The partonic structure of the electron at the next-to-leading logarithmic accuracy in QED

in collaboration with
V. Bertone, M. Cacciari, S. Frixione
based on: arXiv:1909.03886, arXiv:1911.12040

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Introduction

- Not unreasonable to assume that the future of high-energy physics will involve an e^+e^- collider $\rightarrow$ enlarge the legacy of LEP
- Presence in the matrix elements of logarithmic terms numerically large and thus prevent the perturbative series from being well behaved:

 $\propto \alpha \log\left(\frac{E}{m_e}\right)$

- A large class of them is process-independent and can be accounted in a universal manner $\rightarrow$ structure-function approach: collect these terms in QED PDFs and then resum them by means of evolution equations.
- Crucial difference w.r.t. hadronic PDFs: QED PDFs are entirely calculable with perturbative techniques!
- QED PDFs known at leading-logarithmic (LL) accuracy i.e. resummation of $(\alpha \log(E/m_e))^k$ terms. Goal here: reach NLL accuracy i.e. extend the resummation to the $\alpha(\alpha \log(E/m_e))^k$ terms.

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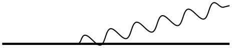

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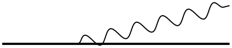

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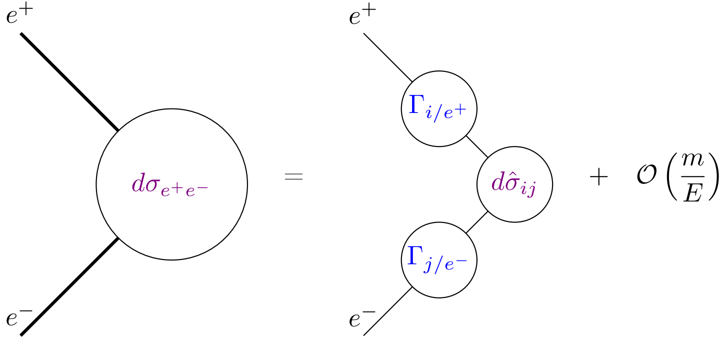
QED factorisation formula

particle-level (with mass) vs. *parton*-level (massless) cross section

$$d\sigma_{e^+e^-}(p_{e^+}, p_{e^-}) = \sum_{ij=e^\pm,\gamma} \int dz_+ dz_- \, \Gamma_{i/e^+}(z_+, \mu^2, m^2)\, \Gamma_{j/e^-}(z_-, \mu^2, m^2)$$

$$\times\, d\hat{\sigma}_{ij}(z_+ p_{e^+}, z_- p_{e^-}, \mu^2) + \mathcal{O}\left(\frac{m}{E}\right)$$



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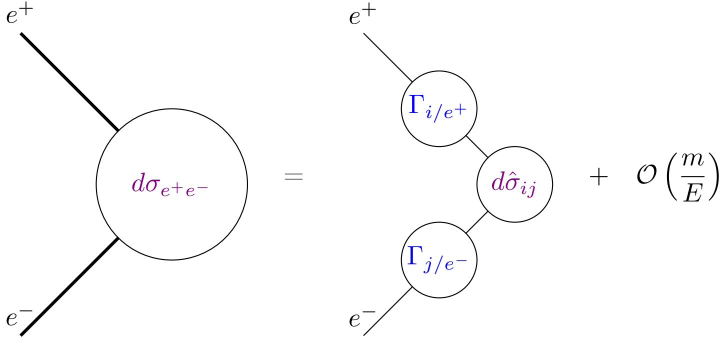

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Initial conditions for the PDFs of the electron

By explicit computation at NLO [Frixione (2019)], one founds Γ up to $\mathcal{O}(\alpha)$:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

This result is interpreted as the initial condition at a scale $\mu_0 \sim m$.

$$\Gamma_i^{[0]}(z,\mu_0^2) = \delta_{ie^-}\delta(1-z)$$
$$\Gamma_{e^-}^{[1]}(z,\mu_0^2) = \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z)(= 0 \text{ in } \overline{\text{MS}})$$
$$\Gamma_{\gamma}^{[1]}(z,\mu_0^2) = \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z)(= 0 \text{ in } \overline{\text{MS}})$$
$$\Gamma_{e^+}^{[1]}(z,\mu_0^2) = 0$$

Electron PDF, apart from factors, is the same as the initial condition for b-quark fragmentation function obtained in [Mele and Nason (1991)].

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Evolution operator formalism

Gribov-Lipatov equation in z-space:

$$\frac{\partial \Gamma(z, \mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} [\mathbb{P} \otimes \Gamma](z, \mu^2) \tag{1}$$

In Mellin space, we can introduce the evolution operator:

$$\Gamma_N(\mu^2) = \mathbb{E}_N(\mu^2, \mu_0^2)\, \Gamma_{0,N}, \quad \mathbb{E}_N(\mu_0^2, \mu_0^2) = \mathbb{I} \tag{2}$$

Convenient to introduce a variable t:

$$t = \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} = \frac{\alpha(\mu)}{2\pi} \log \frac{\mu^2}{\mu_0^2} + \mathcal{O}(\alpha^2) \tag{3}$$

i.e. LL terms: t^k and NLL terms: $(\alpha/(2\pi))\, t^k$. We obtain:

$$\frac{\partial \mathbb{E}_N(t)}{\partial t} = \left[\mathbb{P}_N^{[0]} + \frac{\alpha(\mu)}{2\pi} \left(\mathbb{P}_N^{[1]} - \frac{2\pi b_1}{b_0} \mathbb{P}_N^{[0]} \right) \right] \mathbb{E}_N(t) + \mathcal{O}(\alpha^2) \tag{4}$$

which can be solve analytically (NS) or numerically (S/γ).

ePDF

Public numerical code, written in `C++`, available here:

`https://github.com/gstagnit/ePDF`

- Evolution equation solved in Mellin space by means of a discretised path-ordered product (see e.g. [Bonvini (2012)]) or adopting the U-matrix formalism (see e.g. [Vogt (2005)])
- Numerical inverse Mellin transform with an algorithm based on an optimized path in the complex plane (Talbot path)

In the code you can also find:

- a routine for the evolution of α at NLL
- all the analytical solutions (recursive, asymptotic and matched)

Outline

Analytical solutions

Why?

- benchmark of the numerical result
- rapid growth of the electron PDF at $z \to 1$ (initial condition at $\mathcal{O}(\alpha^0)$ is $\delta(1-z)$) $\to$ analytical knowledge crucial in the context of numerical computations of cross sections

How?

- solving the evolution equations order by order in perturbation theory directly in the z-space $\to$ recursive solution (calculated up to $\mathcal{O}(\alpha^3)$)
- going to Mellin space, using the large-N behaviour of the evolution operator E_N and then analytically anti-trasforming back to z-space $\to$ asymptotic large-z solution (all order in α)
- combining the two to obtain predictions in the whole z range.

