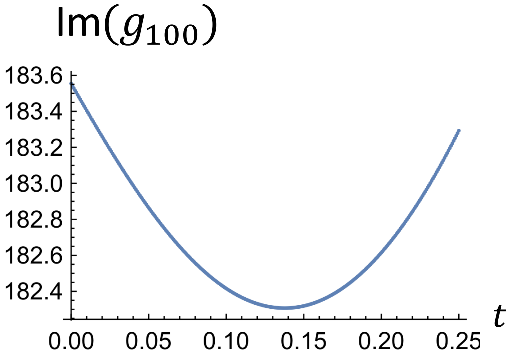
Numerical evaluation in the physical region

$g_{100} =$ 



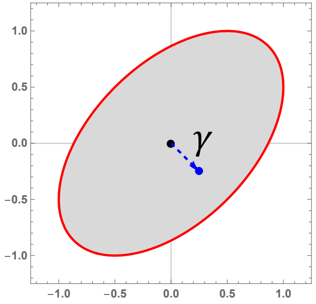
- Written in terms of pentagon functions
- Result in terms of Goncharov polylogs

$$\gamma(t) = \left(3, -1 + \frac{t}{t^2+1}, 1, 1, -1 - \frac{t}{t^2+1}\right)$$



- Numerical evaluations (GiNaC)

  Checks using SecDec ✓



Color decomposition

The amplitudes are vectors in color space $\boldsymbol{SU(N_c)}$

Color relations

$$\mathcal{A}_5^{(1)} = \sum_{\lambda=1}^{12} N_c A_\lambda^{(1,0)} \mathrm{T}_\lambda + \sum_{\lambda=13}^{22} A_\lambda^{(1,1)} \mathrm{T}_\lambda$$

$$\mathcal{A}_5^{(2)} = \sum_{\lambda=1}^{12} \left(N_c^2 A_\lambda^{(2,0)} + A_\lambda^{(2,2)} \right) \mathrm{T}_\lambda + \sum_{\lambda=13}^{22} N_c A_\lambda^{(2,1)} \mathrm{T}_\lambda$$

NEW

Basis of single and double traces:

$$\mathrm{T}_1 = \mathrm{Tr}(12345) - \mathrm{Tr}(15432)$$

$$\mathrm{T}_{13} = \mathrm{Tr}(12)[\mathrm{Tr}(345) - \mathrm{Tr}(543)]$$

Generators of $\boldsymbol{SU(N_c)}$

and permutations thereof

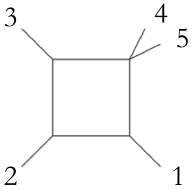
The planar two-loop hard function

$$\mathcal{H}_1^{(2,0)} = \sum_{\text{cyclic}} \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\left\{-\kappa\, \frac{s_{51}}{s_{24}}\, \mathrm{tr}_-(4512)\, I_{234;51} + \kappa^2\left[5\, s_{12}s_{23} + s_{12}s_{34} + \frac{\mathrm{tr}_+^2(1245)}{s_{12}s_{45}}\right]\right\}$$

[Gehrmann, Henn, Lo Presti '15]

Finite part of the one-mass box

$$I_{123;45} =$$



$$= \mathrm{Li}_2\left(1 - \frac{s_{12}}{s_{45}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{23}}{s_{45}}\right) + \log^2\left(\frac{s_{12}}{s_{23}}\right) + \frac{\pi^2}{6}$$

$\kappa = \dfrac{d_s - 2}{6}$ with $d_s = g^{\mu}{}_{\mu}$ gluon spin dimension

$$\mathrm{tr}_\pm(ijkl) = \mathrm{tr}\left[\frac{1 \pm \gamma_5}{2}\, p_i p_j p_k p_l\right]$$

Rational factors from the planar two-loop hard function

$$\mathcal{H}_1^{(2,0)} = \sum_{\text{cyclic}} \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\left\{-\kappa\, \frac{s_{51}}{s_{24}}\,\mathrm{tr}_-(4512)\, I_{234;51} + \kappa^2\left[5\, s_{12}s_{23} + s_{12}s_{34} + \frac{\mathrm{tr}_+^2(1245)}{s_{12}s_{45}}\right]\right\}$$

They generate a 76-dimensional space upon permutations of the external legs

$$\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \Rightarrow$$ 6-dimensional subspace

- ❖ Manifestly conformally invariant
- ❖ Related to the one-loop amplitude

$$A_1^{(1,0)} \propto \kappa\left(\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} + \frac{[23]^2}{\langle 45\rangle\langle 51\rangle\langle 14\rangle} + \frac{[52]^2}{\langle 41\rangle\langle 13\rangle\langle 34\rangle}\right)$$

[Henn, Power, S.Z. '19]

Analytic calculation of the master integrals via differential equations

IBPs

$$\frac{\partial g_i(s,\varepsilon)}{\partial s} = \text{linear combination of } I \overset{\text{IBPs}}{=} \sum_j c_j(s,\varepsilon) g_j(s,\varepsilon)$$

$$\Longrightarrow \boxed{d\vec{g}(s,\varepsilon) = dA(s,\varepsilon) \cdot \vec{g}(s,\varepsilon)}$$

"Messy" solution:

algebraic functions

w-fold iterated integral

$$g(s,\varepsilon) = \frac{1}{\varepsilon^4} \sum_{p=0}^{\infty} \varepsilon^p \sum_k r_k(s) \sum_{w=0}^{p} h_{p,k}^{(w)}(s)$$

Differential equations in the canonical form

Change of master integral basis s.t.

$$\boxed{d\vec{g}(s,\varepsilon) = \varepsilon\, d\tilde{A}(s)\ \cdot \vec{g}(s,\varepsilon)}$$