Recursive solutions

Approach already known at LL (see e.g. [Skrzypek and Jadach (1991), Cacciari et al. (1992)]), we extended it at NLL with α running.
Starting point: rewriting the evolution equation eq. (1) in an integral form:

$$\frac{\partial \mathcal{F}(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi}[\mathbb{P}\,\overline{\otimes}\,\mathcal{F}](z,\mu^2) \tag{5}$$

where:

$$\mathcal{F}(z,\mu^2) = \int_z^1 dy\,\Gamma(y,\mu^2) \quad \Longrightarrow \quad \Gamma(z,\mu^2) = -\frac{\partial}{\partial z}\mathcal{F}(z,\mu^2) \tag{6}$$

and

$$g\,\overline{\otimes}_z h = \int_z^1 dx\, g(x)\, h\left(\frac{z}{x}\right) \tag{7}$$

$\mathcal{F}$ represented as a power series:

$$\mathcal{F}(z,t) = \sum_{k=0}^{\infty}\left(\frac{t^k}{k!}\mathcal{J}_k^{\mathrm{LL}}(z) + \frac{\alpha(t)}{2\pi}\frac{t^k}{k!}\,\mathcal{J}_k^{\mathrm{NLL}}(z)\right) \tag{8}$$

By replacing eq. (8) in eq. (5), we find the following recurrence relations:

$$\begin{aligned}
\mathcal{J}_k^{\text{LL}} &= \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\text{LL}} \\
\mathcal{J}_k^{\text{NLL}} &= (-)^k (2\pi b_0)^k \mathcal{J}_0^{\text{NLL}} \\
&\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p < \left(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\text{NLL}} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\text{LL}} - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\text{LL}} \right)
\end{aligned}$$

with the $\mathcal{J}_0^{\text{LL}}$ and $\mathcal{J}_0^{\text{NLL}}$ terms related to the integral of the initial conditions. The recursive solutions is then:

$$\Gamma(z, \mu^2) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(J_k^{\text{LL}}(z) + \frac{\alpha(t)}{2\pi} J_k^{\text{NLL}}(z) \right) \tag{9}$$

with

$$J_k^{\text{LL}}(z) = -\frac{d}{dz} \mathcal{J}_k^{\text{LL}}(z) \,, \qquad J_k^{\text{NLL}}(z) = -\frac{d}{dz} \mathcal{J}_k^{\text{NLL}}(z) \,. \tag{10}$$

We calculated J_k^{LL} up to $k = 3$ and J_k^{NLL} up to $k = 2$ for three electron PDFs.

Non-singlet recursive solution up to $\mathcal{O}(\alpha^2)$: LL + NLL terms

$$\Gamma(z,t) = \frac{\alpha(t)}{2\pi}\left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right] + t\left(\frac{1+z^2}{1-z}\right)$$

$$+ t\frac{\alpha(t)}{2\pi}\left[-12\frac{\log^2(1-z)}{1-z} + 12\log^2(1-z)\right.$$

$$+ (8\pi b_0 + 8L_0 - 14)\frac{\log(1-z)}{1-z} + (-8\pi b_0 - 8L_0 + 10)\log(1-z)$$

$$+ \frac{1}{1-z}\left(-\frac{4\pi b_1}{b_0} + (6-4\pi b_0)L_0 + 4\pi b_0 - \frac{20N_F}{9} + \frac{4\pi^2}{3} + 1\right)$$

$$\left. + \frac{4\pi b_1}{b_0} + (4\pi b_0 - 2)L_0 - 4\pi b_0 + \frac{32N_F}{9} - \frac{4\pi^2}{3} - 2 + \hat{J}_1^{\mathrm{NLL}}(z)\right]$$

$$+ \frac{t^2}{2}\left[\frac{1}{1-z}\left(4\left(z^2+1\right)\log(1-z) + (z+4)z - (1+3z^2)\log z + 1\right)\right] + \mathcal{O}(\alpha^3)$$

where $L_0 = \log\frac{\mu_0^2}{m^2}$, $\hat{J}_1^{\mathrm{NLL}}(z)$ are terms vanishing in the $z \to 1$ limit.

Full non-singlet recursive solution up to $\mathcal{O}(\alpha^3)$

$$\hat{J}^{\mathrm{LL}}_{\mathrm{NS},0} = 1 - z$$

$$\hat{J}^{\mathrm{LL}}_{\mathrm{NS},1} = \frac{1}{z-1} \times \Big(3z^2\log(z) + (1-z)(z+3) - 4(z-1)^2\log(1-z) + \log(z)\Big)$$

$$\hat{J}^{\mathrm{LL}}_{\mathrm{NS},2} = \frac{1}{4(z-1)} \times \Big(-24(z^2-1)\mathrm{Li}_2(z) + 24(-z^2+2(z^2+1)\log(z)+3)\log(1-z) - 2\log(z)(-9z^2+7z^2\log(z)+$$
$$\log(z)-3) + (z-1)(4\pi^2(3z-1)-3(5z+19)) - 48(z-1)^2\log^2(1-z) - 48z(\log(2-2z)-\log(2z))\Big)$$

$$\hat{J}^{\mathrm{NLL}}_{\mathrm{NS},0} = (1-z)(L_0 - 2\log(1-z) - 1)$$

$$\hat{J}^{\mathrm{NLL}}_{\mathrm{NS},1} = \frac{1}{18b_0(z-1)(z+1)} \times \Big(36\pi b_0^2L_0z^3 - 36\pi b_0^2L_0z^2 - 36\pi b_0^2L_0z + 36\pi b_0^2L_0 - 36\pi b_0^2z^3 - 72\pi b_0^2z^3\log(1-$$
$$z) + 36\pi b_0^2z^2 + 72\pi b_0^2z^2\log(1-z) + 36\pi b_0^2z + 72\pi b_0^2z\log(1-z) - 72\pi b_0^2\log(1-z) - 36\pi b_0^2 - 18b_0L_0z^3 -$$
$$72b_0L_0z^3\log(1-z) + 54b_0L_0z^3\log(z) - 54b_0L_0z^2 + 72b_0L_0z^2\log(1-z) + 54b_0L_0z^2\log(z) + 18b_0L_0z +$$
$$72b_0L_0z\log(1-z) + 18b_0L_0z\log(z) - 72b_0L_0\log(1-z) + 18b_0L_0\log(z) + 54b_0L_0 + 44b_0N_Fz^3 +$$
$$12b_0N_Fz^3\log(z) - 68b_0N_Fz^2 + 12b_0N_Fz^2\log(z) - 44b_0N_Fz + 12b_0N_Fz\log(z) + 12b_0N_F\log(z) +$$
$$68b_0N_F + 72b_0(z^3-z^2-2z+2)\mathrm{Li}_2(-z) + 216b_0z\mathrm{Li}_2\left(\frac{1}{z+1}\right) - 36b_0(z-1)(z+1)^2\mathrm{Li}_2(1-z) -$$
$$216b_0\mathrm{Li}_2\left(\frac{1}{z+1}\right) + 153b_0z^3 - 6\pi^2b_0z^3 + 108b_0z^3\log^2(1-z) - 27b_0z^3\log^2(z) + 54b_0z^3\log(1-z) -$$
$$90b_0z^3\log(z) - 72b_0z^3\log(1-z)\log(z) + 72b_0z^3\log(z)\log(z+1) - 135b_0z^2 + 6\pi^2b_0z^2 - 108b_0z^2\log^2(1-$$
$$z) + 9b_0z^2\log^2(z) + 18b_0z^2\log(1-z) - 90b_0z^2\log(z) - 72b_0z^2\log(1-z)\log(z) - 72b_0z^2\log(z)\log(z+$$
$$1) - 153b_0z - 18\pi^2b_0z - 108b_0z\log^2(1-z) - 9b_0z\log^2(z) + 108b_0z\log^2(z+1) + 108b_0\log^2(1-z) +$$
$$27b_0\log^2(z) - 108b_0\log^2(z+1) - 54b_0z\log(1-z) + 72b_0z\log(z) - 144b_0z\log(z)\log(z+1) - 18b_0\log(1-$$
$$z) + 72b_0\log(z) + 144b_0\log(z)\log(z+1) + 135b_0 + 18\pi^2b_0 + 36\pi b_1z^3 - 36\pi b_1z^2 - 36\pi b_1z + 36\pi b_1\Big)$$

$$\hat{J}^{\mathrm{NLL}}_{\mathrm{NS},2} = \frac{1}{36b_0z(z^2-1)} \times \Big((1-z)z(144(L_0-1)\pi^2(z^2-1)b_0^3 - 8\pi(18L_0(z+1)(z+3) - 2N_F(z+1)(11z-$$
$$17) + 3(3(-8+\pi^2)z^2 - 15z + \pi^2 + 9))b_0^2 + (2(-2N_F(4\pi^2(z-1) - 19z + 63)(z+1) + 9(57z-73)(z+$$
$$1) + 6\pi^2(z(17z+3) + 12b_1(z^2-1) - 2)) - 9L_0(z+1)(8\pi^2(z-1) - 3(5z+19)))b_0 - 144b_1\pi(z+1)(z+$$
$$3)) + 2(6b_0\log^3(z)z^4 - 18b_0\log^2(z)z^4 - 63b_0L_0\log^2(z)z^4 - 36b_0N_F\log^2(z)z^4 + 54b_0^2\pi\log^2(z)z^4 -$$
$$216b_0^2\pi(\log(1-z) - \log(z))z^4 - 342b_0\log(z)z^4 + 81b_0L_0\log(z)z^4 - 156b_0N_F\log(z)z^4 +$$
$$72b_0\log^2(2)\log(z)z^4 - 72b_0\log(2)\log(z)z^4 - 18b_0\pi^2\log(z)z^4 - 216b_1\pi\log(z)z^4 - 216b_0^2L_0\pi\log(z)z^4 -$$
$$24b_0^2N_F\pi\log(z)z^4 + 18b_0\log^3(z)z^3 + 144b_0\log^2(z)z^3 - 63b_0L_0\log^2(z)z^3 - 36b_0N_F\log^2(z)z^3 -$$
$$18b_0^2\pi\log^2(z)z^3 + 234b_0\log(z)z^3 + 297b_0L_0\log(z)z^3 - 12b_0N_F\log(z)z^3 - 72b_0\log^2(2)\log(z)z^3 +$$
$$144b_0\log(2)\log(z)z^3 + 42b_0\pi^2\log(z)z^3 + 288b_0^2\pi\log(z)z^3 - 216b_1\pi\log(z)z^3 - 216b_0^2L_0\pi\log(z)z^3 -$$
$$24b_0^2N_F\pi\log(z)z^3 - 6b_0\log^3(z)z^2 + 54b_0\log^2(z)z^2 - 9b_0L_0\log^2(z)z^2 - 12b_0N_F\log^2(z)z^2 +$$
$$18b_0^2\pi\log^2(z)z^2 + 306b_0\log(z)z^2 + 243b_0L_0\log(z)z^2 + 68b_0N_F\log(z)z^2 - 72b_0\log^2(2)\log(z)z^2 +$$
$$288b_0\log(2)\log(z)z^2 + 66b_0\pi^2\log(z)z^2 - 36b_0^2\pi\log(z)z^2 - 72b_1\pi\log(z)z^2 - 72b_0^2L_0\pi\log(z)z^2 -$$
$$24b_0^2N_F\pi\log(z)z^2 + 288b_0(z-1)^2(z+1)\log^3(1-z)z - 18b_0\log^3(z)z - 48b_0(z-4)(z-1)(z+4)\log^3(z+1)z -$$
$$144b_0\log^2(z)z - 9b_0L_0\log^2(z)z - 12b_0N_F\log^2(z)z - 54b_0^2\pi\log^2(z)z - 270b_0\log(z)z + 27b_0L_0\log(z)z -$$
$$76b_0N_F\log(z)z + 72b_0\log^2(2)\log(z)z - 144b_0\log(2)\log(z)z - 42b_0\pi^2\log(z)z - 108b_0^2\pi\log(z)z -$$
$$72b_1\pi\log(z)z - 72b_0^2L_0\pi\log(z)z - 24b_0^2N_F\pi\log(z)z - 36b_0(z+1)\log^2(1-z)((6L_0(z-1) + 12b_0\pi(z-1) -$$
$$5z-7)(z-1) + (11z^2+1)\log(z))z + \log(1-z)(144\pi^2(z-1)^2(z+1)b_0^3 + 72\pi(4L_0(z+1)(z-1)^2 - 3z(z-1)+$$
$$3)b_0^2 - (z-1)(108L_0(z+1)(z+3) + 32N_F((3-8z)z+11) + 3(-27z(7z+4) + 4\pi^2(9z^2+z-6) + 120\log^2(2)+$$
$$81))b_0 + 12(3(z-1)(z(2z+5)+1)\log^2(z) + (z+1)((30\pi b_0 + 27L_0 + 4N_F - 30)z^2 - 24z + 9L_0 + 4N_F +$$
$$6b_0\pi + 27)\log(z) + 24(z-1)(2\log(2) - \log(z))\log(z+1))b_0 + 288b_1\pi(z-1)^2(z+1))z - 108b_0\log^2(z) -$$
$$18b_0(-16\log(2)z^4 + 8(\log(z) + 2\log(2))z^3 + 4(15z-13)\log(z)z + 8(z-1)(\log(1-z) + \log(z) + 2\log(2))z+$$
$$(z-1)(z(12\pi b_0 + z(14z+29) - 23) - 12))\log^2(z+1) + 12b_0\log(2)(-4\pi^2z(z^3+2z-1) + (z-1)(z+$$
$$1)(z(9z+7)+3)\log(8) + 2z(z(4z^2+z+16) - 11)\log^2(2)) - 216b_0\log(2)\log(z) - 12b_0(6z(z+1)(z(2z-$$
$$3)+2)\log^2(z) + 6(z-1)z(-4z + 2b_0\pi(z^2-2) - 3)\log(z) + (z-1)(12z\log^2(2)(z^2+4) + \pi^2(z^3+z)+$$
$$6(z((-1+2\log(2))z^2 + z + 5) - 2z(\log(1-z) + \log(2)) + 3)(\log(z) + \log(2))))\log(z+1)) + 18b_0(-$$
$$16z(z+1)\mathrm{Li}_3\left(\frac{1-z}{2}\right)(z-1)^2 + 16z(z+1)\mathrm{Li}_3\left(\frac{z-1}{2z}\right)(z-1)^2 - 56z(z+1)\mathrm{Li}_3\left(\frac{z}{z+1}\right)(z-1)^2 + 16z(z+$$
$$1)\mathrm{Li}_3\left(\frac{2z}{z+1}\right)(z-1)^2 - 16z((\log(z+1) - \log(1-z))z^2 + \log(1-z) - \log(z+1) - 5\log(2))\mathrm{Li}_2\left(\frac{1-z}{2}\right)(z-$$
$$1) + 8z(z+1)(7z+4)\mathrm{Li}_3(1-z)(z-1) + 4z(z+1)(7z+5)\mathrm{Li}_3(z)(z-1) - 16z(z^2+4)\mathrm{Li}_3\left(\frac{z+1}{2}\right)(z-1)+$$
$$4z(z+1)((4\pi b_0 + 3L_0 - 8)z^2 + 2(z(-8\log(1-z) + \log(z) + 2\log(z+1)) - 4(\log(z+1) - \log(1-z)))z - 3L_0+$$
$$\log((z-1)^8) - 2\log(z) + 4\log(z+1) - 4b_0\pi + 11)\mathrm{Li}_2(1-z) + 8(z^2-1)(z(2\log(2)(z-1) - z + 2) - 2(z-$$
$$1)z\log(1-z) + 3)\mathrm{Li}_2\left(\frac{z-1}{2z}\right) - 4((z-1)(z(z(15z-1) + 4b_0\pi(z^2-2) - 20) - 6) + 4z(\log(z)z^3 + \log(1-$$
$$z)z - \log(1-z) + \log(z) + (z(z+6)-5)\log(z+1)))\mathrm{Li}_2(-z) + 8z((z-1)(4z^2 - 2z + 6\log(1-z) - 6b_0\pi -$$
$$3) + 2(z^3+5z-4)\log(z+1))\mathrm{Li}_2\left(\frac{1}{z+1}\right) - 4(z^2-1)(z(5z-8) + 10(z-1)z\log(z+1) - 6)\mathrm{Li}_2\left(\frac{z}{z+1}\right)+$$
$$8(z^2-1)(z(2\log(2)(z-1) - z + 2) - 2(z-1)z\log(z+1) + 3)\mathrm{Li}_2\left(\frac{2}{z+1} - 1\right) - 8z^2(z^2-1)\mathrm{Li}_3\left(\frac{z-1}{z}\right) -$$
$$8z(z+1)(z(7z-16)+7)\mathrm{Li}_3(-z) + 16z(-3z^3+z^2+z-3)\mathrm{Li}_3\left(\frac{1}{z+1}\right) + z(z(z(61z-69)+59)+29)\zeta(3))\Big)$$