[Henn ’13]

$$d\tilde{A}(s) = \sum_i c_i\, d\log W_i(s)$$

Letters, algebraic functions

Constant matrices

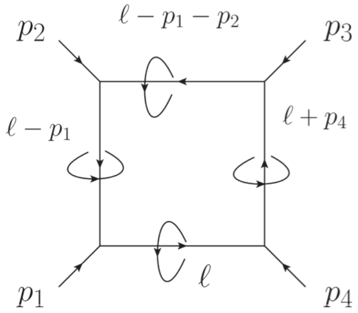
Solution has uniform transcendentality

w-fold iterated integral

$$\vec{g}(s,\varepsilon) = \mathbb{P}\mathrm{exp}\left(\varepsilon \int_\gamma d\tilde{A}\right) \vec{g}(s_0,\varepsilon) \to \frac{1}{\varepsilon^4} \sum_{w=0}^{\infty} \varepsilon^w h^{(w)}(s)$$

Four-dimensional leading singularities

$$\int \frac{d^4\ell}{P_1P_2P_3P_4} = \int_{-\infty}^{\infty} d\ell^0 \int_{-\infty}^{\infty} d\ell^1 \int_{-\infty}^{\infty} d\ell^2 \int_{-\infty}^{\infty} d\ell^3 \, \frac{1}{P_1P_2P_3P_4} \longrightarrow$$



$$\frac{1}{(2\pi i)^4} \oint_{P_1=0} d\ell^0 \oint_{P_2=0} d\ell^1 \oint_{P_3=0} d\ell^2 \oint_{P_4=0} d\ell^3 \, \frac{1}{P_1P_2P_3P_4} = \frac{1}{s\,t}$$

$$\boxed{\int_{\mathbb{R}^{1,3}} (\text{4D integrand}) \longrightarrow \int_{\mathbb{T}^4} (\text{4D integrand}) \;= \text{leading singularity}}$$

[Arkani-Hamed, Cachazo, Cheung, Kaplan ‘09]

Integrands with unit leading singularity can be cast into dlog form

$$\frac{d^4\ell \;\, s\, t}{\ell^2\,(\ell-p_1)^2(\ell-p_1\;-p_2)^2(\ell+p_4)^2} =$$

$$d\log\frac{\ell^2}{(\ell-\ell^*)^2}\wedge d\log\frac{(\ell-p_1)^2}{(\ell-\ell^*)^2}\wedge d\log\frac{(\ell-p_1-p_2)^2}{(\ell-\ell^*)^2}\wedge d\log\frac{(\ell+p_4)^2}{(\ell-\ell^*)^2}$$

where ℓ^* is the solution of the maximal cut

$$(\ell^*)^2 = (\ell^*\; - \; p_1)^2 = \; (\ell^* \; - \; p_1 - p_2)^2 = (\ell^* + \; p_4)^2 = 0$$

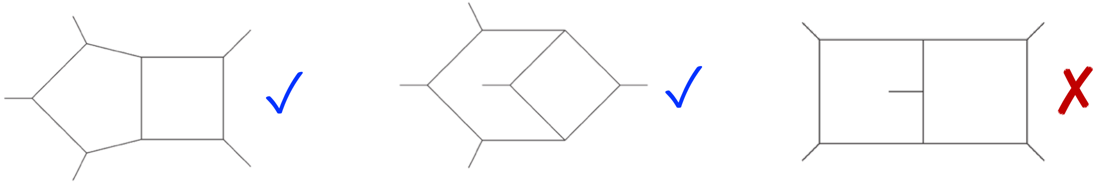
Integrands with dlog integrand evaluate to $\mathbb{Q}$-linear combinations of iterated integrals of uniform weight

[Arkani-Hamed, Bourjaily, Cachazo, Trnka ’10]

Algorithmic construction of canonical bases

Algorithm to find all 4D dlog integrands [Wasser '16]

$\Longrightarrow$ Naïvely upgrade to D dimensions and extract a canonical basis



Subtlety: 4D analysis blind to integrands that vanish in 4D

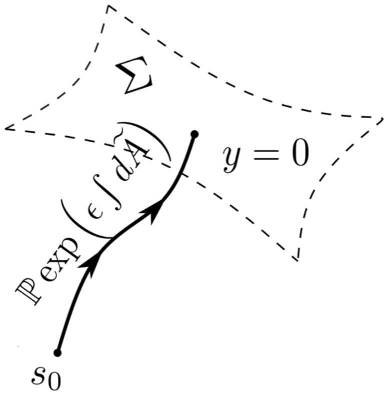
$\Longrightarrow$ D-dimensional leading singularity analysis based on Baikov representation of the integrands

[Chicherin, Gehrmann, Henn, Wasser, Zhang, S.Z. '18]

The boundary constants are constrained by physical requirements

Canonical basis $\vec{g}(s,\varepsilon)$ is UV finite

$\Longrightarrow \vec{g}(s,\varepsilon)$ are **finite** at small $\varepsilon < 0$



Spurious singularities at $W_i = 0$

$$d\vec{g}(s,\varepsilon) = \varepsilon\, d\left[\sum_i c_i\, d\log W_i(s)\right] \cdot \vec{g}(s,\varepsilon)$$

Asymptotic solution near $y = W_1 = 2\,p_1 \cdot p_2 = 0$

$$d\tilde{A}(s) = \tilde{c}\; d\log(y) + \mathcal{O}(y) \quad \Longrightarrow \quad \vec{g}(s,\varepsilon) \sim \exp(\varepsilon\, \tilde{c}\, \log y) \cdot \vec{J}$$

Finiteness of $\vec{g}(s,\varepsilon)$ at $y = 0 \implies$ constraints on $\vec{J}$

Transport constraints to base point s_0