Similar results for the singlet and the photon PDFs

Recursive solution valid in the whole z range, but not good near $z = 1$.

Full non-singlet recursive solution up to $\mathcal{O}(\alpha^3)$

$$\hat{J}^{\rm LL}_{\rm NS,0} = 1 - z$$

$$\hat{J}^{\rm LL}_{\rm NS,1} = \frac{1}{z-1} \times \left(3z^2\log(z) + (1-z)(z+3) - 4(z-1)^2\log(1-z) + \log(z)\right)$$

$$\begin{aligned}\hat{J}^{\rm LL}_{\rm NS,2} = \frac{1}{4(z-1)} \times \Big(&-24\left(z^2-1\right)\mathrm{Li}_2(z)+24\left(-z^2+2\left(z^2+1\right)\log(z)+3\right)\log(1-z)-2\log(z)\Big(-9z^2+7z^2\log(z)+\\ &\log(z)-3\Big)+(z-1)\left(4\pi^2(3z-1)-3(5z+19)\right)-48(z-1)^2\log^2(1-z)-48z(\log(2-2z)-\log(2z))\Big)\end{aligned}$$

$$\hat{J}^{\rm NLL}_{\rm NS,0} = (1-z)(L_0 - 2\log(1-z) - 1)$$

$$\begin{aligned}\hat{J}^{\rm NLL}_{\rm NS,1} = \frac{1}{18b_0(z-1)(z+1)} \times \Big(&36\pi b_0^2 L_0 z^3 - 36\pi b_0^2 L_0 z^2 - 36\pi b_0^2 L_0 z + 36\pi b_0^2 L_0 - 36\pi b_0^2 z^3 - 72\pi b_0^2 z^3\log(1-\\ &z) + 36\pi b_0^2 z^2 + 72\pi b_0^2 z^2\log(1-z) + 36\pi b_0^2 z + 72\pi b_0^2 z\log(1-z) - 72\pi b_0^2\log(1-z) - 36\pi b_0^2 - 18b_0L_0z^3 -\\ &72b_0L_0z^3\log(1-z) + 54b_0L_0z^3\log(z) - 54b_0L_0z^2 + 72b_0L_0z^2\log(1-z) + 54b_0L_0z^2\log(z) + 18b_0L_0z +\\ &72b_0L_0z\log(1-z) + 18b_0L_0z\log(z) - 72b_0L_0\log(1-z) + 18b_0L_0\log(z) + 54b_0L_0 + 44b_0N_F z^3 +\\ &12b_0N_F z^3\log(z) - 68b_0N_F z^2 + 12b_0N_F z^2\log(z) - 44b_0N_F z + 12b_0N_F z\log(z) + 12b_0N_F\log(z) +\\ &68b_0N_F + 72b_0\left(z^3 - z^2 - 2z + 2\right)\mathrm{Li}_2(-z) + 216b_0 z\,\mathrm{Li}_2\left(\frac{1}{z+1}\right) - 36b_0(z-1)(z+1)^2\mathrm{Li}_2(1-z) -\\ &216b_0\mathrm{Li}_2\left(\frac{1}{z+1}\right) + 153b_0z^3 - 6\pi^2b_0z^3 + 108b_0z^3\log^2(1-z) - 27b_0z^3\log^2(z) + 54b_0z^3\log(1-z) -\\ &90b_0z^3\log(z) - 72b_0z^3\log(1-z)\log(z) + 72b_0z^3\log(z)\log(z+1) - 135b_0z^2 + 6\pi^2b_0z^2 - 108b_0z^2\log^2(1-\\ &z) + 9b_0z^2\log^2(z) + 18b_0z^2\log(1-z) - 90b_0z^2\log(z) - 72b_0z^2\log(1-z)\log(z) - 72b_0z^2\log(z)\log(z+\\ &1) - 153b_0z - 18\pi^2b_0z - 108b_0z\log^2(1-z) - 9b_0z\log^2(z) + 108b_0z\log^2(z+1) + 108b_0\log^2(1-z) +\\ &27b_0\log^2(z) - 108b_0\log^2(z+1) - 54b_0z\log(1-z) + 72b_0z\log(z) - 144b_0z\log(z)\log(z+1) - 18b_0\log(1-\\ &z) + 72b_0\log(z) + 144b_0\log(z)\log(z+1) + 135b_0 + 18\pi^2b_0 + 36\pi b_1z^3 - 36\pi b_1z^2 - 36\pi b_1 z + 36\pi b_1\Big)\end{aligned}$$

$$\begin{aligned}\hat{J}^{\rm NLL}_{\rm NS,2} = \frac{1}{36b_0z\left(z^2-1\right)} \times \Big(&(1-z)z\Big(144(L_0-1)\pi^2\left(z^2-1\right)b_0^3 - 8\pi\Big(18L_0(z+1)(z+3) - 2N_F(z+1)(11z-\\ &17)+3\left(3\left(-8+\pi^2\right)z^2-15z+\pi^2+9\right)\Big)b_0^2+\Big(2\Big(-2N_F\left(4\pi^2(z-1)-19z+63\right)(z+1)+9(57z-73)(z+\\ &1)+6\pi^2\left(z(17z+3)+12b_1\left(z^2-1\right)-2\right)\Big)-9L_0(z+1)\left(8\pi^2(z-1)-3(5z+19)\right)\Big)b_0-144b_1\pi(z+1)(z+\\ &3)\Big)+2\Big(6b_0\log^3(z)z^4-18b_0\log^2(z)z^4-63b_0L_0\log^2(z)z^4-36b_0N_F\log^2(z)z^4+54b_0^2\pi\log^2(z)z^4-\\ &216b_0^2\pi(\log(1-z)-\log(z))z^4-342b_0\log(z)z^4+81b_0L_0\log(z)z^4-156b_0N_F\log(z)z^4+\\ &72b_0\log^2(2)\log(z)z^4-72b_0\log(2)\log(z)z^4-18b_0\pi^2\log(z)z^4-216b_1\pi\log(z)z^4-216b_0^2L_0\pi\log(z)z^4-\\ &24b_0^2N_F\pi\log(z)z^4+18b_0\log^3(z)z^3+144b_0\log^2(z)z^3-63b_0L_0\log^2(z)z^3-36b_0N_F\log^2(z)z^3-\\ &18b_0^2\pi\log^2(z)z^3+234b_0\log(z)z^3+297b_0L_0\log(z)z^3-12b_0N_F\log(z)z^3-72b_0\log^2(2)\log(z)z^3+\\ &144b_0\log(2)\log(z)z^3+42b_0\pi^2\log(z)z^3+288b_0^2\pi\log(z)z^3-216b_1\pi\log(z)z^3-216b_0^2L_0\pi\log(z)z^3-\\ &24b_0^2N_F\pi\log(z)z^3-6b_0\log^3(z)z^2+54b_0\log^2(z)z^2-9b_0L_0\log^2(z)z^2-12b_0N_F\log^2(z)z^2+\\ &18b_0^2\pi\log^2(z)z^2+306b_0\log(z)z^2+243b_0L_0\log(z)z^2+68b_0N_F\log(z)z^2-72b_0\log^2(2)\log(z)z^2+\\ &288b_0\log(2)\log(z)z^2+66b_0\pi^2\log(z)z^2-36b_0^2\pi\log(z)z^2-72b_1\pi\log(z)z^2-72b_0^2L_0\pi\log(z)z^2-\\ &24b_0^2N_F\pi\log(z)z^2+288b_0(z-1)^2(z+1)\log^3(1-z)z-18b_0\log^3(z)z-48b_0(z-4)(z-1)(z+4)\log^3(z+1)z-\\ &144b_0\log^2(z)z-9b_0L_0\log^2(z)z-12b_0N_F\log^2(z)z-54b_0^2\pi\log^2(z)z-270b_0\log(z)z+27b_0L_0\log(z)z-\\ &76b_0N_F\log(z)z+72b_0\log^2(2)\log(z)z-144b_0\log(2)\log(z)z-42b_0\pi^2\log(z)z-108b_0^2\pi\log(z)z-\\ &72b_1\pi\log(z)z-72b_0^2L_0\pi\log(z)z-24b_0^2N_F\pi\log(z)z-36b_0(z+1)\log^2(1-z)\Big((6L_0(z-1)+12b_0\pi(z-1)-\\ &5z-7)(z-1)+\left(11z^2+1\right)\log(z)\Big)z+\log(1-z)\Big(144\pi^2(z-1)^2(z+1)b_0^3+72\pi\Big(4L_0(z+1)(z-1)^2-3z(z-1)+\\ &3\Big)b_0^2-(z-1)\Big(108L_0(z+1)(z+3)+32N_F((3-8z)z+11)+3\Big(-27z(7z+4)+4\pi^2\left(9z^2+z-6\right)+120\log^2(2)+\\ &81\Big)\Big)b_0+12\Big(3(z-1)(z(2z+5)+1)\log^2(z)+(z+1)\Big((30\pi b_0+27L_0+4N_F-30)z^2-24z+9L_0+4N_F+\\ &6b_0\pi+27\Big)\log(z)+24(z-1)(2\log(2)-\log(z))\log(z+1)\Big)b_0+288b_1\pi(z-1)^2(z+1)\Big)z-108b_0\log^2(z)-\\ &18b_0\Big(-16\log(2)z^4+8(\log(z)+2\log(2))z^3+4(15z-13)\log(z)z+8(z-1)(\log(1-z)+\log(z)+2\log(2))z+\\ &(z-1)(z(12\pi b_0+z(14z+29)-23)-12)\Big)\log^2(z+1)+12b_0\log(2)\Big(-4\pi^2z\left(z^3+2z-1\right)+(z-1)(z+\\ &1)(z(9z+7)+3)\log(8)+2z\left(z\left(4z^2+z+16\right)-11\right)\log^2(2)\Big)-216b_0\log(2)\log(z)-12b_0\Big(6z(z+1)(z(2z-\\ &3)+2)\log^2(z)+6(z-1)z\left(-4z+2b_0\pi\left(z^2-2\right)-3\right)\log(z)+(z-1)\Big(12z\log^2(2)\left(z^2+4\right)+\pi^2\left(z^3+z\right)+\\ &6\Big(z\left((-1+2\log(2))z^2+z+5\right)-2z(\log(1-z)+\log(2))+3\Big)(\log(z)+\log(2))\Big)\Big)\log(z+1)\Big)+18b_0\Big(-\\ &16z(z+1)\mathrm{Li}_3\left(\frac{1-z}{2}\right)(z-1)^2+16z(z+1)\mathrm{Li}_3\left(\frac{z-1}{2z}\right)(z-1)^2-56z(z+1)\mathrm{Li}_3\left(\frac{z}{z+1}\right)(z-1)^2+16z(z+\\ &1)\mathrm{Li}_3\left(\frac{2z}{z+1}\right)(z-1)^2-16z\Big((\log(z+1)-\log(1-z))z^2+\log(1-z)-\log(z+1)-5\log(2)\Big)\mathrm{Li}_2\left(\frac{1-z}{2}\right)(z-\\ &1)+8z(z+1)(7z+4)\mathrm{Li}_3(1-z)(z-1)+4z(z+1)(7z+5)\mathrm{Li}_3(z)(z-1)-16z\left(z^2+4\right)\mathrm{Li}_3\left(\frac{z+1}{2}\right)(z-1)+\\ &4z(z+1)\Big((4\pi b_0+3L_0-8)z^2+2(z(-8\log(1-z)+\log(z)+2\log(z+1))-4(\log(z+1)-\log(1-z)))z-3L_0+\\ &\log\left((z-1)^8\right)-2\log(z)+4\log(z+1)-4b_0\pi+11\Big)\mathrm{Li}_2(1-z)+8\left(z^2-1\right)(z(2\log(2)(z-1)-z+2)-2(z-\\ &1)z\log(1-z)+3)\mathrm{Li}_2\left(\frac{z-1}{2z}\right)-4\Big((z-1)\left(z\left(z(15z-1)+4b_0\pi\left(z^2-2\right)-20\right)-6\right)+4z\Big(\log(z)z^3+\log(1-\\ &z)z-\log(1-z)+\log(z)+(z(z+6)-5)\log(z+1)\Big)\Big)\mathrm{Li}_2(-z)+8z\Big((z-1)\Big(4z^2-2z+6\log(1-z)-6b_0\pi-\\ &3\Big)+2\left(z^3+5z-4\right)\log(z+1)\Big)\mathrm{Li}_2\left(\frac{1}{z+1}\right)-4\left(z^2-1\right)(z(5z-8)+10(z-1)z\log(z+1)-6)\mathrm{Li}_2\left(\frac{z}{z+1}\right)+\\ &8\left(z^2-1\right)(z(2\log(2)(z-1)-z+2)-2(z-1)z\log(z+1)+3)\mathrm{Li}_2\left(\frac{2}{z+1}-1\right)-8z^2\left(z^2-1\right)\mathrm{Li}_3\left(\frac{z-1}{z}\right)-\\ &8z(z+1)(z(7z-16)+7)\mathrm{Li}_3(-z)+16z\left(-3z^3+z^2+z-3\right)\mathrm{Li}_3\left(\frac{1}{z+1}\right)+z(z(z(61z-69)+59)+29)\zeta(3)\Big)\Big)\end{aligned}$$

Similar results for the singlet and the photon PDFs

Recursive solution valid in the whole z range, but not good near $z=1$.

Asymptotic solutions

Key fact: the large-z region corresponds to the large-N region in the Mellin space. Then:

- calculation of E_N in the large-N region;
- analytical Mellin inverse transform: $\Gamma(z,\mu^2) = M^{-1}[E_N\,\Gamma_{0,N}]$.

LL solution for non-singlet [Gribov and Lipatov (1972)]

$$P_N^{[0]} \xrightarrow{N\to\infty} -2\log\bar{N} + 2\lambda_0\,, \quad \bar{N} = N\,e^{\gamma_{\mathrm{E}}}, \quad \lambda_0 = \frac{3}{4} \tag{11}$$

$$\Gamma^{\mathrm{LL}}(z,\mu^2) = \frac{e^{-\gamma_{\mathrm{E}}\eta_0}\,e^{\lambda_0\eta_0}}{\Gamma(1+\eta_0)}\,\eta_0(1-z)^{-1+\eta_0}, \quad \eta_0 = \frac{\alpha}{\pi}\log\frac{\mu^2}{\mu_0^2} \tag{12}$$

[α is supposed as fixed here (since at LL we are entitled to neglect it)]
We are resumming the $\log(1-z)/(1-z)$ divergent terms to all order in α

NLL solution for non-singlet

Convenient to perform the convolution with initial condition in the z-space:

$$\Gamma^{\rm NLL}(z,\mu^2) = \left(\delta(1-x) + \frac{\alpha(\mu_0^2)}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-x) - 1\right)\right]_+\right) \otimes_z \; M^{-1}\big[\exp\big(\log E_N\big)\big] \tag{13}$$

By exploiting:

$$P_N^{[1]} \overset{N\to\infty}{\longrightarrow} \frac{20}{9}N_F \log\bar{N} + \lambda_1\,, \quad \lambda_1 = \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{N_F}{18}(3+4\pi^2) \tag{14}$$

one obtains:

$$M^{-1}\big[\exp\big(\log E_N\big)\big] = \frac{e^{-\gamma_{\rm E}\xi_1}e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\,\xi_1(1-z)^{-1+\xi_1} \tag{15}$$

Same structure of the LL result, with $\xi_1 = 2t + \mathcal{O}(\alpha^2)$ and $\hat{\xi}_1 = \frac{3}{2}t + \mathcal{O}(\alpha^2)$.

After having performed the convolution we obtain:

$$\Gamma^{\rm NLL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)} \xi_1 (1-z)^{-1+\xi_1} \times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(\log\frac{\mu_0^2}{m^2} - 1 \right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} + \left(\log\frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1-z) - \log^2(1-z) \right] \right\} \tag{16}$$

with:

$$A(\xi_1) = \frac{1}{\xi_1} + \mathcal{O}(\xi_1)\,, \quad B(\xi_1) = -\frac{\pi^2}{6} + 2\zeta_3\xi_1 + \mathcal{O}(\xi_1^2) \tag{17}$$

- NLL still very peaked towards $z = 1$, with behavior worse than LL
- if $\mu_0 \simeq m_e$ and $\mu \simeq 100$ GeV, then $\xi_1 \simeq 0.05$
  $\rightarrow$ the $\log(1-z)$ term is much larger than the $\log^2(1-z)$ one, even for z values *extremely* close to one.

Singlet and photon cases

Dominant term of the splitting matrices in the large-N region are:

$$\mathbb{P}_{\mathrm{S},N} \xrightarrow{N\to\infty} \begin{pmatrix} -2\log\bar{N} + 2\lambda_0 & 0 \\ 0 & -\frac{2}{3}N_F \end{pmatrix} + \frac{\alpha}{2\pi}\begin{pmatrix} \frac{20}{9}N_F\log\bar{N} + \lambda_1 & 0 \\ 0 & -N_F \end{pmatrix} + \mathcal{O}(\alpha^2) \tag{18}$$

This is a diagonal matrix → independent evolution

- Singlet solution = non-singlet solution (i.e. the mixing with the photon does not affect the electron large-z behaviour)
- Photon solution:

$$\Gamma_\gamma(z,\mu^2) = \frac{\alpha(\mu_0)}{\alpha(\mu)}\left[\frac{\alpha(\mu_0)}{2\pi}\frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right)\right] \tag{19}$$

Unfortunately, eq. (19) does not work . . . here mixing is important!

Improvement of photon large-z PDF

Solving the evolution equations by including off-diagonal elements implies a significant increase in complexity. Main idea: solve the matrix differential equation by treating the off-diagonal subdominant terms ($\mathcal{O}(1/N)$) as a “perturbation” of the “LO” diagonal result ($\log \bar{N}$ and constants):

$$\mathbb{E}_N(t) = \mathbb{E}_N^{(0)}(t)\, \mathbb{E}_N^{(1)}(t).$$

Then convolve with initial conditions and perform the Mellin anti-trasform. The final result is rather involved, but dominant terms in the $z \to 1$ limit are:

$$\Gamma_\gamma(z) \stackrel{z\to 1}{\longrightarrow} \frac{\alpha(\mu_0)}{\alpha(\mu)} \left[\left(\frac{\alpha(\mu_0)}{2\pi} \right) \frac{3}{\xi_{1,0}} \log(1-z) - \left(\frac{\alpha(\mu_0)}{2\pi} \right)^2 \frac{1}{2\,\xi_{1,0}} \log^3(1-z) \right]$$

where $\xi_{1,0} = 2 + \mathcal{O}(\alpha)$. Formally dominant term suppressed w.r.t the subdominant one by a factor proportional to α.

Consistency checks

Once we have both the recursive and the asymptotic solutions:

Expansion of the asymptotic solutions should reproduce order by order the most singular terms of the recursive solutions.

This is indeed the case, the expansion generate:

- ✓ non-singlet, singlet: all the $\dfrac{\log(1-z)}{1-z}$ terms;
- ✓ photon: all the $\log(1-z)$ terms and the constant terms (i.e. all the non-vanishing terms in the $z \to 1$ limit).

In particular, for $\mu \to \mu_0$ they asymptotic solutions reproduce the initial conditions (this has been verified at the distributional level).

Matching

Combine the recursive and the asymptotic solution by means of an additive formula:

$$\Gamma_{\text{mtc}}(z) = \Gamma_{\text{rec}}(z) + \Big(\Gamma_{\text{asy}}(z) - \Gamma_{\text{subt}}(z)\Big)G(z)\,, \quad \lim_{z\to 1} G(z) = 1 \tag{20}$$

Choice of subtraction term Γ_{subt} and matching function G dictated by:

$$\Gamma_{\text{mtc}} \sim \Gamma_{\text{asy}} \quad z \simeq 1 \tag{21}$$

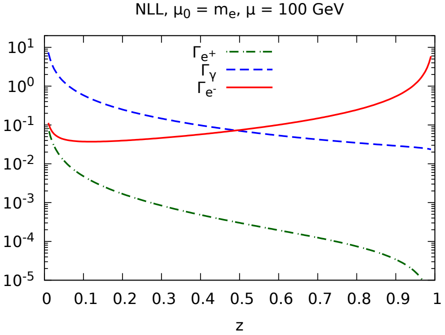
$$\Gamma_{\text{mtc}} \sim \Gamma_{\text{rec}} \quad \text{small- and intermediate-}z \tag{22}$$

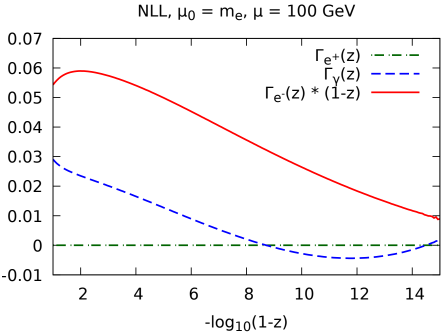
After technical studies:

- Γ_{subt} chosen as $\mathcal{O}(\alpha^3)$ expansion of Γ_{asy}
- different strategy for G:
  - NS/S: $G(z) \equiv 1$ ($\Gamma_{\text{asy}}(z) - \Gamma_{\text{subt}}(z)$ cancel very well in the small-z region)
  - γ: non trivial G needed (Γ_{asy} problematic in the small-z region)! $G(\hat{z}_0, \hat{z}_1, p)$ (transition between Γ_{rec} and Γ_{asy} in the region $\hat{z}_0 < -\log_{10}(1-z) < \hat{z}_1$ with p used to adjust the abruptness of the transition)

Outline

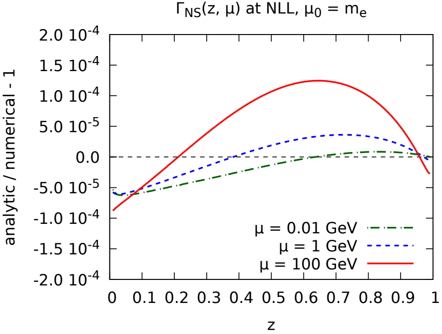
NLL electron PDFs

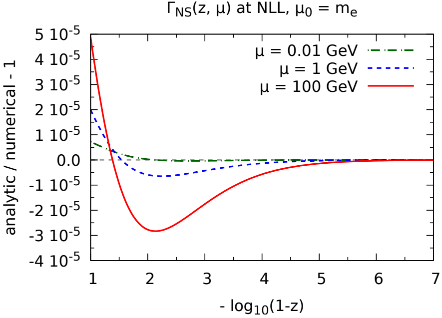




- Electron still dominates at large z, while photon at small z (however remember the constraint $z_+ z_- \geq M^2/s$)
- At large z, Γ_γ is smaller than Γ_{e^-} by $[-\log_{10}(1-z)]$ orders of magnitude.

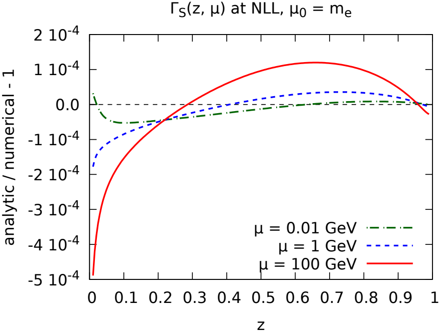
Analytical vs. numerical

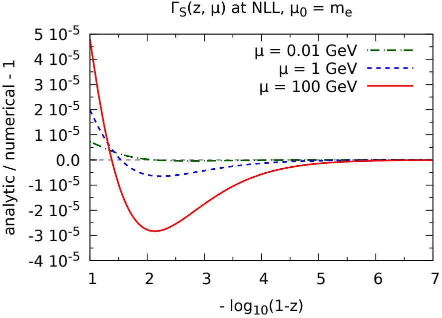




- Worst-case scenario ($\mu = 100$ GeV): agreement at the $10^{-4} - 10^{-5}$ level
- On the linear scale, largest discrepancy at small z's for the singlet
- Photon problematic on the log scale, but small in absolute value

Analytical vs. numerical





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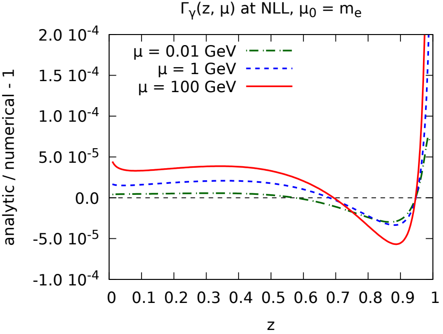

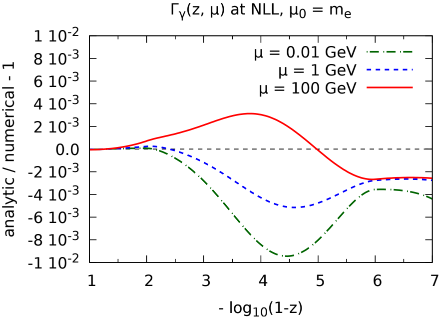

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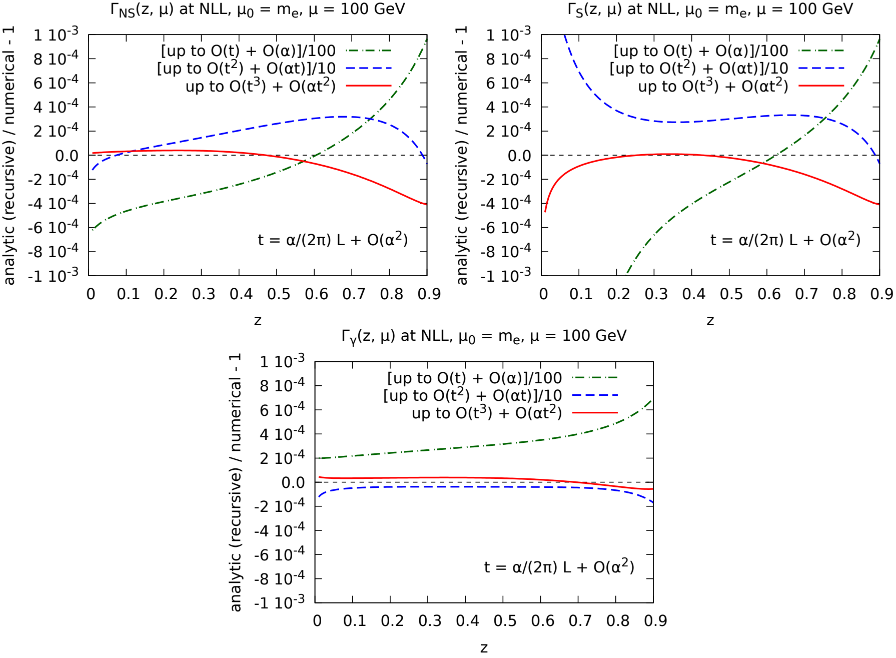
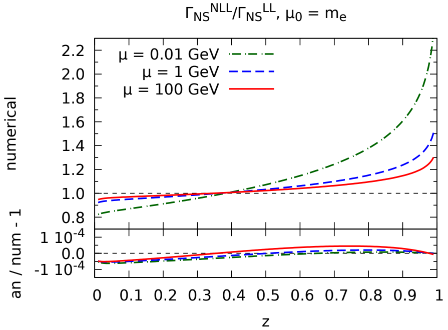
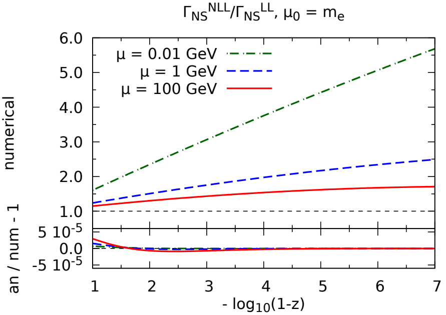

$\Gamma_{NS}(z, \mu)$ at NLL, $\mu_0 = m_e$, μ = 100 GeV
$\Gamma_S(z, \mu)$ at NLL, $\mu_0 = m_e$, μ = 100 GeV
$\Gamma_\gamma(z, \mu)$ at NLL, $\mu_0 = m_e$, μ = 100 GeV
analytic (recursive) / numerical - 1
[up to O(t) + O(α)]/100
[up to O(t^2) + O(αt)]/10
up to O(t^3) + O(αt^2)
t = α/(2π) L + O(α^2)
z

NLL vs. LL





$$\Gamma^{\mathrm{NLL}}_{\mathrm{NS/S}}(z,\mu^2) \sim \mathrm{LL}\left(1 + \frac{\alpha(\mu_0)}{\pi}\left[a + \frac{b}{\alpha(\mu)\log(\mu^2/\mu_0^2)}\log(1-z) - \log^2(1-z)\right]\right)$$

$$\text{Insets:}\quad \left(\frac{\mathrm{PDF}_{\mathrm{NLL}}}{\mathrm{PDF}_{\mathrm{LL}}}\right)_{\mathrm{an}} \Bigg/ \left(\frac{\mathrm{PDF}_{\mathrm{NLL}}}{\mathrm{PDF}_{\mathrm{LL}}}\right)_{\mathrm{num}} - 1$$

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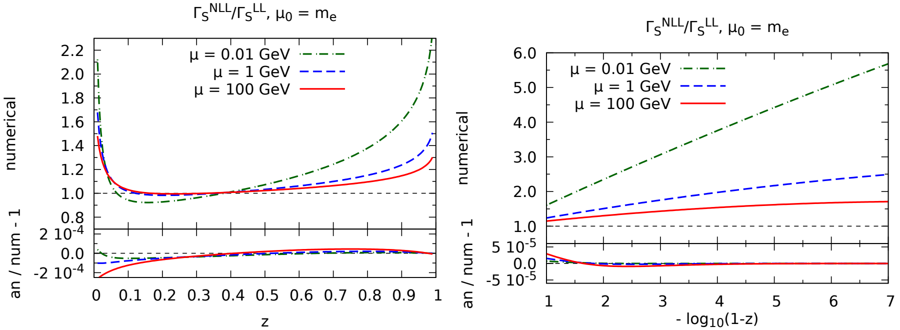

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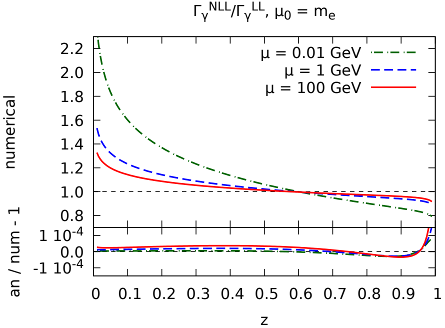

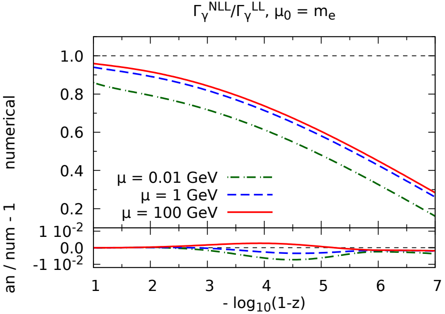

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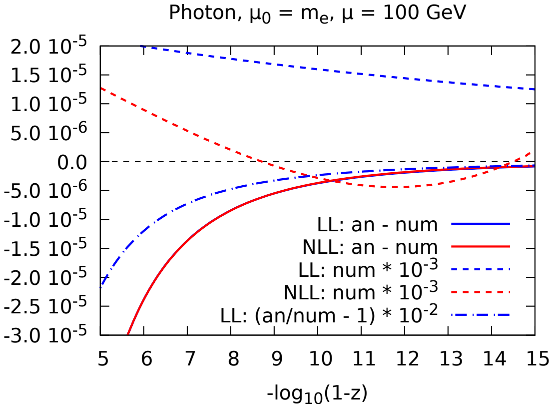
Asymptotic photon behaviour

$$\Gamma_\gamma(z) \xrightarrow{z\to1} \frac{\alpha(\mu_0)}{\alpha(\mu)}\left[\left(\frac{\alpha(\mu_0)}{2\pi}\right)\frac{3}{\xi_{1,0}}\log(1-z) - \left(\frac{\alpha(\mu_0)}{2\pi}\right)^2\frac{1}{2\,\xi_{1,0}}\log^3(1-z)\right]$$



Onset of the true asymptotic regime occurs at much larger z values!

Outline

Conclusions

- We computed the electron, positron and photon PDFs of the unpolarised e^- at NLL (and by charge conjugation also the ones of the incoming e^+). By similar techniques one can deal with the PDFs of the photon.
- They are obtained by means of both numerical and analytical methods, which agree extremely well in the region relevant for phenomenology.
- Analytical results stem from an additive matching between a recursive solution and an asymptotic $z \to 1$ one.
- At NLL the large-z peak is even more pronunced than at LL $\to$ this is in part an artefact of the $\overline{\text{MS}}$ scheme and in future works will explore the adoption of alternative subtraction schemes
- NLL PDFs are the first ingredients towards a full NLO framework for theoretical predictions for e^+e^- collider. Next step: what is the impact of NLL PDFs on observables?

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